Problem 1  [6 points] Given \( r'(t) = \langle 4e^t, 9\cos(3t), 5t \rangle \) and \( r(0) = \langle 1, 2, 0 \rangle \), answer the following.

a) Find \( r(t) \).

Solution:

\[
\begin{align*}
\mathbf{r}(t) &= \left\langle \int 4e^t \, dt, \int 9\cos(3t) \, dt, \int 5t \, dt \right\rangle \\
&= \left\langle 4e^t + C_1, 3\sin(3t) + C_2, \frac{5t^2}{2} + C_3 \right\rangle \\
\end{align*}
\]

Since \( \mathbf{r}(0) = \langle 1, 2, 0 \rangle \), we have

\[
\langle 1, 2, 0 \rangle = \mathbf{r}(0) = \left\langle 4e^0 + C_1, 3\sin(3 \cdot 0) + C_2, \frac{5 \cdot 0^2}{2} + C_3 \right\rangle = \langle 4 + C_1, C_2, C_3 \rangle.
\]

Thus \( C_1 = -3 \), \( C_2 = 2 \), and \( C_3 = 0 \). So

\[
\mathbf{r}(t) = \left\langle 4e^t - 3, 3\sin(3t) + 2, \frac{5t^2}{2} \right\rangle.
\]

b) Find an equation of the line tangent to the curve described by \( \mathbf{r}(t) \) at \( t = 0 \). [Note: The line tangent to \( \mathbf{r}(t) \) at \( t = t_0 \) is the line parallel to the tangent vector \( \mathbf{r}'(t_0) \) that passes through \( \mathbf{r}(t_0) \). You may represent your line either using a vector-valued function or using parametric equations.]

Solution: To find the tangent line, we need a point and a direction. The point is given in the statement of part a) as \( \mathbf{r}(0) = \langle 1, 2, 0 \rangle \). Additionally \( \mathbf{r}'(0) = \langle 4e^0, 9\cos(3 \cdot 0), 5 \cdot 0 \rangle = \langle 4, 9, 0 \rangle \). Thus our tangent line is given by

\[
R(t) = \langle 1, 2, 0 \rangle + t\langle 4, 9, 0 \rangle.
\]
Problem 1 [4 points] Consider the curve described by \( \mathbf{r}(t) = \langle a \cos t, b \cos t, c \sin t \rangle \) where \( a, b, c > 0 \) and \( a^2 + b^2 = c^2 \). Find the following and simplify your answers.

a) \( | \mathbf{r}(t) | \).

Solution:

\[
| \mathbf{r}(t) | = \sqrt{(a \cos t)^2 + (b \cos t)^2 + (c \sin t)^2} \\
= \sqrt{(a^2 + b^2) \cos^2 t + c^2 \sin^2 t} \\
= \sqrt{a^2 \cos^2 t + b^2 \cos^2 t} \\
= c.
\]

b) \( \mathbf{r'}(t) \).

Solution:

\[
\mathbf{r'}(t) = \langle -a \sin t, -b \sin t, c \cos t \rangle.
\]

c) \( \mathbf{r}(t) \cdot \mathbf{r'}(t) \).

Solution:

\[
\mathbf{r}(t) \cdot \mathbf{r'}(t) = \langle a \cos t, b \cos t, c \sin t \rangle \cdot \langle -a \sin t, -b \sin t, c \cos t \rangle \\
= -a^2 \cos t \sin t - b^2 \cos t \sin t + c^2 \cos t \sin t \\
= (a^2 + b^2) \cos t \sin t \\
= 0.
\]

d) \( \mathbf{r}(t) \times \mathbf{r'}(t) \).

Solution:

\[
\mathbf{r}(t) \times \mathbf{r'}(t) = \langle a \cos t, b \cos t, c \sin t \rangle \times \langle -a \sin t, -b \sin t, c \cos t \rangle \\
= \langle bc \cos^2 t + bc \sin^2 t, -(ac \cos^2 t + ac \sin^2 t), -ab \sin t \cos t + ab \sin t \cos t \rangle \\
= \langle bc, -ac, 0 \rangle.
\]

e) \( | \mathbf{r}(t) \times \mathbf{r'}(t) | \).

Solution:

\[
| \mathbf{r}(t) \times \mathbf{r'}(t) | = \sqrt{(bc)^2 + (-ac)^2 + 0^2} \\
= c \sqrt{a^2 + b^2}.
\]

f) \( \frac{d}{dt} | \mathbf{r}(t) \times \mathbf{r'}(t) | \).

Solution:

\[
\frac{d}{dt} | \mathbf{r}(t) \times \mathbf{r'}(t) | = \frac{d}{dt} c \sqrt{a^2 + b^2} \\
= 0.
\]