Mass of an linear object of variable density: Suppose we have a thin strait object like a bar or a wire with variable density. Further suppose that we can model the object by a line segment on the interval $a \leq x \leq b$ with density function $\rho(x)$. Then the mass of the object is

$$\text{Mass} = \int_{a}^{b} \rho(x) \, dx.$$ 

1. Find the mass of the thin bar with density function $\rho(x) = \begin{cases} 1 & 0 \leq x \leq 2 \\ 1 + x & 2 < x \leq 4. \end{cases}$

2. Two bars of length $L$ have densities $\rho_1(x) = 5e^{-x}$ and $\rho_2(x) = 7e^{-2x}$ (Assume that the ends of the bars are at $x = 0$ and $x = L$). For what values of $L$ is the first bar heavier? As the lengths increase, do their masses increase without bound?

3. Find the mass of the thin bar over the interval $0 \leq x \leq \pi$ with the density function $\rho(x) = \frac{4x}{3 + x^2}$.

Work done over varying force: Let us move an object from $x = a$ to $x = b$ in the presence of a force dependent on $x$, $F(x)$. Then the total work done is

$$\text{Work} = \int_{a}^{b} F(x) \, dx.$$ 

Hooke’s Law: The force required to keep a spring compressed or stretched in a position $x$ units from the equilibrium position is $F(x) = kx$.

4. (Test Question) True or false: Given a spring that obeys Hooke’s Law, the work required to stretch the spring from equilibrium to 1 cm is the same as the work required to stretch the spring from 1 cm to 2 cm.
5. How much work is required to move an object from \( x = 1 \) to \( x = 3 \) in the presence of a force (in N) given by \( F(x) = \frac{2}{x^2} \) acting along the \( x \)-axis.

6. Calculate the work required to stretch a spring 0.5 m from its equilibrium position if it requires a force of 50N to hold the spring in a stretched position 0.2 m from its equilibrium position.

**Emptying tanks:** Suppose a liquid of density \( \rho \) in kg/m\(^3\) is in a tank s.t. the area of horizontal cross sections is given by \( A(y) \) on the interval \([a, b]\) (here \( y \) represents the height of the horizontal cross section). Additionally assume that the horizontal cross section at a height \( y \) is is lifted a distance \( D(y) \). Then the work done lifting all of the water is

\[
W = \int_{a}^{b} \rho \cdot g \cdot A(y)D(y)dy,
\]

where \( g \approx 9.8 \) is the force of gravity.

7. A water tank is shaped like an inverted cone with a height 6 m and a base radius 1.5. If the tank is full, how much work is required to pump the water to the level of the top of the tank? Second, is it true that it takes half as much work to pump the water out of the tank when it is filled to half its depth as when it is full?
8. A water trough has a semicircular vertical cross section with a radius of .25 m and a length of 3m. How much work is required to pump the water out when it is full?

9. (Test Question) A large vase is constructed by revolving the portion of the graph $x = 2y^2 + 4y$, $0 \leq y \leq 6$ around the $y$-axis. (Assume that distances are measured in meters.) Find the total volume that the vase can hold. Next find the work required to pump all of the water in the vase over the top when the vase is filled to half its depth.
10. (Test Question) A tank is shaped like a parabolic bowl. It is formed by revolving the graph of $y = 4x^2$ for $0 \leq x \leq 3$ (in meters) about the $y$-axis. The tank is filled with water to a height of 30 meters. How much work is required to pump all of the water to an exit pipe at the top of the tank? [Note: The density of water is 1000 $kg/m^3$.]

11. A 60-m long, 9.4 mm diameter rope hangs freely from a ledge. The density of the rope is 55g/m. How much work is needed to lift the entire rope to the ledge?