Problem 1  [6 points] Evaluate \[ \int \frac{-x^2 + 3x + 4}{x^3 + 2x^2 + 2x} \, dx. \]

**Solution:** The denominator factors as \( x(x^2 + 2x + 2). \) We know \( x^2 + 2x + 2 \) is irreducible since \( 2^2 - 4 \cdot 1 \cdot 2 = -4 < 0. \) Thus the general form of the partial fraction decomposition for \( \frac{-x^2 + 3x + 4}{x^3 + 2x^2 + 2x} \) is

\[
\frac{-x^2 + 3x + 4}{x^3 + 2x^2 + 2x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 2x + 2},
\]

where \( A, B, C \) are unknown constants. After multiplying through by the denominator, we have that

\[
-x^2 + 3x + 4 = A(x^2 + 2x + 2) + (Bx + C)x.
\]

By plugging in \( x = 0, \) we have

\[
-0^2 + 3 \cdot 0 + 4 = A(0^2 + 2 \cdot 0 + 2) + (B \cdot 0 + C)0.
\]

In other words \( 4 = 2A \) and \( A = 2. \) In order to find \( B \) and \( C \) we will need to pass to a system of linear equations. Here is the expanded version of the equation relating \( A, B, \) and \( C. \)

\[
-x^2 + 3x + 4 = Ax^2 + 2Ax + 2A + Bx^2 + Cx.
\]

From here we deduce that

\[
\begin{align*}
(x^2) & \quad -1 \quad = \quad A + B \\
(x) & \quad 3 \quad = \quad 2A + C \\
(1) & \quad 4 \quad = \quad 2A.
\end{align*}
\]

So \( B = -3 \) and \( C = -1. \) Thus

\[
\int \frac{-x^2 + 3x + 4}{x^3 + 2x^2 + 2x} \, dx = \int \frac{2}{x} + \frac{-3x - 1}{(x + 1)^2 + 1} \, dx
\]

\[
= \int \frac{2}{x} + \frac{-3x - 3}{x^2 + 2x + 2} + \frac{2}{x^2 + 2x + 2} \, dx
\]

\[
= 2 \ln |x| - \frac{3}{2} \ln |x^2 + 2x + 2| + 2 \arctan(x + 1) + C.
\]
Problem 1  [2 points]  Evaluate the following improper integral or show that it diverges. Assume $b > a > 0$ so that the function is bounded on the interval $[b, \infty)$.

\[ \int_b^\infty \frac{1}{x^2 - a^2} \, dx \]

**Solution:** Let us perform partial fraction decomposition on $\frac{1}{x^2 - a^2}$. The Denominator factors as $(x - a)(x + a)$. Thus there exists constants $A$ and $B$ such that

\[ \frac{1}{x^2 - a^2} = \frac{A}{x - a} + \frac{B}{x + a}. \]

After multiplying through

\[ 1 = A(x + a) + B(x - a). \]

By plugging in $a$ and $-a$ into the equation we have that $A = 1/(2a)$ and $B = -1/(2a)$. Thus

\[ \int \frac{1}{x^2 - a^2} = \int \frac{1/(2a)}{x - a} + \frac{-1/(2a)}{x + a} = 1/(2a) (\ln(x - a) - \ln(x + a)) = 1/(2a) \ln \left( \frac{x - a}{x + a} \right). \]

So

\[ \int_b^\infty \frac{1}{x^2 - a^2} \, dx = \lim_{t \to \infty} \int_b^t \frac{1}{x^2 - a^2} \, dx \]

\[ = \lim_{t \to \infty} 1/(2a) \ln \left( \frac{x - a}{x + a} \right) \bigg|_b^t \]

\[ = \lim_{t \to \infty} 1/(2a) \ln \left( \frac{t - a}{t + a} \right) - 1/(2a) \ln \left( \frac{b - a}{b + a} \right) \]

\[ = 1/(2a) \ln \left( \lim_{t \to \infty} \frac{t - a}{t + a} \right) - 1/(2a) \ln \left( \frac{b - a}{b + a} \right) \]

\[ = 1/(2a) \ln (1) - 1/(2a) \ln \left( \frac{b - a}{b + a} \right) \]

\[ = -1/(2a) \ln \left( \frac{b - a}{b + a} \right). \]

Problem 2  [2 points]  Sequences and Series. Consider the sequence given by the follow-
ing recurrence relation.

\[ a_{n+1} = \frac{1}{3}a_n + 4; \ a_1 = 12 \]

a) Find the first four terms of the sequence.

**Solution:**

\[ a_2 = \frac{1}{3} \times 12 + 4 = 8, \]
\[ a_3 = \frac{1}{3} \times 8 + 4 = \frac{20}{3}, \]
\[ a_3 = \frac{120}{3} + 4 = \frac{56}{9}. \]

b) Consider the infinite series \( \sum_{k=1}^{\infty} a_k \), where the terms \( a_k \) are as in the sequence given above.

Find the first four terms of the sequence of **partial sums** for the series.

**Solution:**

\[ S_1 = a_1 = 12 \]
\[ S_2 = a_1 + a_2 = 12 + 8 \]
\[ S_3 = a_1 + a_2 + a_3 = 12 + 8 + \frac{20}{3} \]
\[ S_4 = a_1 + a_2 + a_3 + a_4 = 12 + 8 + \frac{20}{3} + \frac{56}{9} \]