Problem 1  [6 points]  Find the area of the entire region that is bounded by the graphs of \( y = x^2 \) and \( y = x^3 + x^2 - 4x \). Give the exact answer (you do not need to simplify the numerical value.)

\[
\begin{align*}
\text{Intersection points: } x^2 &= x^3 + x^2 - 4x \\
0 &= x^3 - 4x = x(x - 2)(x + 2).
\end{align*}
\]

Thus we have intersection points at \( x = -2, 0, 2 \).

At \( x = 1 \),
\[
1^2 = 1 \quad \text{and} \quad 1^3 + 1^2 - 4(1) = -2.
\]

Thus \( x^2 \) is the top function the interval \([0, 2]\). At \( x = -1 \),
\[
(-1)^2 = 1 \quad \text{and} \quad (-1)^3 + (-1)^2 - 4(-1) = 4.
\]

So \( x^3 + x^2 - 4x \) is the top function on the interval \([-2, 0]\). Thus

\[
\begin{align*}
\text{Area} &= \int_{-2}^{0} x^3 + x^2 - 4x - x^2 \, dx + \int_{0}^{2} x^2 - (x^3 + x^2 - 4x) \, dx \\
&= \int_{-2}^{0} x^3 - 4x \, dx + \int_{0}^{2} -x^3 + 4x \, dx \\
&= \left. \frac{x^4}{4} - 2x^2 \right|_{-2}^{0} + \left. -\frac{x^4}{4} + 2x^2 \right|_{0}^{2} \\
&= \left( \frac{0^4}{4} - 2(0)^2 \right) - \left( \frac{(-2)^4}{4} - 2(-2)^2 \right) + \left( \frac{-2^4}{4} + 2(2)^2 \right) - \left( \frac{-0^4}{4} + 2(0)^2 \right) \\
&= 8.
\end{align*}
\]
Problem 1  [4 points]  Consider the shaded region $R$ in the image below.

![Diagram of the shaded region](image)

a) Set up an integral that represents the area of the region $R$ using the $y$-axis.

**Solution:** Let us solve the first equation to be a function of $y$.

\[ x = \pm \sqrt{4 - y}. \]

There are two branches to this square root and the one of interest is the positive one corresponding to $x = \sqrt{4 - y}$. Thus

\[ \text{Area} = \int_{-1}^{4} (y - 2)^2 + 2 - \sqrt{4 - y} \, dy. \]

b) Set up a sum of integrals that represents the area of the region $R$ using the $x$-axis.

**Solution:** Let us start by calculating the intersections of our curves: The curves $y = -1$ and $x = \sqrt{4 - y}$ intersect at $(\sqrt{5}, -1)$. The intersection of $y = -x^2 + 4$ and $y = 4$ occurs at $(0, 4)$. The intersection of $y = 4$ and $x = (y - 2)^2 + 2$ occurs at $(6, 4)$. Finally, $y = -1$ and $x = (y - 2)^2 + 2$ intersect at $(11, -1)$.

The curve $x = (y - 2)^2 + 2$ can be expressed as the union of two functions of $x$. The upper branch is given by $y = \sqrt{x - 2} + 2$, and the lower branch is given by $y = -\sqrt{x - 2} + 2$. 
The left most $x$ value for which these functions are defined is $x = 2$. We then add the orange line which is $x = 2$ and the black line which is $x = \sqrt{5}$ to decompose the region into 4 smaller regions:

**Area of $R_1$:** The region $R_1$ corresponds to the region bounded by $y = 4$, $y = -x^2 + 4$ and $x = 2$. Thus its area is given by the following integral

$$I_1 = \int_0^2 4 - (-x^2 + 4) \, dx.$$

**Area of $R_2$:** The region $R_2$ corresponds to the region bounded by $x = \sqrt{5}$, $x = 2$, $y = -x^2 + 4$, and $y = -\sqrt{x - 2} + 2$. Thus its area is given by the following integral

$$I_2 = \int_2^{\sqrt{5}} -\sqrt{x - 2} + 2 - (-x^2 + 4) \, dx.$$

**Area of $R_3$:** The region $R_3$ corresponds to the region bounded by $x = 2$, $y = 4$, and $y = \sqrt{x - 2} + 2$. Thus its area is given by the following integral

$$I_3 = \int_2^6 4 - (\sqrt{x - 2} + 2) \, dx.$$

**Area of $R_4$:** The region $R_4$ corresponds to the region bounded by $x = \sqrt{5}$, $y = -1$, and $y = -\sqrt{x - 2} + 2$. Thus its area is given by the following integral

$$I_4 = \int_{\sqrt{5}}^{11} -\sqrt{x - 2} + 2 - (-1) \, dx.$$

Thus the volume is the sum of the four integrals $I_1$, $I_2$, $I_3$, and $I_4$. 

3
c) Compute and compare the values for the area of region $R$ from parts a and b.

**Solution:** First let’s integrate over $y$:

\[
\text{Area} = \int_{-1}^{4} (y - 2)^2 + 2 - \sqrt{4 - y} \, dy
\]

\[
= \int_{-1}^{4} y^2 - 4y + 4 + 2 - (4 - y)^{\frac{1}{2}} \, dy
\]

\[
= \left[ \frac{y^3}{3} - 2y^2 + 4y + 2y + \frac{2}{3}(4 - y)^{\frac{3}{2}} \right]_{-1}^{4}
\]

\[
= \left( \frac{(4)^3}{3} - 2(4)^2 + 4(4) + 2(4) + \frac{2}{3}(4 - (4))^{\frac{3}{2}} \right) - \left( \frac{(-1)^3}{3} - 2(-1)^2 + 4(-1) + 2(-1) + \frac{2}{3}(4 - (-1))^{\frac{3}{2}} \right)
\]

\[
\approx 14.2131067417.
\]

Now integrating over $x$

\[
\text{Area} = I_1 + I_2 + I_3 + I_4
\]

\[
= \int_{0}^{2} 4 - (-x^2 + 4) \, dx + \int_{2}^{\sqrt{5}} -\sqrt{x - 2} + 2 - (-x^2 + 4) \, dx
\]

\[
+ \int_{2}^{6} 4 - (\sqrt{x - 2} + 2) \, dx + \int_{\sqrt{5}}^{11} -\sqrt{x - 2} + 2 - (-1) \, dx
\]

\[
= \int_{0}^{2} x^2 \, dx + \int_{2}^{\sqrt{5}} -(x - 2)^{\frac{1}{2}} - 2 + x^2 \, dx
\]

\[
+ \int_{2}^{6} 2 - (x - 2)^{\frac{1}{2}} \, dx + \int_{\sqrt{5}}^{11} -(x - 2)^{\frac{3}{2}} + 3 \, dx
\]

\[
= \left[ \frac{x^3}{3} \right]_{0}^{2} + \left[ -\frac{2}{3}(x - 2)^{\frac{3}{2}} - 2x + \frac{x^3}{3} \right]_{2}^{\sqrt{5}}
\]

\[
+ \left[ 2x - \frac{2}{3}(x - 2)^{\frac{3}{2}} \right]_{2}^{6} + \left[ -\frac{2}{3}(x - 2)^{\frac{3}{2}} + 3x \right]_{\sqrt{5}}^{11}
\]

\[
= \left( \frac{2^3}{3} \right) - \left( \frac{0^3}{3} \right) + \left( -\frac{2}{3}(\sqrt{5} - 2)^{\frac{3}{2}} - 2\sqrt{5} + \frac{\sqrt{5}^3}{3} \right) - \left( -\frac{2}{3}(2 - 2)^{\frac{3}{2}} - 2 \cdot 2 + \frac{2^3}{3} \right)
\]

\[
+ \left( 2 \cdot 6 - \frac{2}{3}(6 - 2)^{\frac{3}{2}} \right) - \left( 2 \cdot 2 - \frac{2}{3}(2 - 2)^{\frac{3}{2}} \right)
\]

\[
+ \left( -\frac{2}{3}(11 - 2)^{\frac{3}{2}} + 3 \cdot 11 \right) - \left( -\frac{2}{3}(\sqrt{5} - 2)^{\frac{3}{2}} + 3\sqrt{5} \right)
\]

\[
\approx 14.2131067417.
\]