Problem 1  [6 points]  Use the general slicing method to find the volume of the following solid. The solid whose base is the region bounded by the ellipse \( \frac{x^2}{17} + \frac{y^2}{8} = 1 \) and whose cross sections perpendicular to the base and \textbf{parallel} to the \( x \)-axis are semicircles.

Solution:  Since the cross section parallel to the \( x \)-axis are semicircles and perpendicular to the base, we should integrate cross-sectional area as a function of \( y \) with respect to \( y \) to find the volume.

The area of a semicircle with radius \( r \) is given by \( \frac{1}{2} \pi r^2 \). Thus we need to find a formula for the radius of the cross-section at a given value of \( y \). After solving the equation of our curve for \( x \) as a function of \( y \), we have the following two branches of the ellipse:

\[
x = \sqrt{17 - \frac{17}{8} y^2}, \quad \text{and} \quad x = -\sqrt{17 - \frac{17}{8} y^2}.
\]

The diameter of our cross-section travels from the left branch of our ellipse to the right branch of our ellipse. Thus the radius is given by \( \sqrt{17 - \frac{17}{8} y^2} \), and

\[
CSA(y) = \frac{\pi}{2} \left( \sqrt{17 - \frac{17}{8} y^2} \right)^2 = \frac{\pi}{2} (17 - \frac{17}{8} y^2).
\]

By plugging \( x = 0 \) into our curve, we can find the max \( y \) value and min \( y \) value of our curve. \( y^2/8 = 1 \), or \( y = \pm \sqrt{8} \). Thus

\[
Volume = \int_{-\sqrt{8}}^{\sqrt{8}} \frac{\pi}{2} \left( 17 - \frac{17}{8} y^2 \right)
\]

\[
= \frac{\pi}{2} \left[ 17x - \frac{17}{24} y^3 \right]_{-\sqrt{8}}^{\sqrt{8}}
\]

\[
= \frac{\pi}{2} \left[ 17(\sqrt{8}) - \frac{17}{24}(\sqrt{8})^3 \right] - \frac{\pi}{2} \left[ 17(-\sqrt{8}) - \frac{17}{24}(-\sqrt{8})^3 \right].
\]
Problem 1  [4 points]  Consider the region \( R \) bounded by the graph \( y = -x^2 + 2x \) and the \( x \)-axis.

a) Find the volume of the solid of revolution formed by revolving the region \( R \) about the line \( x = 4 \).

Solution: Our functions are functions of \( x \) and we are revolving around \( x = 4 \) which is parallel to the \( y \)-axis. Thus we should employ the shell method.

The function intersects the \( x \)-axis when

\[
0 = -x^2 + 2x = x(-x + 2), \quad \text{or}
\]

when \( x = 0, 2 \). At \( x = 1 \), \( y = -(1)^2 + 2 \cdot 1 = 1 \). Thus \( y = -x^2 + 2x \) is the top function of our region and the height of the region is as a function of \( x \) is \(-x^2 + 2x\). For a value \( x \) in \([0,2]\) the distance from that point to 4 is \( 4 - x \). Thus

\[
Volume = 2\pi \int_{0}^{2} (4 - x)(-x^2 + 2x) \, dx
\]

\[
= 2\pi \int_{0}^{2} 8x - 6x^2 + x^3 \, dx
\]

\[
= 2\pi \left[ 4x^2 - 2x^3 + \frac{1}{4}x^4 \right]_{0}^{2}
\]

\[
= 2\pi \left[ 4(2)^2 - 2(2)^3 + \frac{1}{4}(2)^4 \right] - 2\pi \left[ 4(0)^2 - 2(0)^3 + \frac{1}{4}(0)^4 \right]
\]

\[
= 8\pi.
\]

b) Find the volume of the solid of revolution formed by revolving the region \( R \) about the line \( y = a \) where \( a > 1 \) is a constant.
Solution: In this case we still wish to integrate with respect to $x$, but as we revolve around the line $y = a$, we will need to use the washer/disk method. On the interval $[0, 2]$, $-x^2 + 2x$ takes on values between 0 and 1. Thus this curve is closer to the axis of rotation than the $x$-axis. Hence the distance from this curve to the axis of rotation will be our inner radius and the distance from the $x$-axis to our axis of rotation will be our outer function. So the inner function is
\[ \text{in} = a - (-x^2 + 2x), \quad \text{and} \]
the outer function is
\[ \text{out} = a - 0 = a. \]
Thus
\[
Vol = \pi \int_0^2 a^2 - (a - (-x^2 + 2x))^2 \, dx
\]
\[
= \pi \int_0^2 a^2 - (a^2 + x^4 + 4x^2 + 2ax^2 - 4ax - 4x^3) \, dx
\]
\[
= \pi \int_0^2 4x^3 - x^4 - (4 + 2a)x^2 - 2ax^2 + 4ax \, dx
\]
\[
= \pi \left[ x^4 - \frac{1}{5}x^5 - \frac{4 + 2a}{3}x^3 + 2ax^2 \right]_0
\]
\[
= \pi \left( 16 - 32/5 - \frac{4 + 2a}{3} \cdot 8 + 8a \right)
\]
\[
= \pi \left( \frac{8}{3}a - \frac{16}{15} \right).
\]

\[ \text{c) Find any values of } a \text{ for which the volume of the solid in part (b) is the same as the volume of the solid in part (a). You may use a calculator and approximate any answers to two decimal places.} \]

Solution:
\[
\pi \left( \frac{8}{3}a - \frac{16}{15} \right) = 8\pi
\]
\[
\frac{8}{3}a - \frac{16}{15} = 8
\]
\[
\frac{8}{3}a = \frac{136}{15}
\]
\[
a = \frac{17}{5}.
\]