SHOW ALL WORK!!! Unsupported answers might not receive full credit.

**Problem 1**  [6 points] An arch is to be modeled by the function $y = e^{-2x} + \frac{1}{16}e^{2x}$ on the interval $[-1, 1]$. Find the length of the arch. You do not need to simplify your final answer.

**Solution:** Recall that the arc length formula of a differentiable curve, $f(x)$, from $a$ to $b$:

$$
Length = \int_a^b \sqrt{1 + f'(x)^2} \, dx.
$$

In this case $a = -1$, and $b = 1$, and $f(x) = e^{-2x} + \frac{1}{16}e^{2x}$.

$$
f'(x) = -2e^{-2x} + \frac{1}{8}e^{2x}
$$

and

$$
1 + f'(x)^2 = 1 + \left(-2e^{-2x} + \frac{1}{8}e^{2x}\right)^2
$$

$$
= 1 + \left(-2e^{-2x}\right)^2 - \frac{1}{2} + \left(\frac{1}{8}e^{2x}\right)^2
$$

$$
= (-2e^{-2x})^2 + \frac{1}{2} + \left(\frac{1}{8}e^{2x}\right)^2
$$

$$
= \left(2e^{-2x} + \frac{1}{8}e^{2x}\right)^2.
$$

Thus

$$
L = \int_{-1}^{1} \sqrt{\left(2e^{-2x} + \frac{1}{8}e^{2x}\right)^2} \, dx
$$

$$
= \int_{-1}^{1} 2e^{-2x} + \frac{1}{8}e^{2x} \, dx
$$

$$
= -e^{-2} + \frac{1}{16}e^{2x} \bigg|_{-1}^{1}
$$

$$
= -e^{-2} + \frac{1}{16}e^{2} - \left(-e^{2} + \frac{1}{16}e^{-2}\right)
$$

$$
= \frac{17}{16}(e^{2} - e^{-2}).
$$
Problem 1  [4 points] A tank is formed by revolving the graph of \( y = 4x^2 \) for \( 0 \leq x \leq 3 \) (in meters) about the \( y \)-axis.

a) If the tank is filled with water to the level (height) of \( h \) meters, find the volume of the water in terms of \( h \).

**Solution:** Let’s use the washer method to find the volume. After solving the function defining the graph for \( x \) we find that

\[
x = \sqrt{\frac{y}{4}}.
\]

The liquid exists from a \( y \) value of 0 to a \( y \) value of \( h \). Thus

\[
Volume = \pi \int_{0}^{h} \left( \sqrt{\frac{y}{4}} \right)^2 \, dy
\]

\[
= \pi \int_{0}^{h} \frac{y}{4} \, dy
\]

\[
= \pi \left[ \frac{y^2}{8} \right]_{0}^{h}
\]

\[
= \pi \frac{h^2}{8}.
\]

b) If the tank is losing water at the rate of \( 2 \frac{m^3}{s} \), at what rate is the level of the water falling when the level is at 1 meter? (Approximate to 2 decimal places.)
Solution: By taking a derivative of the relation between volume and height with respect to time, we find that

\[ \frac{dV}{dt} = \frac{\pi}{4} h \frac{dh}{dt}. \]

Thus if \( \frac{dV}{dt} = -2 \) and \( h = 1 \),

\[ \frac{dh}{dt} = -2 \cdot \frac{4}{\pi} = \frac{-8}{\pi} \approx -2.55. \]

So the water is falling at 2.55 m/s.

c) Given that the density of water is \( 1000 \frac{kg}{m^3} \), find the level of the water when there is 20,000 kg of water in the tank. (Approximate to 2 decimal places.)

Solution: Since volume times density is mass, 20,000 kg of water is 20 cubic meters of water. Then by part a),

\[ 20 = \frac{\pi}{8} h^2 \]

\[ h = \sqrt{\frac{20 \cdot 8}{\pi}} \approx 7.14. \]

d) If the tank contains 20,000 kg of water, how much work is done pumping all of the water to an exit pipe at the top of the tank? (Approximate to 2 decimal places.)

Solution: Since we limit ourselves to values of \( x \) in \([0, 3]\) to form the tank, the top of the tank has a height of 36 m. Thus the distance the \( y \)th cross section must travel is \( 36 - y \). Thus

\[
\text{Work} \approx \int_0^{\sqrt{\frac{20 \cdot 8}{\pi}}} 1000 \cdot 9.8 \cdot \frac{\pi y}{4} (36 - y) \, dy
\]

\[
= \frac{9800\pi}{4} \int_0^{\sqrt{\frac{20 \cdot 8}{\pi}}} 36y - y^2 \, dy
\]

\[
= \frac{9800\pi}{4} \left[ 18y^2 - \frac{y^3}{3} \right]_0^{\sqrt{\frac{20 \cdot 8}{\pi}}}
\]

\[
= \frac{9800\pi}{4} \left[ 18 \left( \sqrt{\frac{20 \cdot 8}{\pi}} \right)^2 - \frac{\sqrt{\frac{20 \cdot 8}{\pi}}^3}{3} \right]
\]

\[
\approx 6123497.80 \, J
\]