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# A Manifold which does not admit any Differentiable Structure

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An example of a triangulable closed manifold  $M_0$  of dimension 10 will be constructed. It will be shown that  $M_0$  does not admit any differentiable structure. Actually,  $M_0$  does not have the homotopy type of any differentiable manifold.

Also, a 9-dimensional differentiable manifold  $\Sigma^9$  is obtained.  $\Sigma^9$  is homeomorphic but not diffeomorphic to the standard 9-sphere  $S^9$ .

Use is made of a procedure for killing the homotopy groups of differentiable manifolds studied by J. MILNOR in [6]. I am indebted to J. MILNOR for sending me a copy of the manuscript of his paper.

Although much of the constructions (in particular the construction of  $M_0$ ) generalizes to higher dimensions, I did not succeed disproving the existence of a differentiable structure on the higher dimensional analogues of  $M_0$ . A more general case of some of the constructions below will be published in a subsequent paper, with other applications.<sup>1)</sup>

## § 1. Construction of an invariant

Let  $M^{10}$  be a closed triangulable manifold. Assume that  $M^{10}$  is 4-connected. ( $M^{10}$  is connected, and  $\pi_i(M) = 0$  for  $1 \leq i \leq 4$ .) It follows from POINCARÉ duality and the universal coefficient theorem that  $H^q(M; \mathbf{G}) = 0$  for  $5 < q < 10$ , and  $H^5(M)$  is free abelian of even rank  $2s$ , say. (If no coefficients are mentioned, integer coefficients are understood.)

Let  $\Omega = \Omega S^6$  be the loop-space on the 6-sphere. It is well known that  $H^5(\Omega) = \mathbf{Z}$ ,  $H^{10}(\Omega) = \mathbf{Z}$ , and if  $\pi: \Omega \times \Omega \rightarrow \Omega$  is the map given by the product of loops, then

$$\begin{aligned}\pi^*(e_1) &= e_1 \otimes 1 + 1 \otimes e_1, \quad \text{and} \\ \pi^*(e_2) &= e_2 \otimes 1 + 1 \otimes e_2 + e_1 \otimes e_1,\end{aligned}$$

where  $e_1, e_2$  are the generators of  $H^5(\Omega)$  and  $H^{10}(\Omega)$  respectively, and  $H^*(\Omega \times \Omega)$  is identified with  $H^*(\Omega) \otimes H^*(\Omega)$  by the KÜNNETH formula. (Compare R. BOTT and H. SAMELSON [1], Theorem 3.1.B.)

**Lemma 1.1.** *Let  $X \in H^5(M)$  be given. There exists a map  $f: M \rightarrow \Omega$  such that  $f^*(e_1) = X$ .*

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<sup>1)</sup> This paper was presented at the International Colloquium on Differential Geometry and Topology, Zurich, June 1960.

*Proof.* Let  $K$  be a triangulation of  $M$ . Define  $f$  by stepwise extension on the skeletons  $K^{(q)}$  using obstruction theory.  $f|K^{(4)}$  is taken to be the constant map into a base point on  $\Omega$ . Let  $X_0$  be a representative cocycle of  $X$ . For every 5-dimensional simplex  $s_5$  of  $K$ , define  $f|s_5$  to be a representative of  $X_0[s_5]$ -times the generator of  $\pi_5(\Omega) \cong \pi_6(S^6) \cong Z$ . The obstruction cocycle to extend  $f|K^{(5)}$  in dimension 6 is zero. The next obstruction is in dimension 10 with values in  $\pi_9(\Omega) \cong \pi_{10}(S^6) = 0$ . (See [9], § 41.) Thus the lemma is proven.

Define a function  $\varphi_0: H^5(M) \rightarrow Z_2$  by the following device. For every  $X \in H^5(M)$ , take a map  $f: M \rightarrow \Omega$  such that  $f^*(e_1) = X$ . Then,  $\varphi_0(X) = f^*(u_2)[M]$ , where  $u_2 \in H^{10}(\Omega; Z_2)$  is the reduction modulo 2 of  $e_2 \in H^{10}(\Omega)$ , and  $f^*(u_2)[M]$  is the value of the cohomology class  $f^*(u_2)$  on the generator of  $H_{10}(M^{10}; Z_2)$ .

**Lemma 1.2.** *The function  $\varphi_0: H^5(M) \rightarrow Z_2$  is well defined, i.e.,  $\varphi_0(X)$  does not depend on the choice of the map  $f: M \rightarrow \Omega$  such that  $f^*(e_1) = X$ .*

*Proof.* Let  $f, g: M \rightarrow \Omega$  be two maps such that  $f^*(e_1) = g^*(e_1)$ . We have to show that  $f^*(u_2) = g^*(u_2)$ . Let  $K$  again be a triangulation of  $M$ . Since  $f^*(e_1) - g^*(e_1) = 0$ , it follows that  $f$  and  $g$  are 5-homotopic. (See S. T. HU [2], Chap. VI.) Since  $H^q(M; \pi_q(\Omega)) = 0$  for  $5 < q < 10$ , it follows that  $f$  and  $g$  are 9-homotopic. Hence, we may assume that  $f|K^{(9)} = g|K^{(9)}$ . Let  $\omega^{10}(f, g) \in C^{10}(K; \pi_{10}(\Omega))$  be the difference cochain. Then,

$$(f^*(u_2) - g^*(u_2))[s_{10}] = u_2[h\omega^{10}(f, g)[s_{10}]] ,$$

for every 10-simplex  $s_{10}$ , where  $h: \pi_{10}(\Omega) \rightarrow H_{10}(\Omega)$  is the HUREWICZ homomorphism. According to J. P. SERRE,  $u_2[h\alpha]$  is the mod. 2 HOPF invariant of the element in  $\pi_{11}(S^6)$  represented by  $\alpha \in \pi_{10}(\Omega S^6)$ . (Compare [8], Lemme 2.) Since no element of odd HOPF invariant occurs in  $\pi_{11}(S^6)$ , it follows that  $f^*(u_2) = g^*(u_2)$ , and the proof is complete.

**Lemma 1.3.** *Let  $X, Y \in H^5(M)$  be two integer cohomology classes of  $M$ . Then,*

$$\varphi_0(X + Y) = \varphi_0(X) + \varphi_0(Y) + x \cdot y ,$$

where  $x \cdot y$  is the value on the generator of  $H_{10}(M^{10}; Z_2)$  of the cup-product  $x \smile y$ . ( $x, y$  are the mod. 2 reductions of  $X$  and  $Y$  respectively.)

*Proof.* Let  $f, g: M \rightarrow \Omega$  be maps such that  $f^*(e_1) = X$  and  $g^*(e_1) = Y$ . By definition,  $\varphi_0(X) = f^*(u_2)[M]$ , and  $\varphi_0(Y) = g^*(u_2)[M]$ .

Let  $f \times g: M \times M \rightarrow \Omega \times \Omega$  be the product of  $f$  and  $g$ . (I.e.,  $f \times g(u, v) = (f(u), g(v))$ .) Let  $D: M \rightarrow M \times M$  be the diagonal map. Define  $F: M \rightarrow \Omega$

by  $F = \pi \circ (f \times g) \circ D$ , where  $\pi: \Omega \times \Omega \rightarrow \Omega$  is given by the multiplication of loops. Since  $D^*$  maps the tensor product of cohomology classes into their cup-product, we have  $F^*(e_1) = D^*(X \otimes 1 + 1 \otimes Y) = X + Y$ . Therefore,

$$\varphi_0(X + Y) = F^*(u_2)[M].$$

On the other hand,

$$\begin{aligned} F^*(u_2) &= D^*(f^*(u_2) \otimes 1 + 1 \otimes g^*(u_2) + f^*(u_1) \otimes g^*(u_1)) \\ &= f^*(u_2) + g^*(u_2) + f^*(u_1) \smile g^*(u_1) \\ &= f^*(u_2) + g^*(u_2) + x \smile y. \end{aligned}$$

( $u_1$  is the reduction modulo 2 of  $e_1$ .) This proves Lemma 1.3.

The function  $\varphi_0: H^5(M) \rightarrow \mathbb{Z}_2$  induces a function  $\varphi: H^5(M; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$  satisfying  $\varphi(x + y) = \varphi(x) + \varphi(y) + x \cdot y$ . Indeed, if  $X$  is an integer class whose reduction modulo 2 yields  $x \in H^5(M; \mathbb{Z}_2)$ , we define  $\varphi(x) = \varphi_0(X)$ . It follows from

$$\varphi_0(2Y) = \varphi_0(Y) + \varphi_0(Y) + y \cdot y = y \cdot y = 0,$$

that  $\varphi(x) \in \mathbb{Z}_2$  depends only on  $x \in H^5(M; \mathbb{Z}_2)$ .

The function  $\varphi: H^5(M; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$  is then used to construct the number  $\Phi(M)$  as follows. A basis  $x_1, \dots, x_s, y_1, \dots, y_s$  of  $H^5(M; \mathbb{Z}_2)$  as a vector space over  $\mathbb{Z}_2$  will be called *symplectic* if  $x_i \cdot x_j = y_i \cdot y_j = 0$ , and  $x_i \cdot y_j = \delta_{ij}$  for all  $i, j = 1, \dots, s$ . Clearly, symplectic bases always exist. Moreover, it is well known that since the function  $\varphi: H^5(M; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$  satisfies the equation

$$\varphi(x + y) = \varphi(x) + \varphi(y) + x \cdot y,$$

the remainder modulo 2

$$\Phi(M) = \sum_1^s \varphi(x_i) \cdot \varphi(y_i)$$

is independent of the symplectic basis  $x_1, \dots, x_s, y_1, \dots, y_s$ .

The rest of the paper is devoted to investigating the properties of the invariant  $\Phi$ .

Clearly,  $\Phi$  is an invariant of the homotopy type of 4-connected closed manifolds of dimension 10.

Our objective is the proof of the following theorems.

**Theorem 1.** *If  $M^{10}$  has the homotopy type of a  $C^1$ -differentiable 4-connected closed manifold, then  $\Phi(M) = 0$ .*

(It can be shown that the converse of this theorem would follow from the conjecture that the cohomology ring  $H^*(M)$  and  $\Phi(M)$  are a complete set of invariants of the homotopy type of the triangulable 4-connected closed manifold  $M$  of dimension 10.)

**Theorem 2.** *There exists a closed 4-connected combinatorial manifold  $M_0$  of dimension 10 for which  $\Phi(M_0) = 1$ .*

(In fact a specific example will be constructed.)

In § 2, the proof of Theorem 1 will be carried out taking Lemmas 4.2 and 5.1 for granted. (Lemma 4.2 is used in the proof of Lemma 2.2, and Lemma 5.1 is used to deduce Theorem 1 from Lemma 2.4.) The Lemmas 4.2 and 5.1 are proved at the end of the paper, in § 4 and § 5. Theorem 2 will be proved in § 3.

## § 2. Proof of Theorem 1

Let  $M^{10}$  be a closed  $C^1$ -differentiable manifold which is 4-connected.

**Lemma 2.1.**  *$M^{10}$  is a  $\pi$ -manifold.*

*Proof.* Let  $M^{10} \subset R^{n+10}$  be an imbedding with  $n$  large. We have to show that the normal bundle  $\nu$  is trivial. This is done by constructing a field of normal  $n$ -frames  $f_n$ . Let  $K$  be a triangulation of  $M^{10}$ . Since  $\pi_4(SO_n) = 0$ , and  $M^{10}$  is 4-connected, it follows that  $H^{q+1}(M; \pi_q(SO_n)) = 0$  for  $0 \leq q < 9$ . Thus, there is only one possibly non-vanishing obstruction  $\mathfrak{o}(\nu, f_n) \in H^{10}(M; \pi_9(SO_n)) \cong \pi_9(SO_n)$  to the construction of the field  $f_n$  of normal  $n$ -frames. By Lemma 1 of [7],  $\mathfrak{o}(\nu, f_n)$  is in the kernel of the HOPF-WHITEHEAD homomorphism  $J_9: \pi_9(SO_n) \rightarrow \pi_{n+9}(S^n)$ . But  $J_9$  is a monomorphism. (Compare proof of Lemma 1.2 of [4].) Hence,  $\mathfrak{o}(\nu, f_n) = 0$ , and the lemma is proved. (Recall that the proof of the assertion:  $J_9$  is a monomorphism, was based on Corollary 2.6 of J. F. ADAMS paper *On the structure and applications of the STEENROD algebra*, Comm. Math. Helv. 32 (1958), 180–214. This statement also follows from the portion of the POSTNIKOV decomposition mod. 2 of  $S^n$  given below in § 5.)

The THOM construction associates with every framed manifold  $(M; f_n)$ , where  $M \subset R^{n+\dim M}$ , an element  $\alpha(M; f_n) \in \pi_{n+\dim M}(S^n)$ . We say that  $(M^{10}; f_n)$  is *homotopic to zero* if the corresponding element  $\alpha(M; f_n)$  is the neutral element of  $\pi_{n+10}(S^n)$ .

**Lemma 2.2.** *If  $(M^{10}; f_n)$  is homotopic to zero, where  $M^{10}$  is 4-connected, then  $\Phi(M) = 0$ .*

*Proof.* The assumption that  $(M; f_n)$  is homotopic to zero implies the existence of a framed manifold  $(V^{11}; F_n)$  with boundary  $M^{10}$ . (Compare R. THOM [10].) We may assume that  $V$  is connected, and hence has a trivial tangent bundle. We can therefore apply to  $V - M$  the procedure for killing the homotopy groups of a differentiable manifold studied by J. MILNOR. Specifically, using Theorem 3 of [6], we obtain a new 11-dimensional differen-

able manifold with boundary  $M^{10}$  which is also 4-connected. This new 4-connected manifold will again be denoted by  $V^{11}$ . We can now forget about the fields of normal frames.

We proceed to compute  $\Phi(M)$ . Consider the cohomology exact sequence of the pair  $(V, M)$  with coefficients in  $Z_2$ ,

$$\dots \rightarrow H^5(V) \xrightarrow{i^*} H^5(M) \xrightarrow{\delta} H^6(V, M) \rightarrow \dots .$$

Using relative POINCARÉ-LEFSCHETZ duality (over  $Z_2$ ), and the formula

$$u \cup \delta x[V, M] = i^*(u) \cup x[M],$$

where  $u \in H^5(V)$ ,  $x \in H^5(M)$  and  $[V, M]$ ,  $[M]$  are the generators of  $H_{11}(V, M; Z_2)$  and  $H_{10}(M; Z_2)$  respectively, it follows that  $H^5(M; Z_2)$  has a symplectic basis  $x_1, \dots, x_s, y_1, \dots, y_s$  say, such that  $x_1, \dots, x_s$  is a vector basis of  $\text{Ker } \delta$ . Consequently, in order to prove  $\Phi(M) = 0$ , it is sufficient to show that  $\varphi(x) = 0$  for every  $x \in \text{Ker } \delta$ .

Let  $X \in H^5(M)$  be an integer class whose reduction modulo 2 is  $x$ , and let  $f: M^{10} \rightarrow \Omega = \Omega S^6$  be a map such that  $f^*(e_1) = X$ . We have to show that  $f^*(u_2) = 0$ , where  $u_2$  generates  $H^{10}(\Omega; Z_2)$ . Let  $\Omega^*$  be the space obtained from  $\Omega$  by attaching a cell of dimension 6 by a map  $S^5 \rightarrow \Omega$  of degree 2. By Lemma 4.2 in § 4, below, for every map  $g: S^{10} \rightarrow \Omega^*$ , one has  $g^*(u_2) = 0$ , where we denote by  $u_2 \in H^{10}(\Omega^*; Z_2)$  again the class corresponding to the old  $u_2 \in H^{10}(\Omega; Z_2)$  under the canonical isomorphism  $H^{10}(\Omega; Z_2) \cong H^{10}(\Omega^*; Z_2)$ .

We attempt to extend  $f: M \rightarrow \Omega^*$  to a map of  $V$  into  $\Omega^*$ . Let  $(K, L)$  be a triangulation of  $(V, M)$ . The stepwise extension of  $f$  on the skeletons  $K^{(q)} \cup L$  leads to obstructions in the groups  $H^{q+1}(K, L; \pi_q(\Omega^*))$ . For  $q < 5$ ,  $\pi_q(\Omega^*) = 0$ . We meet a first obstruction for  $q = 5$  in  $H^6(K, L; Z_2)$ . By the HOPF theorem, this obstruction is  $\delta x$ . (See S. T. HU [2].) Since  $\delta x = 0$ , it is possible to extend  $f$  on  $K^{(6)} \cup L$ . Using  $H^{q+1}(K, L; G) = 0$  for  $5 < q < 10$  (since  $V$  is 4-connected), it follows that there exists a map  $F: K - \tau \rightarrow \Omega^*$ , where  $\tau$  is some 11-dimensional simplex in  $K - L$ , such that  $F|L = f$ . Let  $S^{10}$  denote the boundary of  $\tau$ , and let  $g: S^{10} \rightarrow \Omega^*$  be the restriction of  $F$  on  $S^{10}$ . Since  $\partial(K - \tau) = L - S^{10}$ , and  $g^*(u_2) = 0$ , it follows that  $f^*(u_2) = 0$ . The proof of Lemma 2.2 is complete.

**Corollary 2.3.** *If two 4-connected framed manifolds  $(M; f_n)$  and  $(M'; f'_n)$  of dimension 10 define the same element  $\alpha = \alpha(M; f_n) = \alpha(M'; f'_n)$  by the THOM construction, then  $\Phi(M) = \Phi(M')$ .*

This is obtained by observing that  $\Phi$  is additive with respect to the connected sum of manifolds.

It follows that  $\Phi$  provides a homomorphism from a subgroup of  $\pi_{n+10}(S^n)$

into  $Z_2$ . We denote this homomorphism by  $\Phi$  again. Actually,  $\Phi$  is defined on every element of  $\pi_{n+10}(S^n)$ . Indeed, using spherical modifications [6], it is easy to see that every element  $\alpha \in \pi_{n+10}(S^n)$  is obtainable from a 4-connected framed manifold by the THOM construction. This remark will not be used in the present paper.

It follows from Corollary 2.3 that Theorem 1 is equivalent to the statement that  $\Phi(\alpha) = 0$  for every  $\alpha \in \pi_{n+10}(S^n)$ , provided  $\Phi(\alpha)$  is defined.

Since  $\Phi(\alpha)$  is obviously zero for every element  $\alpha$  of odd order, and by J. P. SERRE's results  $\pi_{n+10}(S^n)$  contains no element of infinite order, it is sufficient to show that  $\Phi$  annihilates the 2-component of the group  $\pi_{n+10}(S^n)$ . By Lemma 5.1 in § 5 below, every element  $\alpha$  in the 2-component of  $\pi_{n+10}(S^n)$  is representable in the form

$$\alpha = \beta \circ \eta,$$

where  $\eta \in \pi_{n+10}(S^{n+9})$  is the generator of the stable 1-stem, and  $\beta \in \pi_{n+9}(S^n)$ . Hence, Theorem 1 will follow from the

**Lemma 2.4.** *Every element  $\alpha \in \pi_{n+10}(S^n)$  of the form  $\alpha = \beta \circ \eta$ , with  $\eta \in \pi_{n+10}(S^{n+9})$ , and  $\beta \in \pi_{n+9}(S^n)$  is obtainable by the THOM construction from a framed manifold  $(\Sigma^{10}; f_n)$ , where  $\Sigma^{10}$  has the homotopy type of the 10-sphere  $S^{10}$ .*

*Proof.* We first show that  $\beta \in \pi_{n+9}(S^n)$  is obtainable by the THOM construction from a framed manifold  $(\Sigma^9; f_n)$ , where  $\Sigma^9$  has the homotopy type of the 9-sphere.

It is well known that  $\beta$  is obtainable by the THOM construction from some framed manifold  $(M^9; f_n)$ . We have to show that  $(M^9; f_n)$  is homotopic to a framed manifold  $(\Sigma^9; f_n)$ , where  $\Sigma^9$  is a homotopy sphere. This is done by simplifying  $M^9$  by a series of spherical modifications. (See J. MILNOR [6].)

Assuming by induction that  $M^9$  is  $(p-1)$ -connected ( $0 \leq p \leq 4$ ), we have to prove that  $(M; f_n)$  is homotopic to a  $p$ -connected framed manifold  $(M'; f'_n)$ . Recall that a spherical modification of type  $(p+1, q+1)$  applied to a class  $\lambda \in \pi_p(M^9)$  consists of the following construction. Represent  $\lambda$  by an imbedding

$$f: S^p \times D^{q+1} \rightarrow M^9,$$

with  $p+q+1=9$ . (This is possible for  $p \leq 4$  since  $M^9$  is a  $\pi$ -manifold and the normal bundle of any imbedding  $S^p \rightarrow M^9$  is stable in this range of dimensions.) The manifold  $M$  is then replaced by

$$M' = (M - f(S^p \times D^{q+1})) \cup (D^{p+1} \times S^q),$$

under identification of  $f(S^p \times S^q)$  regarded as the boundary of  $f(S^p \times D^{q+1})$  with  $S^p \times S^q$  regarded as the boundary of  $D^{p+1} \times S^q$ . By Theorem 2 of

[6], the manifolds  $M$  and  $M'$  bound a 10-dimensional differentiable manifold  $\omega = \omega(M, f)$ , and  $f: S^p \times D^{q+1} \rightarrow M^9$  can be chosen such that the field  $f_n$  (over  $M$ ) is extendable over  $\omega$  as a field of normal  $n$ -frames. (We can think of  $\omega$  as imbedded in  $R^{n+10}$  with  $M \subset R^{n+9} \times (0)$  and  $M' \subset R^{n+9} \times (1)$  since  $n$  can be taken as large as we please.) Hence spherical modifications of type  $(p + 1, q + 1)$  with  $0 \leq p \leq 4$  can be performed so as to carry  $(M; f_n)$  into a homotopic framed manifold. It is known (Theorem 3 of [6]) that for  $p < 4$ , spherical modifications simplify the manifold. More precisely  $\pi_p(M')$  is isomorphic to the quotient of  $\pi_p(M)$  by the subgroup generated by  $\lambda$ , and  $\pi_i(M) \cong \pi_i(M') = 0$  for  $i < p$ . Hence, it is easy, using [6], to obtain a 3-connected framed manifold homotopic to  $(M^9; f_n)$ . The case  $p = 4$  requires special care. If  $\lambda \in \pi_4(M^9)$  is the class we want to kill, there exists an imbedding  $f: S^4 \times D^5 \rightarrow M^9$  such that  $f|S^4 \times (0)$  represents  $\lambda$ . Let  $M' = \chi(M, f)$  be the 9-dimensional manifold obtained from  $M$  and  $f$  by spherical modification. ( $f$  is supposed to be chosen so that  $(M'; f'_n)$  with some  $f'_n$  is homotopic to  $(M; f_n)$ .) In general, however,  $f|x_0 \times (bdry D^5)$  represents a *non-zero* element of  $\pi_4(M')$ . Thus, it is not clear a priori that a series of spherical modifications of type  $(5, 5)$  will carry  $M$  into a 4-connected manifold, and hence a homotopy sphere.

If  $\lambda$  is a generator of the free part of  $\pi_4(M) \cong H_4(M)$ , there exists by POINCARÉ duality a class  $\mu \in H_6(M)$  whose intersection coefficient with  $\lambda$  (or  $h\lambda$  rather, where  $h$  is the HUREWICZ homomorphism) is 1. It follows that in this case the cycle given by  $f|x_0 \times (bdry D^5)$  is homologous to zero in  $M - f(S^4 \times D^5)$ , and hence in  $M'$ . Thus  $H_4(M') \cong \pi_4(M')$  has strictly smaller rank than  $H_4(M) \cong \pi_4(M)$ , and the torsion subgroup is unchanged.

I claim that if  $\lambda \in \pi_4(M)$  is a torsion element, the homology class of the cycle  $f|x_0 \times (bdry D^5)$  is of infinite order for any  $f$  representing  $\lambda$ . Hence, one more spherical modification will lead to a manifold with 4-dimensional homology group of not bigger rank than  $H_4(M)$  and with a strictly smaller torsion subgroup. (I.e., a series of spherical modifications will lead to a 4-connected framed manifold homotopic to  $(M^9; f_n)$ . By POINCARÉ duality, a closed 4-connected manifold of dimension 9 has the homotopy type of  $S^9$ .)

Since the BETTI numbers  $p_4, p'_4$  of  $M$  and  $M'$  (in dimension 4) differ at most by 1, and differ indeed by 1 if and only if  $\lambda'$  (represented by  $f|x_0 \times (bdry D^5)$ ) in  $M'$  is of infinite order, it is sufficient to show that  $p'_4 + p_4 \equiv 1 \pmod{2}$ . Since  $p'_i = p_i$  for  $0 \leq i \leq 3$ , this is equivalent to showing that the semi-characteristics  $E^*(M)$  and  $E^*(M')$  of  $M$  and  $M'$  (over the rationals, say) satisfy  $E^*(M') + E^*(M) \equiv 1 \pmod{2}$ . We use the formula

$$E^*(M') + E^*(M) \equiv E(\omega) + r \pmod{2},$$

where  $E(\omega)$  is the EULER characteristic of the manifold  $\omega$  with boundary  $\dot{\omega} = M' - M$ , and  $r$  is the rank of the bilinear form on  $H^5(\omega, \dot{\omega}; \mathbb{Q})$  defined by the cup-product. (Compare M. A. KERVAIRE [3], § 8, formula (8.9).) It is easily seen that  $E(\omega) = 1$ , up to sign, and since  $u \cdot u = 0$  for every  $u \in H^5(\omega, \dot{\omega}; \mathbb{Q})$ , the rank  $r$  must be even:  $r \equiv 0 \pmod{2}$ . Hence,  $E^*(M') + E^*(M) \equiv 1 \pmod{2}$ .

Summarizing, we have proved so far that every  $\beta \in \pi_{n+9}(S^n)$  is obtainable by the THOM construction from a framed manifold  $(\Sigma^9; f_n)$ , where the manifold  $\Sigma^9$  has the homotopy type of  $S^9$ .

Taking a representative  $f: S^{n+10} \rightarrow S^{n+9}$  of  $\eta$  such that  $f^{-1}(S^{n+9} - x_0)$  is diffeomorphic to  $S^1 \times (S^{n+9} - x_0)$ , we obtain that  $\alpha = \beta \circ \eta$  is obtainable by the THOM construction from  $(S^1 \times \Sigma^9; f_n)$ .

It remains to show that  $(S^1 \times \Sigma^9; f_n)$  is homotopic to a framed manifold  $(\Sigma^{10}; f'_n)$ , where  $\Sigma^{10}$  is a homotopy sphere.

Apply once more the spherical modification theorems (Theorems 2 and 3 of [6]), this time to the class  $\lambda \in \pi_1(S^1 \times \Sigma^9)$  represented by  $S^1 \times (z_0)$ . The resulting framed manifold is homotopic to  $(S^1 \times \Sigma^9; f_n)$  and has the homotopy type of the 10-sphere. This completes the proof of Lemma 2.4.

To complete the proof of Theorem 1 it remains to prove the Lemmas 4.2, and 5.1. This is done in § 4 and § 5.

### § 3. Construction of $M_0$

This section relies on J. MILNOR's paper [5]. Let  $f_0: S^4 \rightarrow SO_4$  be a differentiable map whose homotopy class  $(f_0)$  satisfies

$$i_*(f_0) = \partial i_5,$$

where  $\partial: \pi_5(S^5) \rightarrow \pi_4(SO_5)$  is taken from the homotopy exact sequence of  $SO_6/SO_5$ , and  $i: SO_4 \rightarrow SO_5$  is the usual inclusion. Define  $f_1 = f_2 = i \circ f_0$ . Using  $f_1, f_2: S^4 \rightarrow SO_5$ , a diffeomorphism  $f: S^4 \times S^4 \rightarrow S^4 \times S^4$  is given by  $f(x, y) = (x', y')$ , where  $y' = f_1(x) \cdot y$ , and  $x = f_2(y') \cdot x'$ . Let  $M(f_1, f_2)$  be the MILNOR manifold obtained from the disjoint union of  $D^5 \times S^4$  and  $S^4 \times D^5$  by identifying each point  $(x, y)$  in the boundary of  $D^5 \times S^4$  with  $f(x, y)$ , considered as a point on the boundary of  $S^4 \times D^5$ . By Lemma 1 of [5], together with the remark at the bottom of page 963 in the proof of Lemma 1 in [5], it follows that the differentiable manifold  $M(f_1, f_2)$  is homeomorphic to the 9-sphere. It will follow from Theorem 1 in this paper, that  $M(f_1, f_2)$  is not diffeomorphic to the standard  $S^9$ . Let  $W^{10}$  be the differentiable mani-

fold with boundary  $M(f_1, f_2)$  obtained using the construction on page 964 of [5].  $W$  can alternately be described as follows. Let  $U$  be a tubular neighborhood of the diagonal  $\Delta$  in  $S^5 \times S^5$ . It is well known that  $U$  is the space of the fibre bundle  $p: U \rightarrow S^5$  with fibre  $D^5$  associated with the tangent bundle of  $S^5$ . The differentiable manifold  $W$  is obtained by straightening the angles of the quotient space of the disjoint union of two copies  $U'$  and  $U''$  of  $U$  under an identification of  $p'^{-1}(V)$  with  $p''^{-1}(V)$  such that the images of  $\Delta'$  and  $\Delta''$  in  $W$  have intersection number 1. ( $V$  is an imbedded 5-disc on  $S^5$ , and  $p'^{-1}(V) \cong D^5 \times D^5$  is identified with  $p''^{-1}(V) \cong D^5 \times D^5$  under  $(u, v) \leftrightarrow (v, u)$ ,  $u, v \in D^5$ .)

Since  $W$  is a 10-dimensional manifold whose boundary  $M(f_1, f_2)$  is homeomorphic to  $S^9$ , the union of  $W$  with the cone over the boundary is a 10-dimensional closed manifold  $M_0$ . Since  $M(f_1, f_2)$  is combinatorially equivalent to  $S^9$ , it follows that  $M_0$  possesses a combinatorial structure. (Compare J. MILNOR, *On the relationship between differentiable manifolds and combinatorial manifolds*, mimeographed notes 1956, § 4.)

It is easily seen that  $M_0$  is 4-connected.

We proceed to compute  $\Phi(M_0)$ . Let  $x, y \in H^5(M_0; \mathbb{Z}_2)$  be the cohomology classes dual to the homology classes of the imbedded spheres  $j', j'': S^5 \rightarrow M_0$  given by the images in  $W$  of the diagonals  $\Delta'$  and  $\Delta''$  in  $U'$  and  $U''$  respectively. Clearly,  $x, y$  is a symplectic basis of  $H^5(M_0; \mathbb{Z}_2)$ . (I.e.,  $x \cdot x = y \cdot y = 0$ , and  $x \cdot y = 1$ .) To show that  $\varphi(x) = \varphi(y) = 1$ , observe that the normal bundles of  $j'$  and  $j''$  (regarded as imbeddings of  $S^5$  in the differentiable manifold  $W$ ) are non-trivial. These bundles are isomorphic to  $p: U \rightarrow S^5$ . Let  $K$  be the THOM complex of this bundle. (I.e., the space obtained by collapsing the boundary of  $U$  to a point.) It is well known that  $K$  admits a cell decomposition  $S^5 \cup e^{10}$ , where the attaching map  $S^9 \rightarrow S^5$  is a representative of the WHITEHEAD product  $[i_5, i_5]$ . On the other hand, the THOM construction provides a map  $f_0: M_0 \rightarrow K$  such that  $f_0^*(e_1) = X$ , the dual class of  $j': S^5 \rightarrow M_0$ , and  $f_0^*(u_2)[M_0] = 1$ , where  $e_1$  generates  $H^5(K; \mathbb{Z})$  and  $u_2$  generates  $H^{10}(K; \mathbb{Z}_2)$ . A map  $f: M_0 \rightarrow \Omega S^6$  is obtained by composition of  $f_0$  with the usual inclusion  $S^5 \cup e^{10} \rightarrow \Omega S^6$ . (Recall that  $\Omega S^6$  has a cell decomposition  $\Omega S^6 = S^5 \cup e^{10} \cup e^{15} \cup e^{20} \cup \dots$ , where the attaching map of  $e^{10}$  represents  $[i_5, i_5]$ .) Then,  $f: M_0 \rightarrow \Omega S^6$  has the properties  $f^*(e_1) = X$ ,  $f^*(u_2) = 1$ , showing that  $\varphi(x) = 1$ . The same construction applied to  $Y$ , the dual class of  $j'': S^5 \rightarrow M_0$  yields  $\varphi(y) = 1$ . Hence  $\Phi(M_0) = \varphi(x) \cdot \varphi(y) = 1$ .

If  $M(f_1, f_2)$ , with the differentiable structure induced by  $W$  (of which  $M(f_1, f_2)$  is the boundary) were diffeomorphic to  $S^9$  with the standard differentiable structure, the differentiable structure on  $W$  could be extended to a differentiable structure over the cone  $CM(f_1, f_2)$ , providing a differentiable

structure on  $M_0$ . However,  $\Phi(M_0) = 1$  and Theorem 1 show that a differentiable structure on  $M_0$  does not exist. Hence,  $M(f_1, f_2)$ , homeomorphic to  $S^9$ , is not diffeomorphic to  $S^9$ .

§ 4. The auxiliary space  $\Omega^*$

Let  $Y = S^5 \cup_{2i_5} e^6$  be the space obtained by attaching a 6-cell to  $S^5$  by a map  $S^5 \rightarrow S^5$  of degree 2.

**Lemma 4.1.** *Let  $\alpha \in \pi_5(Y) \cong \mathbb{Z}_2$  be the generator, then  $[\alpha, \alpha] \neq 0 \in \pi_9(Y)$ .*

*Proof.* We identify  $Y$  with the STIEFEL manifold  $V_{7,2}$ . Consider the exact sequence

$$\dots \rightarrow \pi_{10}(S^6) \rightarrow \pi_9(S^5) \xrightarrow{i_*} \pi_9(V_{7,2}) \rightarrow \dots$$

Since  $\pi_{10}(S^6) = 0$ , and  $[i_5, i_5]$  is non-zero in  $\pi_9(S^5)$ , it follows that  $i_*[i_5, i_5] = [i_*(i_5), i_*(i_5)] = [\alpha, \alpha] \neq 0$ .

Let  $Y^* = Y \cup e^{10}$  be the space obtained from  $Y$  by attaching a 10-cell  $e^{10}$  using a representative  $f: S^9 \rightarrow Y$  of  $[\alpha, \alpha]$ . Since  $Y$  is 4-connected, the characteristic map  $\hat{f}: (D^{10}, S^9) \rightarrow (Y^*, Y)$  of  $e^{10}$  induces an isomorphism

$$\hat{f}_*: \pi_{10}(D^{10}, S^9) \rightarrow \pi_{10}(Y^*, Y).$$

(Compare J. H. C. WHITEHEAD [12], Theorem 1.) Thus the relative HUREWICZ homomorphism  $h_R: \pi_{10}(Y^*, Y) \rightarrow H_{10}(Y^*, Y) \cong \mathbb{Z}$  is an isomorphism. Consider the homotopy-homology ladder of  $(Y^*, Y)$ :

$$\begin{array}{ccccccc} \dots & \rightarrow & \pi_{10}(Y) & \rightarrow & \pi_{10}(Y^*) & \xrightarrow{j_0} & \pi_{10}(Y^*, Y) & \xrightarrow{\partial} & \pi_9(Y) & \rightarrow & \dots \\ & & \downarrow & & \downarrow h & & \downarrow h_R & & \downarrow & & \\ \dots & \rightarrow & 0 & \rightarrow & H_{10}(Y^*) & \xrightarrow{j_*} & H_{10}(Y^*, Y) & \rightarrow & 0 & \rightarrow & \dots \end{array}$$

Since  $\partial$  sends the generator of  $\pi_{10}(Y^*, Y)$  into  $[\alpha, \alpha] \neq 0$ , and  $2[\alpha, \alpha] = 0$ , it follows that every element in  $\text{Im}\{h: \pi_{10}(Y^*) \rightarrow H_{10}(Y^*)\}$  can be halved.

It follows that for every map  $g_0: S^{10} \rightarrow Y^*$ , the induced homomorphism  $g_0^*: H^{10}(Y^*; \mathbb{Z}_2) \rightarrow H^{10}(S^{10}; \mathbb{Z}_2)$  is zero.

Let  $\Omega$  be the space of loops over  $S^6$ . Up to homotopy type  $\Omega = S^5 \cup e^{10} \cup e^{15} \cup \dots$ , with  $e^{10}$  attached by a map of class  $[i_5, i_5]$ . Let  $\Omega^* = \Omega \cup e^6$ , where  $e^6$  is attached by a map of degree 2 on  $S^5 \subset \Omega$ . There is a natural inclusion  $Y^* \rightarrow \Omega^*$  which induces an isomorphism on cohomology groups in dimension 10. Hence, we have the

**Lemma 4.2.** *Let  $g: S^{10} \rightarrow \Omega^*$  be a map, and let  $u_2$  be the generator of  $H^{10}(\Omega^*; \mathbb{Z}_2) \cong \mathbb{Z}_2$ . Then,  $g^*(u_2) = 0$ .*

§ 5. A lemma on homotopy groups of spheres

**Lemma 5.1.** *The map  $\pi_{n+9}(S^n) \rightarrow \pi_{n+10}(S^n)$ , for  $n \geq 12$ , defined by composition with the generator  $\eta$  of  $\pi_{n+10}(S^{n+9})$  is surjective on the 2-component.*

This lemma was communicated to me without proof by H. TODA who has also proved that the 2-component of  $\pi_{n+10}(S^n)$  is  $Z_2$ . (See H. TODA [11], Corollary to Proposition 4.10.)

We give a sketch of proof by computation of the POSTNIKOV decomposition modulo 2 of  $S^n$  for large  $n$ , up to dimension  $n + 10$ .

We begin with a remark which will yield Lemma 5.1 whenever a long enough portion of the POSTNIKOV decomposition of  $S^n$  is obtained. Let  $X = K(Z_2, n + 9) \times_k K(Z_2, n + 10)$  be the space of the fibration over  $K(Z_2, n + 9)$  associated with the  $k$ -invariant  $k \in H^{n+11}(Z_2, n + 9; Z_2)$ . Let  $f: S^{n+9} \rightarrow X$  be a map representing the generator of  $\pi_{n+9}(X) \cong Z_2$ . Then, the composition

$$f \circ \eta: S^{n+10} \rightarrow X, \text{ where } \eta: S^{n+10} \rightarrow S^{n+9}$$

represents the generator of  $\pi_{n+10}(S^{n+9})$ , is essential if and only if  $k = Sq^2(\varepsilon)$ , where  $\varepsilon$  is the fundamental class of  $H^{n+9}(Z_2, n + 9; Z_2)$ .

Since  $Sq^2(\varepsilon)$  generates  $H^{n+11}(Z_2, n + 9; Z_2)$ , it follows that  $k \neq Sq^2(\varepsilon)$  implies  $k = 0$ . Hence,  $f \circ \eta$  is inessential if  $k \neq Sq^2(\varepsilon)$ .

If  $k = Sq^2(\varepsilon)$ , let  $\hat{f}: S^{n+9} \cup_{\eta} e^{n+11} \rightarrow X \cup_{f \circ \eta} e^{n+11}$  be the map induced by  $f$ . Let  $s \in H^{n+9}(S^{n+9} \cup_{\eta} e^{n+11}; Z_2)$  be the generator. We identify  $H^{n+9}(X \cup e^{n+11}; Z_2)$  and  $H^{n+9}(X; Z_2)$  with  $H^{n+9}(Z_2, n + 9; Z_2)$ . Since  $f^*(\varepsilon) = s$ , and  $Sq^2(s) \neq 0$ , it follows that  $Sq^2(\varepsilon) \neq 0$  in  $H^{n+11}(X \cup e^{n+11}; Z_2)$ . To show that  $f \circ \eta$  is essential, it is therefore sufficient to show that  $Sq^2(\varepsilon) = 0$  in  $H^{n+11}(X; Z_2)$ . This follows from the commutativity of the diagram

$$\begin{array}{ccccc} 0 \leftarrow H^{n+9}(X; Z_2) & \leftarrow & H^{n+9}(Z_2, n + 9; Z_2) & \leftarrow & 0 \\ & \downarrow Sq^2 & \approx & \downarrow Sq^2 & \\ & H^{n+11}(X; Z_2) & \leftarrow & H^{n+11}(Z_2, n + 9; Z_2) & \xleftarrow{\tau} H^{n+10}(Z_2, n + 10; Z_2), \end{array}$$

where the rows are taken from the exact sequence of the fibration defining  $X$  (in the stable range), and  $\tau$  is the transgression.

Let  $Y_{10} \rightarrow Y_9 \rightarrow \dots \rightarrow Y_i \rightarrow Y_{i-1} \rightarrow \dots \rightarrow Y_0 = K(Z, n)$  be the modulo 2 POSTNIKOV decomposition of  $S^n$ . (I.e.,  $p_i: Y_i \rightarrow Y_{i-1}$  is a fibration with fibre  $F_i = K(\pi_i, n + i)$ , where  $\pi_i$  is the 2-component of the stable group  $\pi_{n+i}(S^n)$ , and  $H^*(Y_i; Z_2)$  contains  $Z_2$  in dimension 0 and  $n$ ,  $H^q(Y_i; Z_2) = 0$  for  $0 < q < n$ , and  $H^{n+k}(Y_i; Z_2) = 0$  for  $0 < k < i + 2$ .) By the  $\mathfrak{C}$ -theory with  $\mathfrak{C} =$  the class of finite groups whose order is prime to

2, a map  $S^n \rightarrow Y_i$  inducing an isomorphism  $H^n(Y_i; \mathbb{Z}_2) \cong H^n(S^n; \mathbb{Z}_2)$  induces an isomorphism of the 2-component of  $\pi_{n+k}(S^n)$  with  $\pi_{n+k}(Y_i)$  for  $k \leq i$ . (Compare J. P. SERRE [8].) We have  $\pi_9 \cong \mathbb{Z}_2 + \mathbb{Z}_2 + \mathbb{Z}_2$  and  $\pi_{10} \cong \mathbb{Z}_2$  as will be seen below, thus

$$F_9 = K(\mathbb{Z}_2, n + 9) \times K(\mathbb{Z}_2, n + 9) \times K(\mathbb{Z}_2, n + 9),$$

and Lemma 5.1 follows by showing that the restriction of the fibration  $Y_{10} \rightarrow Y_9$  over one of the factors of  $F_9$  is  $K(\mathbb{Z}_2, n + 9) \times_k K(\mathbb{Z}_2, n + 10)$  with  $k = Sq^2$ . This is equivalent to showing that  $H^{n+11}(Y_9; \mathbb{Z}_2) \cong \mathbb{Z}_2$  is generated by a class  $u_9$  such that  $i_9^*(u_9) = Sq^2(\varepsilon_9)$ , where  $\varepsilon_9$  is one of the fundamental classes of  $H^9(F_9; \mathbb{Z}_2)$ , and  $i_9: F_9 \rightarrow Y_9$  is the inclusion.

In a similar way, it can be read off from the tables below that composition with  $\eta$  provides *injective* maps  $\pi_{n+7}(S^n) \otimes \mathbb{Z}_2 \rightarrow \pi_{n+8}(S^n)$  and  $\pi_{n+8}(S^n) \rightarrow \pi_{n+9}(S^n)$  in the stable range. Using  $\pi_7(SO_n) \cong \mathbb{Z}$ ,  $\pi_8(SO_n) \cong \mathbb{Z}_2$ , and  $\pi_9(SO_n) \cong \mathbb{Z}_2$ , this implies that  $J_9: \pi_9(SO_n) \rightarrow \pi_{n+9}(S^n)$  is a monomorphism.

We proceed to a partial description of the modulo 2 cohomology of the spaces  $Y_7$ .

$H^*(Y_0)$  is given by J. P. SERRE in [9]. This result of J. P. SERRE and the ADEM relations between the STEENROD squares are the essential tools in computing  $H^*(Y_k; \mathbb{Z}_2)$  for  $k > 0$ . Since we stay in the stable range, the spectral sequences of  $p_k: Y_k \rightarrow Y_{k-1}$  reduce to exact sequences

$$\dots \leftarrow H^{n+q+1}(Y_{k-1}) \xleftarrow{\tau} H^{n+q}(F_k) \xleftarrow{i_k^*} H^{n+q}(Y_k) \xleftarrow{p_k^*} H^{n+q}(Y_{k-1}) \leftarrow \dots$$

It is therefore sufficient to determine at each step the kernel and the image of the transgression  $\tau$ . Since the cohomology of  $Y_k$  is independent of  $k$  up to dimension  $n$ , we omit to mention the non-vanishing cohomology groups in dimension  $\leq n$ . The direct sum of the subgroups of  $H^*(Y_k; \mathbb{Z}_2)$  in dimensions  $> n$  is denoted  $H^+(Y_k)$ .

The symbol  $q_k$  stands for the composition  $p_1 \circ p_2 \circ \dots \circ p_k$ , and  $\varepsilon_k$  denotes the fundamental class of  $H^{n+k}(G, n + k; G)$ .

I omit  $Y_1$  and  $Y_2$  whose cohomology is straightforward, but has to be computed up to dimension  $n + 17$  and  $n + 16$  respectively.  $H^{n+4}(Y_2; \mathbb{Z}_2)$  is generated by  $q_2^*(Sq^4 \varepsilon_0)$ , and  $H^{n+5}(Y_2; \mathbb{Z}_2)$  by a class  $u_2$  such that  $i_2^*(u_2) = Sq^3(\varepsilon_2)$ .

$F_3 = K(\mathbb{Z}_8, n + 3)$ , with  $\tau(\varepsilon'_3) = q_2^*(Sq^4 \varepsilon_0)$  and  $\tau(\beta \varepsilon_3) = u_2$ , where  $\beta$  is the BOCKSTEIN operator associated with the sequence of coefficients  $0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_{16} \rightarrow \mathbb{Z}_8 \rightarrow 0$ , and  $\varepsilon'_3$  is the mod. 2 reduction of  $\varepsilon_3$ .

$H^+(Y_3)$  has a basis consisting of

$u_3$  in dimension  $n + 7$ , such that  $i_3^*(u_3) = Sq^4 \varepsilon_3'$ ;  
 $Sq^1(u_3), q_3^*(Sq^8 \varepsilon_0); Sq^2(u_3), v_3$  such that  $i_3^*(v_3) = Sq^5 \beta \varepsilon_3; Sq^3(u_3);$   
 $Sq^4(u_3); Sq^5(u_3), Sq^4 Sq^1(u_3), q_3^*(Sq^{12} \varepsilon_0); Sq^6(u_3), Sq^4 Sq^2(u_3), Sq^4(v_3);$   
 $Sq^6 Sq^1(u_3), Sq^5 Sq^2(u_3), q_3^*(Sq^{14} \varepsilon_0);$   
 $Sq^8(u_3), Sq^7 Sq^1(u_3), Sq^6 Sq^2(u_3), Sq^6(v_3), q_3^*(Sq^{15} \varepsilon_0); \dots$

$$Y_4 = Y_5 = Y_3. \quad (\pi_4 = \pi_5 = 0.)$$

$F_6 = K(Z_2, n + 6)$  with  $\tau(\varepsilon_6) = p_5^* p_4^*(u_3)$ .

$H^+(Y_6)$  has a basis consisting of

$q_6^*(Sq^8 \varepsilon_0); p_6^* p_5^* p_4^*(v_3), u_6$  such that  $i_6^*(u_6) = Sq^2 Sq^1 \varepsilon_6;$   
 $Sq^1(u_6);$  nothing in dimension  $n + 11; q_6^*(Sq^{12} \varepsilon_0), Sq^2 Sq^1(u_6);$   
 $p_6^* p_5^* p_4^*(Sq^4 v_3), Sq^4(u_6), v_6$  such that  $i_6^*(v_6) = Sq^7 \varepsilon_6;$   
 $q_6^*(Sq^{14} \varepsilon_0), Sq^5(u_6); q_6^*(Sq^{15} \varepsilon_0), p_6^* p_5^* p_4^*(Sq^6 v_3), \dots$   
 (and possibly other classes of dimension  $n + 15$ ).

$F_7 = K(Z_{16}, n + 7)$  with  $\tau(\varepsilon_7') = q_6^*(Sq^8 \varepsilon_0)$  and  $\tau(\beta' \varepsilon_7) = p_6^* p_5^* p_4^*(v_3)$ , where  $\beta'$  is the BOCKSTEIN operator of  $0 \rightarrow Z_2 \rightarrow Z_{32} \rightarrow Z_{16} \rightarrow 0$ , and  $\varepsilon_7'$  is the reduction modulo 2 of  $\varepsilon_7$ .

$H^+(Y_7)$  has a basis consisting of

$u_7$  in dimension  $n + 9$ , such that  $i_7^*(u_7) = Sq^2(\varepsilon_7'), p_7^*(u_6);$   
 $Sq^1(u_7), p_7^*(Sq^1 u_6), v_7$  such that  $i_7^*(v_7) = Sq^2 \beta' \varepsilon_7;$   
 $Sq^1(v_7); Sq^2 Sq^1(u_7), p_7^*(Sq^2 Sq^1 u_6), \dots$  ( $Sq^2(v_7) = 0.$ )

$F_8 = K(Z_2 + Z_2, n + 8)$  with  $\tau(\varepsilon_8') = u_7, \tau(\varepsilon_8'') = p_7^*(u_6)$ , where  $\varepsilon_8'$  and  $\varepsilon_8''$  are the two fundamental classes in  $H^{n+8}(F_8; Z_2)$ .

$H^+(Y_8)$  has a basis consisting of

$p_8^*(v_7), u_8, v_8$ , where  $i_8^*(u_8) = Sq^2(\varepsilon_8')$  and  $i_8^*(v_8) = Sq^2(\varepsilon_8'');$   
 $Sq^1(u_8), Sq^1(v_8), p_8^*(Sq^1 v_7);$   
 $Sq^2(u_8), Sq^2(v_8), \dots$

$F_9 = K(Z_2 + Z_2 + Z_2, n + 9)$  with fundamental classes  $\varepsilon_9, \varepsilon_9', \varepsilon_9''$  which are sent by transgression on  $p_8^*(v_7), u_8, v_8$  respectively.

$$H^{n+11}(Y_9; Z_2) \cong Z_2(u_9), \quad \text{where } i_9^*(u_9) = Sq^2(\varepsilon_9).$$

We have seen that this statement implies Lemma 5.1, hence the proof is complete.

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