AN INFINITE DIMENSIONAL VERSION OF SARD'S THEOREM.

By S. SMALE.*

The purpose of this note is to introduce a non-linear version of Fredholm operators and to prove that in this context, Sard's Theorem holds if zero measure is replaced by first category (Section 1). We give applications to local uniqueness in non-linear elliptic equations (Section 2) and to define a notion of degree (Section 3).

Section 1. We recall first some definitions and facts from the linear theory (see [3], [4]) for details. A *Fredholm operator* is a continuous linear map $L: E_1 \to E_2$ from one Banach Space to another with the properties:

a) $\dim \operatorname{Ker} L < \infty$

b) Range L is closed

c) $\operatorname{Coker} L = E_{2/\mathrm{Range}\, L}$ has finite dimension.

If L is Fredholm, then its *index* is $\dim \operatorname{Ker} L - \dim \operatorname{Coker} L$ so that the index of L is an integer.

(1.1) THEOREM. *The set $F(E_1, E_2)$ of Fredholm operators is open in the space of all bounded operators $L(E_1 E_2)$ in the norm topology. Furthermore the index is continuous on $F(E_1, E_2)$.*

For a proof see [4].

The non-linear extension of the preceding notion seems to fit best into the context of differentiable manifolds locally like Banach spaces (see Lang [5]), called here differentiable manifolds and denoted by M, V. We will assume all our manifolds to be connected and to have a countable base.

A *Fredholm map* is a C' map $f: M \to V$ such that for each $x \in M$, the derivative $Df(x): T_x(M) \to T_{f(x)}(V)$ is a Fredholm operator. The *index* of f is defined to be the index of $Df(x)$ for some x. By (1.1), since f is C' and M is connected, the definition doesn't depend on x.

Let $f: M \to V$ be any C' map. A point $x \in M$ is called a *regular point* of f if $Df(x): T_x(M) \to T_{f(x)}(V)$ is surjective and is *singular* if not regular. The images of the singular points under f are called the *singular values* or

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critical values and their complement the *regular values.* Note that if $y \in V$ is not in the image of f it is automatically a regular value. We will need the following.

(1.2) SARD THEOREM. *Let U be an open set of R^p and $f\colon U \to R^q$ be a C^s map where $s > \max(p-q, 0)$. Then the set of critical values in R^q has measure zero.*

For a proof see [8] or [9].

We will say *almost all* instead of "except for a set of first category" etc.

Our main theorem is

(1.3) THEOREM. *Let $f\colon M \to V$ be a C^q Fredholm map with $q > \max(\operatorname{index} f, 0)$. Then the regular values of f are almost all of V.*

The condition that f be Fredholm is necessary from the example of I. Kupka (to be published) of a C^∞ real function on Hilbert space with critical values containing an open set. His example extends easily to give a C^∞ map from one separable Hilbert space to another with critical values possessing an interior point.

Strictly speaking (1.3) is a generalization of a theorem of A. B. Brown, an earlier special case of Sard's Theorem [A. B. Brown, *Transactions of the American Mathematical Society,* vol. 38 (1935), pp. 379-394].

(1.4) COROLLARY. *If $f\colon M \to V$ is a Fredholm map of negative index, its image contains no interior points.*

(1.5) COROLLARY. *If $f\colon M \to V$ is a C^q Fredholm map, $q > \max(\operatorname{Index} f, 0)$, then for almost all $y \in V$, $f^{-1}(y)$ is a submanifold of M whose dimension is equal to index f or is empty.*

We now prove (1.3). Since M has a countable base and first category is closed under countable union, it is sufficient to prove the theorem locally. Thus we can assume given $f\colon U \to E'$, where U is an open set of a Banach space E, and E' is another Banach space.

Let $x_0 \in U$, $A = Df(x_0)\colon E \to E'$. Since $\dim \operatorname{Ker} A < \infty$, E can be written in the form $E_1 \times \operatorname{Ker} A$, E_1 a Banach space and $x_0 = (p_0, q_0)\, p_0 \in E_1$, $q_0 \in \operatorname{Ker} A$. Then the first partial derivative $D_1 f(p,q)\colon E_1 \to E$ maps E_1 injectively onto a closed subspace of E for all (p,q) sufficiently close to (p_0, q_0). Thus using the implicit function theorem, we can choose a product neighborhood $D_1 \times D_2$ of (p_0, q_0) in $E_1 \times \operatorname{Ker} A$ such that D_2 is compact and if $q \in D_2$, f restricted to $D_1 \times q$ is a (differentiable) homeomorphism onto its image.

(1.6) THEOREM. *A Fredholm map is locally proper. In other words,* *if $f: M \to V$ is Fredholm and $x \in M$, there exists a neighborhood N of x such that f restricted to N is proper.*

A map is *proper* if the inverse image of a compact set is compact.

To prove (1.6) choose $N(x) = D_1 \times D_2$ as above and let $f(x_i) = y_i \to y$, $x_i = (p_i, q_i) \in D_1 \times D_2$. It is sufficient to show that the x_i have a convergent subsequence. Since D_2 is compact we may assume $q_i \to q$ and since $f(p_i, q) \to y$ even that $q_i = q$. But f restricted to $D_1 \times q$ is a homeomorphism onto its image, so $p_i \to p$, proving (1.6).

To prove (1.3), let $x_0 \in M$ and choose again $D_1 \times D_2 \subset E_1 \times \operatorname{Ker} A$ as above. The critical points of f are closed and therefore by the preceding theorem it is sufficient, given a neighborhood U_1 of $f(x_0)$ in E', to find a regular value of f in U_1.

Let $\pi: E' \to E'/\operatorname{Range} A$ be the projection. From the hypotheses of (1.3), (1.2) applies to the map $\phi: p_0 \times \operatorname{Ker} A \to E'/\operatorname{Range} A$ defined by $\phi(q) = \pi \cdot f(p_0, q)$ to give as a regular value z of ϕ in πU_1. Let $y \in \pi^{-1}(z) \cap U_1$. Then y is our desired regular value and our proof is finished.

Section 2. We will prove a local uniqueness theorem for the case of non-linear elliptic equations of 2nd order for domain in R^n with Dirichlet boundary conditions. Obviously the proof is valid in much greater generality.

Let Ω be a bounded domain in R^n with boundary, $\partial\bar{\Omega}$ a smooth sub-manifold. Define $C^k(\bar{\Omega}), k$ a non-negative integer to be the Banach space of C^k functions on $\bar{\Omega}$ with the C^k norm (see [1] or [2]). Let α satisfy $0 < \alpha < 1$, and let $C^{k+\alpha}(\bar{\Omega})$ be the space of C^k functions on $\bar{\Omega}$ with k-th derivative Hölder continuous, exponent α, endowed with the corresponding Banach space structure (again see [1] for example). If $f_0 \in C^{2+\alpha}(\bar{\Omega})$ let $C_{f_0}{}^{2+\alpha}(\bar{\Omega})$ denote the affine subspace of $C^{2+\alpha}(\bar{\Omega})$ of maps which agree with f_0 on $\partial\bar{\Omega}$.

Let $J^2(\bar{\Omega}) = \bar{\Omega} \times R \times L(R^n, R) \times L_s{}^2(R^n, R) = \{(x, p_0, p_1, p_2)\}$ where $L(R^n, R)$ denotes the space of linear maps $R^n \to R$, and $L_s{}^2(R^n, R)$ the space of symmetric bilinear maps $R^n \times R^n \to R$. Thus if $f \in C^2(\bar{\Omega})$ we can define $j_2 f: \bar{\Omega} \to J^2(\bar{\Omega})$ by $j_2(f)(x) = (x, f(x), D^1 f(x), D^2 f(x))$, D^1, D^2 denoting the first and second derivative respectively of f at x.

A *(non-linear) partial differential equation*, $\Phi(u) = g$ (2nd order etc.) is defined by a map $F: J^2(\bar{\Omega}) \to R$ which we will assume to be C^∞, as follows. Let $\Phi: S^s(\bar{\Omega}) \to C^{s-2}(\bar{\Omega})$ be the map $\Phi(u)(x) = F(j_2 u)(x)$, $s \geqq 2$. We will consider solutions of $\Phi(u) = f$ for given $f \in C^{s-2}(\bar{\Omega})$. Of course $\Phi u = g$ can be written as $F(x, u(x), Du(x), D^2 u(x)) = g(x)$.

We say that F (or Φ) is *elliptic* if the partial derivative

$$F_{p_2}(x, p_0, p_1, p_2) \in L_s{}^2(R^n; R)$$

is a positive or negative definite bilinear form for each $(x, p_0, p_1, p_2) \in J^2(\bar{\Omega})$.

(2.1) Suppose $F: J^2(\bar{\Omega})\ R$ is elliptic as above and *linear* (i.e., for each x is linear on $x \times R \times L(R^n, R) \times L_s{}^2(R^n, R)$), so $L = \Phi: C_{f_0}{}^{2+\alpha}(\bar{\Omega}) \to C^\alpha(\bar{\Omega})$ defined by F is also linear, $f_0 \equiv 0$. Then L is a Fredholm optrator of index zero.

This is well known and essentially proved in such greater generality in [1] and [2].

As an immediate consequence of (2.1) we obtain

(2.2) LEMMA. *If $F: J^2(\bar{\Omega}) \to R$ is an elliptic non-linear partial differential equation and $f_0 \in C^{2+\alpha}(\bar{\Omega})$, then the induced map $\Phi: C_{f_0}{}^{2+\alpha}(\bar{\Omega}) \to C^\alpha(\bar{\Omega})$ is a Fredholm map of index zero with derivatives at $u \in C_{f_0}{}^{2+\alpha}(\bar{\Omega})$ given by*

$$D\Phi(u)(\eta) = \sum_{i=0}^{2} F_{p_i}(x, u(x), Du(x), D^2u(x)) D^i\eta(x)$$

where $\eta \in C_0{}^{2+\alpha}(\bar{\Omega})$ (i.e. $\eta \in C^{2+\alpha}(\bar{\Omega})$, $\eta(x) = 0$ for $x \in \partial\bar{\Omega}$), and $D^0\eta = \eta$.

From (1.5) and (2.2) we obtain "local uniqueness" for solutions of $\Phi(u) = g$ in the following form.

(2.3) THEOREM. *If $F: J^2(\bar{\Omega}) \to R$ is an elliptic non-linear differential equation and $f_0 \in C^{2+\alpha}(\bar{\Omega})$, for almost all $g \in C^\alpha(\bar{\Omega})$, the set of $u \in C_{f_0}{}^{2+\alpha}(\bar{\Omega})$ such that $F(x, u(x), Du(x), D^2u(x)) = g(x)$ is discrete.*

Section 3. Let $f: M \to V$ be a C' map and $g: W \to V$ be a C' imbedding. We say that f is transversal to g if for each $(x, y) \in M \times W$ such that $f(x) = g(y)$ the spaces range $Df(x)$, range $Dg(y)$, span the tangent space $T_{f(x)}(V)$.

(3.1) THEOREM. *Let $f: M \to V$, be a C^q Fredholm map and $g: W \to V$ a C' imbedding of a finite dimensional manifold W with*

$$q > \max(\operatorname{Index} f + \dim W, 0).$$

Then there exists a C' approximation g' of g such that f is transversal to g'. Furthermore if f is transversal to the restriction of g to a closed set A of W. then g' may be chosen so that $g' = g$ on A.

We remark that the usual finite dimensional version of this theorem [10] gives an approximation f' of f which requires a partition of unity on M, but requires less differentiability. In view of applications in the direction of Section 2, we wish to avoid such an assumption on M.

Since M and W have a countable base, a standard argument reduces the proof of (3.1) to a local lemma, namely the following.

(3.2) LEMMA. *Let* $y \in W$, *M*, *W* *as in* (3.1). *Then there exists a neighborhood* U_1 *of* y *and for any* $\epsilon > 0$ *an* $\epsilon C'$ *approximation* g' *of* g *such that* f *is transversal to* g'/U_1.

Proof of (3.2). We can assume that we have a neighborhood of $g(y)$ in V given as follows. $U_2 \subset R^p$, $V =$ Banach space $F = R^p \times F_1$ and $g: U_2 \to R^p \times 0$ is the identity $\times 0$, $\pi: F \to F_1$ the projection. Let U_1 be a neighborhood of y with $\mathrm{Cl}\, U_1 \subset \mathrm{interior}\, U_2$ and ϕ a C^∞ function 1 on U_1, 0 on exterior U_2. Now by (1.3) let $z \in F_1$ be close to 0 and such that $\pi \cdot f$ has z as a regular value on $f^{-1}(g(U_2))$. Define g' as the translate of g by z on U_1 smoothed by ϕ.

This proves (3.2) and therefore (3.1) (That the last statement of (3.1) is true is clear from the argument).

(3.3) THEOREM. *Let* $f: M \to V$ *be a Fredholm map transversal to the imbedding* $g: W \to V$, $\dim W < \infty$. *Then* $f^{-1}(g(w))$ *is a submanifold of* M, *of dimension equal to the index* $f + \dim W$.

The proof is the same as the finite dimensional case [10], as well as for the modification of (3.3) to the case that $\partial W \neq \phi$.

We wish to define a generalized degree for proper Fredholm maps.

Recall that a *closed* map is one in which the image of a closed set is closed. Quite generally a proper map is closed and a closed map such that the inverse images of points are compact is proper.

A Fredholm map is locally proper (1.6) and the interested reader will be able to verify the following lemma.

(3.4) LEMMA. *If* $f: M \to N$ *is a Fredholm map and is closed where* $\dim M = \infty$, *then* f *is proper.*

Let η be the non-oriented cobordism ring of Thom (see [10]) with η^p consisting of classes of p dimensional manifolds, $p = 0, 1, \cdots$.

Let $f: M \to N$ (reminding the reader that M and N are connected) be a C^q proper Fredholm map, $q > p + 1$ where $\mathrm{p} = \mathrm{index}\, f \geqq 0$. We define an invariant (generalized degree mod 2) $\gamma(f) \in \eta^p$ as the class of $f^{-1}(y)$, y a regular value of f (see 1.5). To see that $\gamma(f)$ is independent of y let y_1 be another regular value of f and suppose $g: I \to V$ is an imbedding of the unit interval with $g(0) = y$, $g(1) = y_1$. By (3.1) we suppose f is transversal to g. Then by (3.3) $f^{-1}(g(I))$ effects a cobordism relation between $f^{-1}(y)$ and $f^{-1}(y_1)$, thus giving us the invariance of $\gamma(f)$. Remembering that a point of V not in the range of f is automatically a regular value, we summarize.

(3.5) THEOREM. *Let* $f: M \to V$ *be a* C^q *proper Fredholm map with*

$p = \operatorname{index} f \geqq 0$, *and* $q > p+1$. *Then there exists an invariant* $\gamma(f) \in \eta^p$ *defined by the non-oriented cobordism class of* $f^{-1}(y)$, y *some regular value of* f. *If* $\gamma(f) \neq 0$, f *is surjective.*

If the index of f is zero then $f^{-1}(y)$ is a finite set of points and $\gamma(f)$ is the ordinary degree reduced mod 2.

This is related to the Leray-Schauder degree [6], [7].

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