A SUMMARY OF MY MATHEMATICS - a personal perspective

My work belongs to the fields:

- *algebraic and geometric topology*,
- *geometry and geometric analysis*,
- *homological algebra (cyclic homology) and K-theory*,
- *applied (computational) topology.*

Much of this work was done in collaboration with mathematicians of various expertise. Here are some of the most known results:

1. *homotopy equivalent infinite dimensional smooth Hilbert manifolds are diffeomorphic and homotopic diffeomorphisms between such manifolds are isotopic*; analogue statements remain true for topological Hilbert manifolds and homeomorphisms, cf. (25),(27),(28),

2. *the structure of cyclic homology of the group rings* [1] and generalizations, cf. (55), (59),

3. *the rank of Waldhausen algebraic K-theory of a 1-connected space X is the same as the rank of the rank of S^1-equivariant homology of the free loop space of X* [2] , cf. (58), (59),

4. *the calculation of the rank of the homotopy groups of the space of diffeomorphisms and homeomorphisms in stability range* for 1-connected compact manifolds [3], cf. (47),(49),(64),

5. *the structure of cyclic homology of CDGAs (commutative differential graded algebras) and applications*, cf. (56), (61),(66),(93),

6. *the BFK (Burghelea-Friedlander-Kappeler) formula for the regularized determinant of elliptic operators* [4] *on a manifold cut in two pieces*, cf. (67),

7. *the calculation of the regularized determinant of an elliptic operator in vector bundles over $1-$dimensional manifolds*, cf. (65),(68),(71),

8. *the equality of the L_2 analytic and the L_2 combinatorial (Reidemeister) torsion*, cf. (73),

9. *the complex-valued extension of the Ray-Singer analytic torsion* [5] cf. (88), (92), (90).

I consider the following contributions, although only partially finalized, not less relevant:

a. *the AMT (alternative to Morse theory) and the AMNT (alternative to Morse-Novikov theory)* [6], cf. (B6), (106), (104),

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[1] $R[G]$, with R a commutative ring, G a group

[2] shifted by one unit

[3] in terms of the rank of S^1-equivariant homology of the free loop space and other familiar algebraic topology invariants of the manifold

[4] zeta regularized determinant

[5] the Ray-Singer torsion is a positive real-valued number

[6] the classical theories apply to smooth manifolds and Morse real-valued smooth function resp.Morse closed differential one form, the alternatives proposed apply to arbitrary ANR's and tame continuous real valued functions resp. tame topological closed one form, and derive almost the same conclusions and are based on computer friendly invariants, bar codes

b. *the algebraic topology of free loop spaces and string cohomology*, cf. (62), (64),(66), (74),(93)

c. *the cornered-manifold structure and the compactification of the stable/unstable sets and of the space of trajectories between rest points of some vector fields* [7], cf. (83), (94), (96),

d. *the homotopy category of spectra*, cf. (17), (20), (24), (15),

e. *the "torsions" viewed as functions on the varieties of representations*, cf. (86), (88), (90),

f. *the "virtually small spectral package" of a Riemannian manifold in the presence of a Morse function*, cf.(105).

I also regard as notable the work in collaboration with R. Lashof on

i. the *refinement of smoothing theory = diffeomorphisms versus homeomorphisms*, cf. (39), (40), motivated by efforts to understand a theorem of C. Morlet, with A. Verona on

ii. the *local homological properties of analytic sets*, cf. (37), motivated by the search of an alternative proof for a result of D. Sullivan on this topic as well as the work with A. Assadi and with R. Schultz, on

iii. the *symmetry of manifolds*, cf. (41), (48), (51).

Bellow are a few comments about the results 1-9, and a-f .

About (1): The diffeomorphism of homotopy equivalent Hilbert manifolds was proven in collaboration with N.Kuiper (under a mild hypotheses subsequently removed), cf.(25) and the isotopy of homotopic diffeomorphisms in cf.(27). The case of topological manifolds was done in collaboration with D. Henderson, cf.(28). Complemented with refinements, the results were applied to nonlinear differential equations, work done in collaboration with N. Saldanha and C. Tomei, cf. (84), (91).

These results imply that the homotopy theory and the differential topology of infinite dimensional Hilbert manifolds are equivalent. As noticed by R. Thom in late 60's, this work has revised the role of infinite dimensional manifolds in global analysis. Part of the results were reported in a Bourbaki seminar [8] and the work was cited in J. Dieudonné's book, *Panorama des mathematiques pures; le choie Bourbachique.*

About (2): This work begins with a conceptual calculation of the cyclic homology of a group ring $R[G]$, R a commutative ring with unit, which decomposes as a direct sum of contributions indexed by the conjugacy classes of elements of the group, cf.(55); there is a difference between the contributions of finite and infinite order elements. Interpreted topologically, this calculation identifies the cyclic homology to the S^1-equivariant homology of the free loop-space of the classifying space BG of the group G. The calculation was extended, in joint work with Z. Foedorowicz, to G a simplicial group implying (for $G = \Omega X$) the isomorphism of the cyclic homology of $\mathbb{R}[\Omega X]$ to the S^1-equivariant homology with coefficients in R of the free loop space of X, cf. (59).

The calculation for G a discrete group was essential in the partial solutions of Bass' conjecture by B. Eckmann and I.Emmanoil, useful in Connes-Moscovici partial solution of

[7] Morse-Smale vector fields

[8] by N. Moulis, 1969/70

Novikov conjecture, and partial answers to a conjecture in geometric group theory (referred to as Burghelea's conjecture) and in representation theory of discrete groups.

About (3): Waldhausen has extended the algebraic K-theory of group rings $Z[G]$ to a topological spaces X, introducing a new homotopy functor $A(X)$[9] which turned out to be highly relevant for geometric topology. Using cyclic homology and rational homotopy theory I have calculated, for any X 1-connected, the rank of the homotopy groups of $A(X)$ in terms of cyclic homology, cf. (58), and then in collaboration with with Z. Fiedorowicz in terms of the S^1-equivariant homology of the free loop space of X, cf. (59). These calculations have inspired the Hermitian algebraic $K-$theory of simplicial rings and topological spaces, proposed and studied in collaboration with Z.Fiedorowicz, cf.(57), and used for a better formulation of the above calculations.

About (4): The word *automorphism* means diffeomorphism for smooth manifolds and homeomorphism for topological manifolds. The calculation is a consequence of four independent pieces of algebraic and geometric topology:
- the proof of the existence of *stability range* for concordances [10],
- the invention of the group of *block automorphisms*, [11], cf. (B1),
- the proof that the homotopy type of the group of automorphisms in stability range and localized away from the prime 2 is the product of the group of block automorphisms and of the *quotient block automorphisms modulo automorphisms*, in collaboration with R. Lashof cf. (49),
- the calculation of the rank of homotopy groups of *block automorphisms modulo automorphisms* ultimately in terms of equivariant homology of the free loop space of the manifold, cf.(47), (64). Parts of this long work were cited in two Bourbaki seminars [12]

About (5): The Hochschild and cyclic homology of a commutative algebra A over a field κ of characteristic zero equipped with the $\lambda-$ operations were extended to CDGAs (commutative differential graded algebras), (A^*, d^*). When (A^*, d^*) is free and finite generated, it was shown that they can be calculated quite effectively, and if in addition is minimal have a simple descriptions, cf. (61),(66), (74), (93).

Indeed, by the work of D. Sullivan, for any finite generated CDGA, one can find free CDGAs called *models* which calculates the same Hochschild cyclic homology and $\lambda-$operations and, when $H^0(A, d) = \kappa$ and $H^1(A, d) = 0$ one can find minimal models all isomorphic.

As applications one obtains :

–in commutative algebra, a simpler proof of the striking properties of eigenspaces of $\lambda-$operations and of the relations between their dimensions,

[9] with values $\infty-$loop space

[10] stability range is an increasing function $\omega(n)$ so far larger than $n/3$ in the smooth case and larger than $n/6$ in topological case; its existence stated by A. Hatcher was finally proven by K. Igusa in smooth case and in topological case concluded by Burghelea-Lashof in (43), while concordance of M means automorphism of $M \times [0, 1]$ which restricts to the identity on $M \times 0$

[11] an extension of the group of automorphism whose homotopy groups are computable by *surgery theory*, work in collaboration with P.Antonelli and P.Kahn

[12] by J.L.Loday on Waldhausen K-theory and P.Cartier on Connes cyclic homology resp.

–in algebraic geometry, the geometric interpretations of the Hochschild and the cyclic homology of the coordinate ring of an affine complete intersection, in particular of a hypersurface, in collaboration with M. Vigué, cf. (61),

–in topology, the isomorphism between the Hochschild homology/ cyclic homology of de-Rham algebra of a smooth manifold M equipped with the $\lambda-$operations with the cohomology/ S^1-equivariant cohomology [13] of the free loop space M^{S^1} equipped with the power maps [14] induced endomorphisms, when M is simply connected, cf. (66) (74).

About (6+7): Regularized determinant means $\zeta-$regularized determinant.

The formula for the *regularized determinant of elliptic differential operator in vector bundles over* S^1, cf. (65), was motivated by the need of applying stationary phase method to some functional integrals on the space of closed curves (closed strings). It turns out to have a remarkable structure which generalizes to regularized determinants on manifolds which are product with S^1. Its extension to pseudo differential operators, cf. (68) and combined with BFK formula helps to calculate regularized determinants of elliptic operators in vector bundles over 2-dimensional manifolds with arbitrary accuracy. A similar formula was also derived for the line segment $I = [a, b]$.

The BFK formula, was the essential ingredient in the proof of equality of analytic and Reidemeister torsion, both for classical torsions [15] and for L_2 torsions, cf.(73). All formulae were obtained in collaboration with L. Friedlander and Th. Kappeler.

About (8): It was observed since early 60's that the linear algebra which emerged from the work of von Neumann in early 40's can be used to produce new numerical invariants for non simply connected compact manifolds with infinite fundamental group, referred to as L_2-topological invariants, These invariants can be defined both analytically and combinatorially and expected and ultimately proven to be equal.

The equality of the L_2-analytic and L_2-Reidemeister = combinatorial torsion was an important conjecture in the field because of the geometric meanings of these numbers but more difficult to verify. The conjecture was proved in 1996, cf. (73), in collaboration with Friedlander, Kappeler and McDonald using analytic tools we have developed or extended [16]. The paper presents, in addition to the proof of the conjecture, the analytic tools necessary for the analytic approach to many L_2 topological invariants. The work led to new concepts like *determinant class for groups, determinant class for elliptic operators in bundles of Hilbert modules* and new conjectures about them.

About (9): In joint work with Stefan Haller we have provided an answer to the following question: *is the Ray-Singer torsion, as a positive real-valued function on the variety of acyclic* [17] *unitary representations of the fundamental group* Γ *of a compact manifold, the absolute value of a holomorphic (or possibly meromorphic function) on the variety of complex acyclic representations of* Γ*? If so, what is the geometric meaning of the phase*

[13] with coefficients in $\mathbb{R}$ or $\mathbb{C}$

[14] the power maps $P_k : M^{S^1} \to M^{S^1}$ are defined by going along the free loop $k-$times faster

[15] as an alternative to the Cheeger and Muller proofs cf.(72)

[16] extension of item 6 above was essential

[17] for a space X a representation ρ of its fundamental group is called acyclic if $H^*(X; \rho) = 0$

function and of the poles of this meromorphic function ?

The answer was: yes locally, and with qualifications globally. The mathematics we use in analytic case was the spectral theory of non self-adjoint operators of Laplace-Beltrami type as opposed to the spectral theory of self adjoint operators of such type used in the study of standard Ray-Singer torsion.

In order to make this function computable via combinatorial topology (Reidemeister torsion) and differential geometry, as in the case of classical Ray-Singer torsion, a number of improvement of previous concepts or new concepts, like Euler and co-Euler structures, and the relation between them, were introduced and analyzed, cf. (86), (87).

About (a): The bar codes [18], are computer friendly [19] refinements of *Betti numbers* defined by a real-valued map, and are the relevant invariants in AMT. We have defined similar refinements of Novikov-Betti numbers for a topologically closed one form = Cech 1-cocycle, abusively still named barcodes, as the relevant invariants in AMNT, cf. (104).

The AMT resp. AMNT weaken the hypotheses of classical MT resp, MNT [20] but retain the existence of the Morse complex resp. the Morse-Novikov complex with their homological implications cf.(B6), (106), (104). In case of classical MT resp. MNT we can derive the Morse complex for a Morse real valued function resp. the Morse-Novikov complex for a Morse closed 1- form from the associated bar codes, but such complexes can be obtained under much weaker hypotheses of AMT resp. AMNT, cf. (B6), (106), (104).

In the context of AMNT we have introduced, in collaboration with Tamal Dey and Stefan Haller, *Jordan blocks* [21], computer friendly invariants for circle-valued maps. In addition to their potential use in data analysis, their calculation also provides algorithms to calculate the *monodromy* in algebraic geometry and the *Alexander polynomial* in knot theory, via the linear algebra of *linear relations*, cf. (98), (102), (A6),(B6).

About (b): My interest in the algebraic topology of the free loop space was motivated by some apparent analogy with algebraic geometry. Precisely:

– I regard a space X, contained in the free loop space X^{S^1} as the fixed points set of the action of the isometry group of S^1 on X^{S^1}, in analogy with an algebraic variety V over a field κ included in $\overline{V} = V \otimes \overline{\kappa}$ [22], with $\overline{\kappa}$ the algebraic closure of κ, as the fixed point set of the action of the Galois group of the extension $\kappa \subset \overline{\kappa}$ on $\overline{V}$.

– When V is a projective variety over a finite field of characteristic p I regard the cohomology/ S^1-equivariant cohomology of X^{S^1} with coefficients in a field of characteristic zero, equipped with the power maps induced endomorphisms, in analogy with the $l-$addic cohomology of $\overline{V}$, $l \neq p$, equipped with the *Geometric Frobenius* induced endomorphisms.

In both situations, topology and algebraic geometry, the endomorphisms have eigenvalues and eigenspaces with remarkable properties and related via remarkable number theoretic functions but apparently not the similar. In topology these facts were new results. The mathematics is based on item (5) above.

[18] intervals with multiplicity $[a, b], a, b \in \mathbb{R}$ of four types, closed, open, closed-open and open-closed

[19] computable by computer implementable algorithms

[20] from manifolds to metrizable spaces, from smooth Morse function resp. Morse closed 1-form to tame continuous real-value map resp. topological closed one form

[21] i.e. $k \times k$ Jordan matrix with λ on diagonal, $\lambda \in \kappa$ a field

[22] completion of V o the algebraic closure of κ

The calculus with differential forms on the Fréchet manifold model of the free loop space of a smooth manifold, based on results in items (1), (3) and (5), permit to define a new $\mathbb{R}/\mathbb{C}-$ vector space valued homotopy functor, the string cohomology which, to some extent, unifies the topological (Atiyah-Hirtzebruch) $K-$theory to the dual of Waldhausen algebraic $K-$theory.

For X 1-connected, the string cohomology is a direct sum of these two $K-$theories tensored by $\mathbb{R}/\mathbb{C}$. The string cohomology, although a homotopy functor defined on all nice topological spaces (i.e. ANR's), was so far defined only using differential forms on infinite dimensional manifolds and based on results in infinite dimensional topology extended to all spaces. In view of results of item 4, it appears to relate *gauge theory* to *gravitation theory* at least at the level of topological invariants.

About (c): The space of trajectories between two rest points and the stable and unstable sets of a Morse-Smale vector fields are smooth manifolds. We have provided a completion of these manifolds to manifolds with corners, with corners precisely described in terms of broken trajectories; when the vector field admits a proper smooth Lyapunov function, in particular it is the gradient of a proper Morse function with respect to a Riemannian metric, these manifolds with corners are compact, cf (A1), (69), (83), (94), (96). The result implies refinements and improvements of Morse theory, Morse-Bott theory, G-Morse theory, cf. (A1), and in combination with WHS (Witten-Hellfer-Sjöstrand)-theory improves on Hodge theorem on closed Riemannian manifold [23].

About (d): Using the semi-simplicial model for *spectra* in homotopy theory introduced by D. Kan and G. Whitehead, we have shown that the homotopy category of spectra can be canonically extended to an abelian category with all basic constructions in the theory of abelian category and the homotopy theory [24] being doable in compatible way. Such thing is not possible in the setting of stable homotopy category of ANRs. Using appropriately this category one can achieve considerable simplifications in a number of construction in algebraic and geometric topology like the cobordism theory, the so called Floer homotopy theory proposed recently and possibly more. The motivation of this work was a conjecture of P. Freyd about the generators in the stable homotopy category of compact ANRs, a full subcategory of the homotopy category of spectra, to our knowledge still unsolved, but solvable in our category. This work, joint with A. Deleanu, has received the Stoilow price in 1967.

About (e): In analogy with linear algebra over the complex numbers, where for a finite dimensional vector space equipped with a linear transformation one associates, using the *determinant of a square matrix*, the characteristic polynomial of the linear transformation, we associate to a closed Riemannian manifold equipped with a Riemannian metric resp. a triangulation resp. a conservative vector field, using *torsion*, a characteristic function type invariant, the complex Ray-Singer torsion, resp. Milnor-Turaev torsion, resp. dynamical

[23] not only the de-Rham cohomology classes have unique representation by differential form *harmonic forms*, but, in the presence of a Morse function, the entire geometric complex defined by g and f has a canonical realization as a sub complex of differential forms

[24] i.e. killing up or down the stable homotopy groups, equivalently, existence of both Postnikov and Cartan-Serre resolutions

torsion. They are meromorphic functions, conjecturally rational functions on affine varieties of complex representations of the fundamental group. They are supposed to carry relevant topological and geometric information through their zeros, their poles and their values, in analogy with the zeros of the characteristic polynomial of a linear transformation. A number of remarkable polynomials (Alexander polynomial of a knot) and rational functions (Lefschetz zeta function of a diffeomorphism) and other counting functions in dynamics appear as restriction to various complex lines in varieties of such representations of such meromorphic functions. So far the mathematics involved is algebraic geometry and spectral theory of non self-adjoint elliptic operators. The paper (90) summarizes part of our work done so far.

About (f): By the spectral package of a closed Riemannian manifold one understands the infinite collection of eigenvalues and eigenvectors =eigenforms of the Laplace-Beltrami operators in all degrees. A Morse function on the manifold decomposes canonically the spectral package as a countable disjoint union of small spectral packages indexed by $\mathbb{Z}_{\geq 0} \cup \infty$. All these packages but the package indexed ∞, which conjecturally is empty, have finite and specified cardinality. The smallest, the ground zero package, of cardinality equal the cardinality of critical points, is referred to as the *virtually small spectral package* and carries, most likely, all topological information which can be derived from the entire spectral package. Being finite, it can be calculated with arbitrary accuracy. The virtually small package does not necessary contain the first smallest eigenvalues but does contain the eigenvalue 0 with the harmonic forms as the eigenforms. Its existence, inside the de-Rham complex of the manifold, represents an interesting extension of the Hodge theorem in Riemannian geometry. The study of the *virtually small spectral package,* its calculation with arbitrary accuracy, as well as of other finite indexed packages are challenging research problems. This research in progress, partly summarized in (105), was presented as a 6 weeks mini-course at IMAR- Bucharest in 2021, intended to be expanded into a book on the spectral package of a Riemannian manifold in the presence of a Morse function.