§9. Oriented Bundles and the Euler Class

Up to this point we have always used the coefficient group $\mathbb{Z}/2$ for our cohomology. This of necessity means that we have overlooked much interesting structure. Now we will take a closer look, using the integers $\mathbb{Z}$ as coefficient group. But in order to do this it will be necessary to impose the additional structure of an orientation on our vector bundles. In particular we will need an orientation in order to construct the fundamental cohomology class $u \in H^n(E, E_0)$ with integer coefficients.

First consider the case of a single vector space.

DEFINITION. An *orientation* of a real vector space V of dimension $n > 0$ is an equivalence class of bases, where two (ordered) bases $v_1, \ldots, v_n$ and $v'_1, \ldots, v'_n$ are said to be equivalent if and only if the matrix $[a_{ij}]$ defined by the equation $v'_i = \sum a_{ij} v_j$ has positive determinant. Evidently every such vector space V has precisely two distinct orientations. Note that the coordinate space R^n has a canonical orientation, corresponding to its canonical ordered basis.

In algebraic topology, it is customary to specify the orientation of a simplex by choosing some ordering of its vertices. Our concept of orientation is related as follows. Let Σ^n be an n-simplex, linearly embedded in the n-dimensional vector space V, with ordered vertices $A_0, A_1, \ldots, A_n$. Then taking the vector from A_0 to A_1 as first basis vector, the vector from A_1 to A_2 as second, and so on, we obtain a corresponding orientation for the vector space V.

Note that a choice of orientation for V corresponds to a choice of one of the two possible generators for the singular homology group $H_n(V, V_0; \mathbb{Z})$. In fact let Δ^n denote the standard n-simplex, with canonically ordered vertices. Choose some orientation preserving linear embedding

$$\sigma : \Delta^n \to V$$

which maps the barycenter of Δ^n to the zero vector (and hence maps the boundary of Δ^n into V_0). Then σ is a singular n-simplex representing an element in the group of relative n-cycles $Z_n(V, V_0; \mathbb{Z})$. The homology class of this n-cycle σ is now the preferred generator μ_V for the homology group $H_n(V, V_0; \mathbb{Z})$.

Similarly the cohomology group $H^n(V, V_0; \mathbb{Z})$ associated with an oriented vector space V has a preferred generator which we denote by the symbol u_V, determined by the equation $\langle u_V, \mu_V \rangle = +1$.

Now consider a vector bundle ξ of fiber dimension $n > 0$.

DEFINITION. An *orientation* for ξ is a function which assigns an orientation to each fiber F of ξ, subject to the following local compatibility condition. For every point b_0 in the base space there should exist a local coordinate system (N, h), with $b_0 \in N$ and $h : N \times R^n \to \pi^{-1}(N)$, so that for each fiber $F = \pi^{-1}(b)$ over N the homomorphism $x \mapsto h(b,x)$ from R^n to F is orientation preserving. (Or equivalently there should exist sections $s_1, \ldots, s_n : N \to \pi^{-1}(N)$ so that the basis $s_1(b), \ldots, s_n(b)$ determines the required orientation of $\pi^{-1}(b)$ for each b in N.)

In terms of cohomology, this means that to each fiber F there is assigned a preferred generator

$$u_F \in H^n(F, F_0; \mathbb{Z}) \ .$$

The local compatibility condition implies that for every point in the base space there exists a neighborhood N and a cohomology class

$$u \in H^n(\pi^{-1}(N),\ \pi^{-1}(N)_0; \mathbb{Z})$$

so that for every fiber F over N the restriction

$$u \mid (F, F_0) \in H^n(F, F_0; \mathbb{Z})$$

is equal to u_F. The proof is straightforward.

The following important result will be proved in § 10. (Compare Theorem 8.1.)

THEOREM 9.1. *Let ξ be an oriented* n*-plane bundle with total space* E*. Then the cohomology group* $H^i(E, E_0; \mathbb{Z})$ *is zero for* $i < n$, *and* $H^n(E, E_0; \mathbb{Z})$ *contains one and only one cohomology class* u *whose restriction*

$$u \mid (F, F_0) \in H^n(F, F_0; \mathbb{Z})$$

is equal to the preferred generator u_F *for every fiber* F *of* ξ. *Furthermore the correspondence* $y \mapsto y \cup u$ *maps* $H^k(E; \mathbb{Z})$ *isomorphically onto* $H^{k+n}(E, E_0; \mathbb{Z})$ *for every integer* k.

In more technical language, this theorem can be summarized by saying that $H^*(E, E_0; \mathbb{Z})$ is a free $H^*(E; \mathbb{Z})$-module on one generator u of degree n. (More generally, any ring with unit could be used as coefficient group.)

It follows of course that $H^{k+n}(E, E_0; \mathbb{Z})$ is isomorphic to the cohomology group $H^k(B; \mathbb{Z})$ of the base space. In fact the *Thom isomorphism*

$$\phi : H^k(B; \mathbb{Z}) \to H^{k+n}(E, E_0; \mathbb{Z})$$

can be defined by the formula

$$\phi(x) = (\pi^* x) \cup u \quad ,$$

just as in §8.

We are now ready to define an important new characteristic class. Given an oriented n-plane bundle ξ, the inclusion (E, empty set) $\subset$ (E, E_0) gives rise to a restriction homomorphism

$$H^*(E, E_0; \mathbb{Z}) \to H^*(E; \mathbb{Z})$$

which we denote by $y \mapsto y \mid E$. In particular, applying this homomorphism to the fundamental class $u \in H^n(E, E_0; \mathbb{Z})$, we obtain a new cohomology class

$$u \mid E \in H^n(E; \mathbb{Z}) \ .$$

But $H^n(E; \mathbb{Z})$ is canonically isomorphic to the cohomology group $H^n(B; \mathbb{Z})$ of the base space.

DEFINITION. The *Euler class* of an oriented n-plane bundle ξ is the cohomology class

$$e(\xi) \in H^n(B; \mathbb{Z})$$

which corresponds to $u \mid E$ under the canonical isomorphism $\pi^* : H^n(B; \mathbb{Z}) \to H^n(E; \mathbb{Z})$.

For the motivation for the name "Euler class," we refer the reader to p. 130. Here are some fundamental properties of the Euler class:

PROPERTY 9.2. (Naturality). *If* $f : B \to B'$ *is covered by an orientation preserving bundle map* $\xi \to \xi'$, *then* $e(\xi) = f^* e(\xi')$.

In particular, if ξ is a trivial n-plane bundle, $n > 0$, then $e(\xi) = 0$. For in this case we can take ξ' to be a bundle over a point.

PROPERTY 9.3. *If the orientation of* ξ *is reversed, then the Euler class* $e(\xi)$ *changes sign.*

The proofs are immediate. ■

PROPERTY 9.4. *If the fiber dimension* n *is odd, then* $e(\xi) + e(\xi) = 0$.

Because of this, we will usually assume that the fiber dimension is even when making use of Euler classes.

First Proof. Any odd dimensional vector bundle possesses an orientation reversing automorphism $(b, v) \mapsto (b, -v)$. The required equation $e(\xi) = -e(\xi)$ now follows from 9.3.

Alternative Proof. The Thom isomorphism $\phi(x) = (\pi^*(x) \cup u$ evidently maps $e(\xi)$ to the cohomology class

$$\pi^* e(\xi) \cup u = (u \mid E) \cup u = u \cup u \ .$$

In other words

$$e(\xi) = \phi^{-1}(u \cup u) \ .$$

But using the identity

$$a \cup b = (-1)^{(\dim a)(\dim b)}\, b \cup a$$

we see that $u \cup u$ is an element of order 2 whenever the dimension n is odd. ∎

PROPERTY 9.5. *The natural homomorphism* $H^n(B; \mathbb{Z}) \to H^n(B; \mathbb{Z}/2)$ *carries the Euler class* $e(\xi)$ *to the top Stiefel-Whitney class* $w_n(\xi)$.

Proof. If we apply this homomorphism (induced by the coefficient surjection $\mathbb{Z} \to \mathbb{Z}/2$) to both sides of the equation $e(\xi) = \phi^{-1}(u \cup u)$, then evidently the integer cohomology class u maps to the mod 2 cohomology class u of §8, and $u \cup u$ maps to $Sq^n(u)$. Hence $\phi^{-1}(u \cup u)$ maps to $\phi^{-1} Sq^n(u) = w_n(\xi)$. ∎

Several important properties of the characteristic class $w_n(\xi)$ apply equally well to $e(\xi)$.

PROPERTY 9.6. *The Euler class of a Whitney sum is given by* $e(\xi \oplus \xi') = e(\xi) \cup e(\xi')$. *Similarly the Euler class of a cartesian product is given by* $e(\xi \times \xi') = e(\xi) \times e(\xi')$.

Here we must specify that the direct sum $F \oplus F'$ of two oriented vector spaces is to be oriented by taking an oriented basis for F followed by an oriented basis for F'.

Proof of 9.6. Let the fiber dimensions be m and n respectively. Taking account of our sign conventions as specified in Appendix A, it is not difficult to check that the fundamental cohomology class of the cartesian product is given by

$$u(\xi \times \xi') = (-1)^{mn} u(\xi) \times u(\xi') \ .$$

(Compare the verification of Axiom 3 in §8. If we used the classical system of sign conventions, as in [Spanier], then there would be no sign here.) Now apply the restriction homomorphism

$$H^{m+n}(E \times E', (E \times E')_0) \to H^{m+n}(E \times E') \approx H^{m+n}(B \times B')$$

to both sides. It follows easily that

$$e(\xi \times \xi') = (-1)^{mn}\, e(\xi) \times e(\xi') \ ;$$

where the sign can be ignored since the right side of this equation is an element of order two whenever m or n is odd.

Now suppose that $B = B'$. Pulling both sides of this equation back to $H^{m+n}(B; \mathbb{Z})$ by means of the diagonal embedding $B \to B \times B$, we obtain the formula $e(\xi \oplus \xi') = e(\xi) \cup e(\xi')$ for the Euler class of a Whitney sum. ∎

REMARK. Although this formula looks very much like the corresponding formula $w(\xi \oplus \xi') = w(\xi) \cup w(\xi')$ for Stiefel-Whitney classes, there is one essential difference. The total Stiefel-Whitney class $w(\xi)$ is a unit in the ring $H^{\Pi}(B; \mathbb{Z}/2)$, hence it is easy to solve for $w(\xi')$ as a function of $w(\xi)$ and $w(\xi \oplus \xi')$. (Compare §4.1.) However the Euler class $e(\xi)$ is certainly not a unit in the integral cohomology ring of B, and in fact it may well be zero or a zero-divisor. So the equation $e(\xi \oplus \xi') = e(\xi) \cup e(\xi')$ cannot usually be solved for $e(\xi')$ as a function of $e(\xi)$ and $e(\xi \oplus \xi')$.

Here is an application of 9.6. Let η be a vector bundle for which $2e(\eta) \neq 0$. Then it follows that η cannot split as the Whitney sum of two oriented odd dimensional vector bundles. As an example, let M be a smooth compact manifold. Suppose that the tangent bundle τ of M is oriented, and that $e(\tau) \neq 0$. Then τ cannot admit any odd dimensional sub vector bundle. For if this sub-bundle ξ were orientable, then the Euler class $e(\tau) = e(\xi) \cup e(\xi^{\perp})$ would have to be an element of order two in the free abelian group $H^n(M; \mathbb{Z})$. (Compare Appendix A.) The case where ξ is not orientable can be handled by passing to a suitable 2-fold covering manifold of M. Details will be left to the reader.

PROPERTY 9.7. *If the oriented vector bundle* ξ *possesses a nowhere zero cross-section, then the Euler class* $e(\xi)$ *must be zero.*

Proof. Let $s : B \to E_0$ be a cross-section, so that the composition

$$B \xrightarrow{s} E_0 \subset E \xrightarrow{\pi} B$$

is the identity map of B. Then the corresponding composition

$$H^n(B) \xrightarrow{\pi^*} H^n(E) \longrightarrow H^n(E_0) \xrightarrow{s^*} H^n(B)$$

is the identity map of $H^n(B)$. By definition the first homomorphism π^* maps $e(\xi)$ to the restriction $u \mid E$. Hence the first two homomorphisms in this composition map $e(\xi)$ to the restriction $(u \mid E) \mid E_0$ which is zero since the composition

$$H^n(E, E_0) \to H^n(E) \to H^n(E_0)$$

is zero. Applying s^*, it follows that $e(\xi) = s^*(0) = 0$. ∎

[If the bundle ξ possesses a Euclidean metric, then an alternative proof can be given as follows: Let ε be the trivial line bundle spanned by the cross-section s of ξ. Then

$$e(\xi) = e(\varepsilon) \cup e(\varepsilon^{\perp})$$

by 9.6, where the class $e(\varepsilon)$ is zero by 9.2.]

To conclude this section we will describe some examples of bundles with non-zero Euler class. (See also §11 and §15.)

Problem 9-A. Recall that γ^n denotes the canonical n-plane bundle over the infinite Grassmann manifold $G_n(R^\infty)$. Show that $\gamma^n \oplus \gamma^n$ is an orientable vector bundle with $w_{2n}(\gamma^n \oplus \gamma^n) \neq 0$, and hence $e(\gamma^n \oplus \gamma^n) \neq 0$. If n is odd, show that $2e(\gamma^n \oplus \gamma^n) = 0$.

Problem 9-B. Now consider the complex Grassmann manifold $G_n(\mathbb{C}^\infty)$, consisting of all complex sub vector spaces of complex dimension n in infinite complex coordinate space. (Compare §14.) Since every complex n-plane can be thought of as a real oriented 2n-plane, it follows that there is a canonical oriented 2n-plane bundle ξ^{2n} over $G_n(\mathbb{C}^\infty)$. Show that the restriction of ξ^{2n} to the real sub-space $G_n(R^\infty)$ is isomorphic to $\gamma^n \oplus \gamma^n$, and hence that $e(\xi^{2n}) \neq 0$. (Remark: The group $H^{2n}(G_n(\mathbb{C}^\infty); \mathbb{Z})$ is actually free abelian, with $e(\xi^{2n})$ as one of its generators. See §14.3.)

Problem 9-C. Let τ be the tangent bundle of the n-sphere, and let $A \subset S^n \times S^n$ be the anti-diagonal, consisting of all pairs of antipodal unit vectors. Using stereographic projection, show that the total space $E = E(\tau)$ is canonically homeomorphic to $S^n \times S^n - A$. Hence, using excision and homotopy, show that

$$H^*(E, E_0) \approx H^*(S^n \times S^n, S^n \times S^n\text{-diagonal}) \approx H^*(S^n \times S^n, A) \subset H^*(S^n \times S^n) .$$

(Compare §11.) Now suppose that n is even. Show that the Euler class $e(\tau) = \phi^{-1}(u \cup u)$ is twice a generator of $H^n(S^n; \mathbb{Z})$. As a corollary, show that τ possesses no non-trivial sub vector bundles.

§10. The Thom Isomorphism Theorem

This section will first give a complete proof of the Thom isomorphism theorem in the unoriented case (compare §8.1), and then describe the changes needed for the oriented case (§9.1). For the first half of this section, the coefficient field $\mathbb{Z}/2$ is to be understood.

We begin by outlining some constructions which are described in more detail in Appendix A. (See in particular A.5.) Let R_0^n denote the set of non-zero vectors in R^n. For $n = 1$ the cohomology group $H^1(R, R_0)$ with mod 2 coefficients is cyclic of order 2. Let e^1 denote the non-zero element. Then for any topological space B a cohomology isomorphism

$$H^j(B) \to H^{j+1}(B \times R, B \times R_0)$$

is defined by the correspondence

$$y \mapsto y \times e^1 \quad ,$$

using the cohomology cross product operation. This is proved by studying the cohomology exact sequence of the triple $(B \times R, B \times R_0, B \times R_-)$, where R_- denotes the set of negative real numbers.

Now let B' be an open subset of B. Then for each cohomology class $y \in H^j(B, B')$ the cross product $y \times e^1$ is defined with

$$y \times e^1 \in H^{j+1}(B \times R, B' \times R \cup B \times R_0) \quad .$$

Using the Five Lemma* it follows that the correspondence $y \mapsto y \times e^1$ defines an isomorphism

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* See for example [Spanier, p. 185].

$$H^j(B, B') \to H^{j+1}(B \times R,\ B' \times R \cup B \times R_0) \ .$$

Therefore it follows inductively that the n-fold composition

$$y \mapsto y \times e^1 \mapsto y \times e^1 \times e^1 \mapsto \ldots \mapsto y \times e^1 \times \ldots \times e^1$$

is also an isomorphism. (See Appendix A for further details.) Setting

$$e^n = e^1 \times \ldots \times e^1 \in H^n(R^n, R^n_0) \ ,$$

this proves the following.

LEMMA 10.1. *For any topological space* B *and any* $n \geq 1$, *a cohomology isomorphism*

$$H^j(B) \to H^{j+n}(B \times R^n,\ B \times R^n_0)$$

is defined by the correspondence $y \mapsto y \times e^n$.

Now recall the statement of Thom's theorem. Let ξ be an n-plane bundle with projection $\pi : E \to B$.

ISOMORPHISM THEOREM 10.2. *There is one and only one cohomology class* $u \in H^n(E, E_0)$ *with* mod 2 *coefficients whose restriction to* $H^n(F, F_0)$ *is non-zero for every fiber* F. *Furthermore the correspondence* $y \mapsto y \cup u$ *maps the cohomology group* $H^j(E)$ *isomorphically onto* $H^{j+n}(E, E_0)$ *for every integer* j.

In particular, taking $j < 0$, it follows that the cohomology of the pair (E, E_0) is trivial in dimensions less than n.

The proof will be divided into four cases.

Case 1. Suppose that ξ is a trivial vector bundle. Then we will identify E with the product $B \times R^n$. Thus the cohomology $H^n(E, E_0) =$

$H^n(B\times R^n, B\times R^n_0)$ is canonically isomorphic to $H^0(B)$ by 10.1. To prove the existence and uniqueness of u, it suffices to show that there is one and only one cohomology class $s \in H^0(B)$ whose restriction to each point of B is non-zero. Evidently the identity element $1 \in H^0(B)$ is the only class satisfying this condition. Therefore u exists and is equal to $1 \times e^n$.

Finally, since every cohomology class in $H^j(B\times R^n)$ can be written uniquely as a product $y \times 1$ with $y \in H^j(B)$, it follows from 10.1 that the correspondence

$$y \times 1 \mapsto (y\times 1) \cup u = (y\times 1) \cup (1\times e^n) = y \times e^n$$

is an isomorphism. This completes the proof in Case 1.

Case 2. Suppose that B is the union of two open sets B' and B'', where the assertion of 10.2 is known to be true for the restrictions $\xi | B'$ and $\xi | B''$ and also for $\xi | B' \cap B''$. We introduce the abbreviation $B^\cap$ for $B' \cap B''$, and the abbreviations E', E'' and $E^\cap$ for the inverse images of B', B'' and $B' \cap B''$ in the total space. The following Mayer-Vietoris sequence will be used:

$$\ldots \to H^{i-1}(E^\cap, E^\cap_0) \to H^i(E, E_0) \to H^i(E', E'_0) \oplus H^i(E'', E''_0) \to H^i(E^\cap, E^\cap_0) \to \ldots .$$

For the construction of this sequence, the reader is referred, for example, to [Spanier, pp. 190, 239].

By hypothesis, there exist unique cohomology classes $u' \in H^n(E', E'_0)$ and $u'' \in H^n(E'', E''_0)$ whose restrictions to each fiber are non-zero. Applying the uniqueness statement for $\xi | B' \cap B''$, we see that these classes u' and u'' have the same image in $H^n(E^\cap, E^\cap_0)$. Therefore they come from a common cohomology class u in $H^n(E, E_0)$. This class u is uniquely defined, since $H^{n-1}(E^\cap, E^\cap_0) = 0$.

Now consider the Mayer-Vietoris sequence

$$\ldots \to H^{j-1}(E^\cap) \to H^j(E) \to H^j(E') \oplus H^j(E'') \to H^j(E^\cap) \to \ldots$$

where $j + n = i$. Mapping this sequence to the previous Mayer-Vietoris sequence by the correspondence $y \mapsto y \cup u$ and applying the Five Lemma, it follows that

$$H^j(E) \xrightarrow{\cong} H^{j+n}(E, E_0) \ .$$

This completes the proof in Case 2.

Case 3. Suppose that B is covered by finitely many open sets $B_1, \ldots, B_k$ such that the bundle $\xi \mid B_i$ is trivial for each B_i. We will prove by induction on k that the assertion of 10.2 is true for the bundle ξ.

To start the induction, the assertion is certainly true when $k = 1$. If $k > 1$, then we can assume by induction that the assertion is true for $\xi \mid (B_1 \cup \ldots \cup B_{k-1})$ and for $\xi \mid (B_1 \cup \ldots \cup B_{k-1}) \cap B_k$. Hence, by Case 2, it is true for ξ.

General Case. Let C be an arbitrary compact subset of the base space B. Then evidently the bundle $\xi \mid C$ satisfies the hypothesis of Case 3. Since the union of any two compact sets is compact* we can form the direct limit

$$\varinjlim H_j(C)$$

of homology groups as C varies over all compact subsets of B, and the corresponding inverse limit $\varprojlim H^j(C)$ of cohomology groups. We recall the following.

LEMMA 10.3. *The natural homomorphism*

$$H^j(B) \to \varprojlim H^j(C)$$

is an isomorphism, and similarly $H^j(E, E_0)$ *maps isomorphically to* $\varprojlim H^j(\pi^{-1}(C),\ \pi^{-1}(C)_0)$.

* Here we are implicitly assuming that the base space B is Hausdorff. This is not actually necessary. The proof goes through perfectly well for non-Hausdorff spaces provided that one substitutes "quasi-compact" (i.e., every open covering contains a finite covering) for "compact" throughout.

Caution. These statements are only true since we are working with field coefficients. The corresponding statements with integer coefficients would definitely be false.

Proof of 10.3. The corresponding homology statement, that $\varinjlim H_j(C)$ maps isomorphically to $H_j(B)$, is clearly true for arbitrary coefficients, since every singular chain on B is contained in some compact subset of B. Similarly, the group $\varinjlim H_j(\pi^{-1}(C),\ \pi^{-1}(C)_0)$ maps isomorphically to $H_j(E, E_0)$. But according to A.1 in the Appendix, the cohomology $H^j(B)$ with coefficients in the field $\mathbb{Z}/2$ is canonically isomorphic to $\mathrm{Hom}\,(H_j(B), \mathbb{Z}/2)$. Together with the easily verified isomorphism

$$\mathrm{Hom}\,(\varinjlim H_j(C), \mathbb{Z}/2) \xrightarrow{\cong} \varprojlim \mathrm{Hom}\,(H_j(C), \mathbb{Z}/2)\ ,$$

this proves 10.3. ∎

In particular, the cohomology group $H^n(E, E_0)$ maps isomorphically to the inverse limit of the groups $H^n(\pi^{-1}(C),\ \pi^{-1}(C)_0)$. But each of the latter groups contains one and only one class u_C whose restriction to each fiber is non-zero. It follows immediately that $H^n(E, E_0)$ contains one and only one class u whose restriction to each fiber is non-zero.

Now consider the homomorphism $\cup u : H^j(E) \to H^{j+n}(E, E_0)$. Evidently, for each compact subset C of B there is a commutative diagram

$$\begin{array}{ccc} H^j(E) & \xrightarrow{\ \cup u\ } & H^{j+n}(E, E_0) \\ \downarrow & & \downarrow \\ H^j(\pi^{-1}(C)) & \longrightarrow & H^{j+n}(\pi^{-1}(C),\ \pi^{-1}(C)_0)\ . \end{array}$$

Passing to the inverse limit, as C varies over all compact subsets, it follows that $\cup u$ is itself an isomorphism. This completes the proof of 10.2. Hence we have finally completed the proof of existence (and uniqueness) for Stiefel-Whitney classes. ∎

Now let us try to carry out analogous arguments with coefficients in an arbitrary ring Λ. (It is of course always assumed that Λ is associative with 1.) Just as in the argument above, the cohomology $H^n(R^n, R^n_0;\Lambda)$ is a free Λ-module, with a single generator $e^n = e^1 \times \ldots \times e^1$. (See A.5 in the Appendix.)

Let ξ be an oriented n-plane bundle. Then for each fiber F of ξ we are given a preferred generator

$$u_F \in H^n(F, F_0; \mathbb{Z}) \ .$$

(Compare §9.) Using the unique ring homomorphism $\mathbb{Z} \to \Lambda$, this gives rise to a corresponding generator for $H^n(F, F_0; \Lambda)$ which will also be denoted by the symbol u_F.

ISOMORPHISM THEOREM 10.4. *There is one and only one cohomology class* $u \in H^n(E, E_0; \Lambda)$ *whose restriction to* (F, F_0) *is equal to* u_F *for every fiber* F. *Furthermore the correspondence* $y \mapsto y \cup u$ *maps* $H^j(E;\Lambda)$ *isomorphically onto* $H^{j+n}(E, E_0; \Lambda)$ *for every integer* j.

If the coefficient ring Λ is a field, then the proof is completely analogous to the proof of 10.2. Details will be left to the reader. Similarly, if the base space B is compact, then the proof is completely analogous to the proof of 10.2. (A similar argument works for any bundle ξ of finite type. Compare Problem 5-E.)

The difficulty in extending to the general case is that Lemma 10.3 is not available for cohomology with non-field coefficients. In fact the inverse limits of 10.3 can be very badly behaved in general. However the construction of the fundamental class u does go through without too much difficulty. We will need the following.

LEMMA 10.5. *The homology group* $H_{n-1}(E, E_0; \mathbb{Z})$ *is zero.*

Assuming this for the present, it follows from A.1 in the Appendix that the cohomology group $H^n(E, E_0; Z)$ is canonically isomorphic to $\mathrm{Hom}\,(H_n(E, E_0; Z), Z)$. Therefore, just as in the proof of 10.3, we see that $H^n(E, E_0; Z)$ is canonically isomorphic to the inverse limit of the groups

$$H^n(\pi^{-1}(C), \pi^{-1}(C)_0; Z)$$

as C varies over all compact subsets of the base space B. Since 10.4 has already been proved for any vector bundle over a compact base space C, it follows that there is a unique fundamental cohomology class $u \in H^n(E, E_0; Z)$.

REMARK. It is important to note that the fundamental class in $H^n(E, E_0; Z)$ corresponds to a fundamental class in $H^n(E, E_0; \Lambda)$ for any ring Λ, under the unique ring homomorphism $Z \to \Lambda$.

To prove that the cup product with u induces cohomology isomorphisms, we will make use of the following formal constructions.

DEFINITION. A *free chain complex* over Z is a sequence of free Z-modules K_n and homomorphisms

$$\cdots \longrightarrow K_n \xrightarrow{\partial} K_{n-1} \xrightarrow{\partial} K_{n-2} \longrightarrow \cdots$$

with $\partial \circ \partial = 0$. A *chain mapping* $f: K \to K'$ of degree d is a sequence of homomorphisms $K_i \to K'_{i+d}$ satisfying $\partial' \circ f = (-1)^d f \circ \partial$.

LEMMA 10.6. *Let* $f: K \to K'$ *be a chain mapping, where* K *and* K' *are free chain complexes over* Z. *If* f *induces a cohomology isomorphism*

$$f^*: H^*(K'; \Lambda) \to H^*(K; \Lambda)$$

for every coefficient field Λ, *then* f *induces isomorphisms of homology and cohomology with arbitrary coefficients.*

Proof. The *mapping cone* K^f is a free chain complex constructed as follows. Let $K_i^f = K_{i-d-1} \oplus K'_i$, with boundary homomorphism $\partial^f: K_i^f \to K_{i-1}^f$ defined by

$$\partial^f(\kappa, \kappa') = ((-1)^{d+1}\partial\kappa,\ f(\kappa) + \partial'\kappa') \ .$$

(Compare [Spanier, p. 166].) Evidently K^f fits into a short exact sequence

$$0 \to K' \to K^f \to K \to 0$$

of chain mappings. Furthermore the boundary homomorphism

$$\partial^f: H_{i-d-1}(K) \to H_{i-1}(K')$$

in the associated homology exact sequence is precisely equal to f_*. Thus the homology $H_*(K^f)$ is zero if and only if f induces an isomorphism $H_*(K) \to H_*(K')$ of integral homology.

In our case, f is known to induce a cohomology isomorphism $H^*(K'; \Lambda) \to H^*(K; \Lambda)$ for every coefficient field Λ. Using the cohomology exact sequence, it follows that $H^*(K^f; \Lambda) = 0$. But the cohomology $H^n(K^f; \Lambda)$ is canonically isomorphic to $\mathrm{Hom}_\Lambda(H_n(K^f \otimes \Lambda), \Lambda)$ by A.1 in the Appendix. Therefore, the homology vector space $H_n(K^f \otimes \Lambda)$ is zero. For otherwise there would exist a non-trivial Λ-linear mapping from this space to the coefficient field Λ.

In particular the rational homology $H_n(K^f \otimes \mathbb{Q})$ is zero. Therefore, for every cycle $\zeta \in Z_n(K^f)$ it follows that some integral multiple of ζ is a boundary. Hence the integral homology $H_n(K^f)$ is a torsion group.

To prove that this torsion group $H_n(K^f)$ is zero, it suffices to prove that every element of prime order is zero. Let $\zeta \in Z_n(K^f)$ be a cycle representing a homology class of prime order p. Then

$$p\zeta = \partial\kappa$$

for some $\kappa \in K_{n+1}^f$. Thus κ is a cycle modulo p. Since the homology $H_{n+1}(K^f \otimes \mathbb{Z}/p)$ is known to be zero, it follows that κ is a boundary mod p, say

$$\kappa = \partial\kappa' + p\kappa'' \ .$$

Therefore $p\zeta = \partial\kappa$ is equal to $p\partial\kappa''$, and hence $\zeta = \partial\kappa''$. Thus ζ represents the trivial homology class, and we have proved that $H_*(K^f) = 0$.

It now follows easily that K^f has trivial homology and cohomology with arbitrary coefficients. (Compare [Spanier, p. 167].) For example since $Z_{n-1}(K^f)$ is free, the exact sequence

$$0 \to Z_n(K^f) \to K_n^f \to Z_{n-1}(K^f) \to 0$$

is split exact, and therefore remains exact when we tensor it with an arbitrary additive group Λ. It follows easily that the sequence

$$\ldots \to K_{n+1}^f \otimes \Lambda \to K_n^f \otimes \Lambda \to K_{n-1}^f \otimes \Lambda \to \ldots$$

is also exact, which proves that $H_*(K^f \otimes \Lambda) = 0$. This completes the proof of 10.6. ■

The proof of 10.4 now proceeds as follows. We will make use of the cap product operation. (For the definition and basic properties, see Appendix A, p. 276.) While proving 10.4, we will simultaneously prove the following. The coefficient ring $\mathbb{Z}$ is to be understood.

COROLLARY 10.7. *The correspondence* $\eta \mapsto u \cap \eta$ *defines an isomorphism from the integral homology group* $H_{n+i}(E, E_0)$ *to* $H_i(E)$.

Proof. Choose a singular cocycle $z \in Z^n(E, E_0)$ representing the fundamental cohomology class u. Then the correspondence $\gamma \mapsto z \cap \gamma$ from $C_{n+i}(E, E_0)$ to $C_i(E)$ satisfies the identity

$$\partial(z \cap \gamma) = (-1)^n z \cap (\partial\gamma) \ .$$

Therefore

$$z\cap : C_*(E, E_0) \to C_*(E)$$

is a chain mapping of degree $-n$. Using the identity

$$<c, z \cap \gamma> = <c \cup z, \gamma>$$

we see that the induced cochain mapping

$$(z\cap)^{\#} : C^*(E;\Lambda) \to C^*(E, E_0;\Lambda)$$

is given by $c \mapsto c \cup z$. Here Λ can be any ring. If the coefficient ring Λ is a field, then this cochain mapping induces a cohomology isomorphism by the portion of 10.4 which has already been proved. Thus we can apply 10.6, and conclude that the homomorphisms

$$u\cap : H_{i+n}(E, E_0;\Lambda) \to H_i(E;\Lambda)$$

and

$$\cup u : H^i(E;\Lambda) \to H^{i+n}(E, E_0;\Lambda)$$

are actually isomorphisms for arbitrary Λ. In particular, using the isomorphism $\cup u : H^0(E;\Lambda) \to H^n(E, E_0;\Lambda)$, the uniqueness of the fundamental cohomology class u with coefficients in Λ can now be verified.

This completes the proof of 10.4 and 10.7 except for one step which has been skipped over. Namely, we must still prove that $H_{n-1}(E, E_0; \mathbb{Z}) = 0$ (Lemma 10.5).

First suppose that the base space B is compact. Then we have already observed that Theorem 10.4 is true independently of 10.5. Similarly the proof of 10.7, in this special case, goes through without making use of 10.5. Thus we are free to make use of 10.7 to conclude that

$$H_{n-1}(E, E_0; \mathbb{Z}) \xrightarrow{\cong} H_{-1}(E; \mathbb{Z}) = 0 \ .$$

The proof of 10.5 in the general case now follows immediately, using the homology isomorphism

$$\varinjlim H_i(\pi^{-1}(C), \pi^{-1}(C)_0; \mathbb{Z}) \xrightarrow{\cong} H_i(E, E_0; \mathbb{Z}) \ ,$$

where C varies over all compact subsets of B. (Compare 10.3.) This completes the proof. ■