REFINMENTS OF TOPOLOGICAL INVARIANTS OF A PAIR $(X, \xi \in H^1(X : \mathbb{Z})$

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For $(X; \xi \in H^1(X : \mathbb{Z}))$, X a compact ANR, κ a field,

the basic topological invariants are:

- **Homology** $H_r(X)$ a $\kappa-$vector space,

  Betti numbers $\beta_r(X) := \dim H_r(X) \in \mathbb{N}_0$

- **Novikov Homology** $H_r^N(X, \xi)$, a $\kappa[t^{-1}, t]-$free module,

  Novikov–Betti numbers
  $\beta_r^N(X; \xi) := \mathrm{rank}\ \mathrm{H_r(X)} \in \mathbb{N}_0$

- **L_2-Homology** $H_r^{L_2}(X;\xi) = H_r^{L_2}(\tilde X)$, a $L^\infty(\mathbb{S}^1)-$Hilbertian module

  L_2-Betti numbers $\beta_r^{L_2}(\tilde X) = \beta_r^N(X;\xi)$

- **Monodromy** $[(V_r, T_r : V_r \to V_r)]$, conjugacy class of linear isomorphisms

  the Jordan cells

  $\mathcal{J}_r(\xi) : \{(\lambda_i^r, n_i^r), 1 \leq i \leq k_r \mid \lambda_i^r \in \overline{\kappa}, n_i^r \in \mathbb{N}_0\}.$

Novikov homology :

- $(X, \xi) \Rightarrow \tilde{X} \to X$ infinite cyclic cover associated to ξ.
- X compact ANR $\Rightarrow H_r(\tilde{X})$ is a f.g. $\kappa[t^{-1}, t]$ module
- X compact ANR $\Rightarrow TH_r(\tilde{X})$ is a $\kappa[t^{-1}, t]-$module which is a f.d. $\kappa-$vector space.
- X compact ANR $\Rightarrow$ $\boxed{H^N(X;\xi) := H_r(\tilde{X})/TH_r(\tilde{X})}$ is a f.g free $\kappa[t^{-1}, t]$ module.

$\boxed{\beta^N_r(X;\xi) := \mathrm{rank}\ H^N(X;\xi)}$ the Novikov Betti numbers.

Explanations 2.

L_2-Homology

If $\kappa = \mathbb{C}$ by "completion" one obtains

- $\mathbb{C}[t^{-1}, t] \rightsquigarrow L^\infty(\mathbb{S}^1)$ a von Neumann algebra.
- $H_r^N(X:\xi) \rightsquigarrow H_r^{L_2}(\tilde{X})$ finite type $L^\infty(\mathbb{S}^1)-$Hilbert module

and $\boxed{\beta^N(X;\xi) = \dim_{vn}(H_r^{L_2}(\tilde{X}))}$.

r-Monodromy:

$$(V_r(\xi), T_r(\xi) : V_r \to V_r) \equiv \begin{cases} V_r = TH_r(\tilde{X}) \\ T = \text{ multiplication by t} \end{cases}$$

1. **Jordan cells of** $T_r(\xi)$,

$$\boxed{\mathcal{J}_r(\xi) \equiv \{(\lambda_1^r, n_1^r), (\lambda_2^r, n_2^r) \cdots (\lambda_k^r, n_k^r) \mid \lambda_i^r \in \overline{\kappa}, n_i^r \in N\}}$$

2. **Alexander polynomial** $A_r^\xi(z)$ =the characteristic polynomial of $T_r(\xi)$,

3. **Alexander function** $\boxed{A^\xi(z) = \prod_r (A_r^\xi(z))^{(-1)^r}}$

A Jordan cell:

$$T(\lambda, k) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{pmatrix}. \tag{1}$$

JORDAN DECOMPOSITION . (with respect to some base in V)
Any $T : V \to V$ is written as

$$\begin{pmatrix} T(\lambda_1, k_1) & 0 & 0 & 0 \\ 0 & T(\lambda_2, k_2) & 0 & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & T(\lambda_r, k_r) \end{pmatrix}. \tag{2}$$

Configurations

Configurations are maps with finite support $\qquad$ Notation

$$\omega : Y \rightsquigarrow \begin{cases} \mathbb{N} \text{ (of cardinality n)} & \mathcal{C}(Y) \quad (\mathcal{C}_n(Y)) \\ \kappa - \text{Vector spaces (subspaces of V)} & \mathcal{C}_\kappa(Y) \quad (\mathcal{C}_V(Y)) \\ \kappa[t^{-1}, t] - \text{free modules (split submodules of M)} & \mathcal{C}_{\kappa[t^{-1},t]}(Y)(\mathcal{C}_M(Y)) \\ \mathcal{A} - \text{Hilbert modules (closed submodules of H)} & \mathcal{C}_{\mathcal{A}}(Y) \quad (\mathcal{C}_{\mathcal{H}}(Y)) \end{cases}$$

$\mathcal{A}-$a finite von Neumann algebra

$$\text{s.t } \omega(y') \cap \omega(y'') = 0$$

with

$$\begin{cases} (\sum_{y \in Y} \omega(y) = n) \\ (\oplus_{y \in Y} \omega(y) \simeq V) \\ (\oplus_{y \in Y} \omega(y) \simeq M) \\ (\oplus_{y \in Y} \omega(y) \simeq \mathcal{H}) \end{cases}$$

- $Y = \mathbb{C} \setminus 0 \Rightarrow \mathcal{C}_n(Y) = \mathbb{C}^{n-1} \times C \setminus 0$ and identifies with degree $n-$ monic polynomials with nonzero constant coefficient,

$$\omega \rightsquigarrow P^\delta(z) = \prod_{z_i \in supp\delta} (z - z_i)^{\delta(z_i)}.$$

Invariants associated with an angle valued maps $f : X \to \mathbb{S}^1$

To an angle value continuous map $f : X \to \mathbb{S}^1$, X a compact ANR we associate :

1. The **configuration** δ^f_r of points located in $\mathbb{C} \setminus 0$
   $\equiv$ **The polynomial** $P^f_r(z) = \prod_{z_i \in \mathrm{supp}\delta^f_r} (z - z_i)^{\delta^f_r(z_i)}$
2. The **configuration** $\hat{\delta}^f_r$ of free $\kappa[t^{-1}, t]-$modules with $\mathrm{supp}\ \hat{\delta}^f_r = \mathrm{supp}\ \delta^f_r$; each $\hat{\delta}^f_r(z)$ is a quotient of free split sub-modules of $H^N_r(X; \xi)$ with $\mathrm{rank}\hat{\delta}^f_r(z) = \delta^f_r(z)$.
3. The collection $\mathcal{J}_r(f)$ of pairs **Jordan cells**
   $\mathcal{J}_r(\xi) := \{(\lambda^r_i, n^r_i), 1 \leq i \leq k_r \mid \lambda^r_i \in \overline{\kappa}, n^r_i \in \mathbb{N}\}$

Main results (Theorem 1)

Theorem

1. $\sharp \mathrm{supp} \delta^f_r \leq \beta^N_r(X)$ *and*

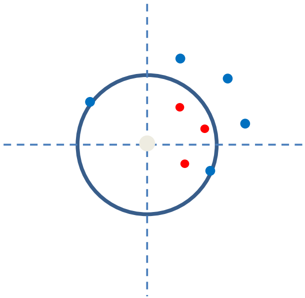
$$\boxed{\sum_{z \in \mathrm{supp} \delta^f_r} \delta^f_r(z) = \beta^N_r(X;\xi)},$$

hence $\deg P^f_r(z) = \beta^N_r(X,\xi)$.

2. $\hat{\delta}^f_r(z)$ *is a* $\kappa[t^{-1},t]-$*free module quotient of two split free submodules of* $H^N(X,\xi)$ *and*

$$\boxed{\oplus_{z_i \in \mathrm{supp} \delta^f_r} \hat{\delta}^f_r(z_i) \simeq H^N(X,\xi)}.$$

3. *For an open and dense set of maps* $f \in C(X;\mathbb{S}^1)$, $\delta^f_r(z) = 0$ *or* $\delta^f_r(z) = 1$.



Main results (Theorem 2)

Observation:

- $f \Rightarrow \xi^f \in \mathbb{H}^1(X; \mathbb{Z})$,
- f_1 and f_2 homotopic $\Rightarrow \xi^{f_1} = \xi^{f_2}$,
- $\xi^f \in \mathbb{H}^1(X; \mathbb{Z})$, $\Rightarrow$ there exists f s.t. $\xi = \xi^{f_1}$

Let $C_\xi(X, \mathbb{S}^1) := \{f \in C(X, \mathbb{S}^1) \mid \xi^f = \xi\}$ with c.o. topology.

Theorem

1. *If $f_1, f_2 : X \to \mathbb{S}^1$ are homotopic then $\mathcal{J}_r(f_1) = \mathcal{J}_r(f_2)$.*
2. $\mathcal{J}(f) = \mathcal{J}_r(\xi^f)$

Main results (Theorems 3,4)

Theorem

The assignment $C_\xi(X, \mathbb{S}^1) \ni f \rightsquigarrow \delta^f_r \in \mathbb{C}^{\beta^N_r - 1} \times \mathbb{C} \setminus 0$ *is continuous.*

The result holds for $\hat{\delta}^f_r$.

Theorem

If M^n *is a closed* $\kappa-$ *orientable topological* $n-$*dimensional manifold then*

$$\delta^f_r(z) = \delta^{-f}_{n-r}(-i\overline{z}).$$

The result holds for the configuration $\hat{\delta}^f_r$

Relation with barcodes and Betti numbers

Proposition

1. *If* $z = e^{ia+(b-a)} \in \mathrm{supp}\delta^f_r$ *then both* $a, b \in Cr(\tilde{f}) = \{t \mid e^{it} \in Cr(f)\}$.
2. z *outside of or on the unit circle implies* $[a, b]$ *is a closed* $r-$*bar code.*
3. z *inside of the unit circle implies* (b, a) *is an open* $(r-1)-$*bar code.*

Proposition

$$\beta_r(X) = \beta^N_r + \begin{cases} \sharp\{(\lambda, k) \in \mathcal{J}_r(f) \mid \lambda = 1\} \\ \sharp\{(\lambda, k) \in \mathcal{J}_{r-1}(f) \mid \lambda = 1\} \end{cases}$$

Description of the invariants (preliminaries).

Suppose $(X, f : X \to \mathbb{S}^1)$

The infinite ciclic cover $\tilde{f} : \tilde{X} \to \mathbb{R}$

is the pullback of the infinite cyclic cover $p : \mathbb{R} \to \mathbb{S}^1$, by by f

The cut at θ

- $\overline{X}^f_\theta$, the two sided compactification of $X \setminus f^{-1}(\theta)$ with sides $f^{-1}(\theta)$.
- One has $f^{-1}(\theta) \xrightarrow{i_1} \overline{X}_\theta \xleftarrow{i_2} f^{-1}(\theta)$.
- It induces in $r-$ homology the linear relation

$$R^f_\theta(r) \equiv \boxed{H_r(f^{-1}(\theta)) \xrightarrow{i_1(r)} H_r(\overline{X}_\theta) \xleftarrow{i_2(r)} H_r(f^{-1}\theta))}.$$

- The configurations δ^f_r and $\hat{\delta}^f_r$ will be derived from the infinite cyclic cover $\tilde{f}$ of f
- The monodromy (i.e. the Jordan cells) will be derived from the linear relations associated with the cut at θ.

The proof of the results will be done in two steps.

- **Step 1:** Establish the results for X a finite simplicial complex and f a simplicial map
- **Step 2.** Use results about the topology of Hilbert cube manifolds and the fact that the statements are true for f iff are true for $\overline{f}_Q$, Q the Hilbert cube to comnclude the results in the stated generality .

The jump functions d^h_r and $\hat{d}^h_r$

For $h : Y \to \mathbb{R}$ a proper map
$a, b \in \mathbb{R}$, κ a field

Denote:

$Y_a := h^{-1}((-\infty, a])$ sub-level
$Y^b := h^{-1}([b, \infty))$ over-level
Define:

- $\mathbb{I}_a(r) := img(H_r(Y_a) \to H_r(Y))$
- $\mathbb{I}^b(r) := img(H_r(Y^b) \to H_r(Y))$
- $\mathbb{F}_r(a, b) := \mathbb{I}_a(r) \cap \mathbb{I}^b(r)$

- Observe
  $a \leq a', b' \leq b$ imply $\mathbb{F}_r(a', b') \subseteq \mathbb{F}_r(a, b)$.
- Prove
  $\mathbb{F}_r(a, b)$ finite dimensional.
- Define $\boxed{\mathbb{F}_r(a, b, \epsilon) := \mathbb{F}_r(a, b)/\mathbb{F}_r(a - \epsilon, b) + \mathbb{F}_r(a, b + \epsilon)}$ for $\epsilon > 0$
- Observe that $\epsilon' > \epsilon''$ induces a surjective map

$$\boxed{\mathbb{F}_r(a, b; \epsilon') \to \mathbb{F}_r(a, b; \epsilon'').}$$

Definition

$$\hat{d}_r^h(a, b) := \varinjlim_{\epsilon \to 0} \mathbb{F}_r(a, b, \epsilon)$$

$$d_r^h(a, b) := \dim \hat{d}_r^f(a, b)$$

The configurations δ^f_r and $\hat{\delta}^f_r$ (definitions)

Apply the above definitions to the proper map $\tilde{f} : \tilde{X} \to \mathbb{R}$.

- Observe that
  1. $t : \hat{d}^{\tilde{f}}_r(a,b) \to \hat{d}^{\tilde{f}}_r(a+2\pi, b+2\pi)$ is an isomorphism and
  2. $d^f_r(a,b) = d^{\tilde{f}}_r(a+2\pi, b+2\pi)$
- Define

$$\boxed{\delta^f_r(z) = d^{\tilde{f}}_r(a,b)}, z = e^{(b-a)+ia}$$

$$\boxed{\hat{\delta}^f_r(z) = \bigoplus_k \hat{d}^{\tilde{f}}_r(a+2\pi k, b+2\pi k)}, z = e^{(b-a)+ia}.$$

- Prove that δ^f_r has finite support and $\hat{\delta}^f_r(z)$ is a split free submodule of the free $\kappa[t^{-1}, t]-$module of $H^N(X;\xi)$.

The abelian category of vector spaces and linear relations

- A **linear relation** $R: V_1 \rightsquigarrow V_2$ is $R \subseteq V_1 \times V_2$.
  write $v_1 R v_2$ iff $(v_1, v_2) \in R$, $v_i \in V_i$.
- The **composition** of two liner relations $R_1 : V_1 \rightsquigarrow V_2$ and $R_2 : V_2 \rightsquigarrow V_2$ defined by
  $(v_1 (R_2 \cdot R_1) v_3$ iff $\exists v_2$ with $v_1 R_1 v_2$ and $v_2 R_2 v_3$.
  The **identity** id_V is the diagonal $\Delta \subset V \times V$.
- The reverse of $R : V_1 \rightsquigarrow V_2$ is $R^\dagger : V_2 \rightsquigarrow V_1$
  $v_2 R^\dagger v_1$ iff $v_1 R v_2$.
  One has $(R_1 \cdot R_2)^\dagger = R_2^\dagger \cdot R_1^\dagger$ and $R^{\dagger\dagger} = R$.

- Direct sum $R' \oplus R'' : V_1' \oplus V_1'' \rightsquigarrow V_2' \oplus V_2''$ of two relations $R' : V_1' \rightsquigarrow V_2'$ and $R'' : V_1'' \rightsquigarrow V_2''$ is defined by: $(v_1', v_1'')(R' \oplus R'')(v_2', v_2'')$ iff $(v_1' R' v_2')$ and $(v_1'' R'' v_2'')$.

Example :

1. $f : V_1 \to V_2 \Rightarrow R(f) = \mathrm{graph\ f} := \{(\mathrm{v}_1, \mathrm{f}(\mathrm{v}_1)) \in \mathrm{V}_1 \times \mathrm{V}_2\}$
2. $\alpha : V_1 \to W, \beta : V_2 \to W$
   $\Rightarrow R(\alpha, \beta) : V_1 \rightsquigarrow V_2 = \{(v_1, v_2 \mid \alpha(v_1) = \beta(v_2)\}$

For a linear relation $R\colon V \rightsquigarrow W$ define:

$$\begin{aligned} \mathrm{dom}\,(R) &:= \{v \in V \mid \exists w \in W : vRw\} = pr_V(R) \\ img\,(R) &:= \{w \in W \mid \exists v \in V : vRw\} = pr_W(R) \\ \ker(R) &:= \{v \in V \mid vR0\} \cong (V \times 0) \cap R \\ \mathrm{mul}\,(R) &:= \{w \in W \mid 0Rw\} \cong (0 \times W) \cap R \end{aligned}$$

For $R: V \rightsquigarrow V$ define

1. $D : \{v \in V \mid \exists v_i \in V, i \in \mathbb{Z}, v_i R v_{i+1}, v_0 = v\}$. The relation R restricts to a relation $R_D : D \rightsquigarrow D$
2. $K_+ := \{v \in V \mid \exists v_i, 0 \leq i \leq N, v_i R v_{i+1}, v_0 = v, v_N = 0\}$
3. $K_- := \{v \in V \mid \exists v_i, -M \leq i \leq 0, v_i R v_{i+1}, v_0 = v, v_{-M} = 0\}$
4. $V_{reg} := \frac{D}{D \cap (K_+ + K_-)}, \pi : D \to \frac{D}{D \cap (K_+ + K_-)}$

and consider the composition of relations

$$R_{reg} := R(\pi) \cdot R_D \cdot R(\pi)^\dagger : V_{reg} \rightsquigarrow V_{reg}.$$

Correcting a linear relations into an isomorphism

Proposition

1. *There exists a linear isomorphism* $T^R : V_{reg} \to V_{reg}$, *s.t.* $R_{reg} = R(T^R)$.
2. *If* $(R\colon V \rightsquigarrow V) \sim (R'\colon V' \rightsquigarrow V')$ *are similar relations* [a] *then* $T^R \sim T^{R'}$ *are similar linear isomorphisms*
3. $(T^R)^{-1} = T^{R^\dagger}$
4. $(R' \oplus R'')_{reg} = R'_{reg} \oplus R''_{reg}$.
5. *Suppose* $R_1 : V_1 \rightsquigarrow V_2, R_2 : V_2 \rightsquigarrow V_1$, *then* $(R_2 \cdot R_1)_{reg} \sim (R_1 \cdot R_2)_{reg}$.

[a] there exists an isomorphism of vector spaces $\omega : V \to V'$ such that $R' = R(\omega) \cdot R \cdot R(\omega^{-1})$

For θ a weakly regular value (i.e. $f^{-1}(\theta)$ is an ANR) consider the relation $R(i_1(r), i_2(r)) := R^f_\theta(r)$ defined by the cut at θ,

$$H_r(f^{-1}(\theta)) \xrightarrow{i_1(r)} H_r(\overline{X}_\theta) \xleftarrow{i_2(r)} H_r(f^{-1}\theta))$$

and then the isomorphism $(R^f_\theta(r))_{\mathrm{reg}}$.

Definition

The $r-$ *topological monodromy* of $f : X \to \mathbb{S}^1$ at $\theta \in \mathbb{S}^1$, (θ a weakly regular value), is the similarity class $[(R^f_\theta(r))_{\mathrm{reg}}]$, denoted by $\boxed{[T^f_\theta(r)]}$. $\mathcal{J}^f_\theta(r)$ denotes the Jordan cells of the linear isomorphism $T^f_\theta(r)$.

Denote by $\bar{f}_K : X \times K \to \mathbb{S}^1$ the composition of f with the projection $X \times K \to X$.

Proposition

1. $[T^f_{\theta_1}(r)] = [T^f_{\theta_2}(r)]$ *for* θ_1, θ_2 *two weakly regular angles* $\therefore [T^f(r)] := [T^f_\theta(r)]$.
2. $f, g : X \to \mathbb{S}^1$ *homotopic* $\Rightarrow [T^f(r)] = [T^g(r)]$.
3. $[T^{\bar{f}_{S^1}}(r)] = [T^f(r) \oplus T^f(r-1)]$ *if* $r > 0$ *and* $[T^{\bar{f}_{S^1}}](0) = [T^f(0)]$.
4. *K acyclic compact ANR implies* $[T^{\bar{f}_K}(r)] = [T^f(r)]$
5. *If* $f_i : X_i \to \mathbb{S}^1$ $i = 1, 2$, $\omega : X_1 \to X_2$ *homeomorphism s.t.* $f_2 \cdot \omega$ *homotopic to* f_1 *then* $[T^{f_1}(r)] = [T^{f_2}(r)]$.

The proof consists in two steps.

Step 1: Establish the results for X a finite simplicial complex and f a simplicial map (or even more general X compact ANR and f tame).

Step 2. Use results about the topology of Hilbert cube manifolds and the fact that the statements are true for f iff are true for $\bar{f}_Q$, Q the Hilbert cube and results about compact Hilbert cube manifolds.

Step 1

- Introduce the following concepts
  A weakly tame map is a proper continuous map $f : X \to \mathbb{R}(\text{ or } \mathbb{S}^1)$ with X an ANR and each level $f^{-1}(t)$ an ANR. Any simplicial map is weakly tame

- For a real valued weakly tame map $f : X \to \mathbb{R}$, $t \in \mathbb{R}$ is a HOMOLOGICAL CRITICAL VALUE if the homology of the levels $f^{-1}(t')$ changes at t when t' varies in an arbitrary small neighborhood of t.

- For $f : X \to \mathbb{S}^1$ a weakly tame map $\tilde{f}$ is weakly tame. Denote by $Cr(f) := \{t \in \mathbb{R} \mid, t \in Cr(\tilde{f})\}$ and define

$$\boxed{\epsilon(f) = \inf |c - c'], c, c' \in Cr(\tilde{f})}$$

Verify:
Fact 1.

1. For any proper tame map $\delta^f(a,b)\neq 0$ implies both $a,b\in Cr(f)$,
2. For and box $B=(a',a]\times[b,b')$ and any proper tame map one has
3. $$\mathbb{F}_r^{\tilde f}(B)=\simeq \oplus_{(a'',b'')\in B\cap \mathrm{supp}\hat{d}^{\tilde f}}\hat{d}_r^{\tilde f}(a'',b''),$$

$\mathbb{F}_r(B)=\mathbb{F}_r(a,b)/\mathbb{F}_r(a',b)+\mathbb{F}_r(a,b')$.

Fact 2 . If $f : X|to\mathbb{S}^1$ weakly tame, $\epsilon < \epsilon(f)$, g continuous and $\epsilon-$close to f then the support of δ^g is in an $2\epsilon-$neighborhood of the support of δ^f and $\sum \delta^g(z) = \sum \delta^f(z)$.

Both results are elementary but tedious.

The above lead to the proof of the first and third theorem.

The proof of the second Theorem is based on the main Proposition about the topological monodromy and tricks plying with different θ's when f_1 and f_2 are close enough.

Checking Poincaré duality involves Borel–Moore homology.

Hilbert cube manifolds

$Q = I^\infty$ the product of infinitely many copies of I.

Theorem

1. *(R Edwards) X is a compact ANR iff $X \times Q$ is a Hilbert cube manifold.*
2. *(T Chapman) If $\omega : X \to Y$ is a homotopy of equivalence between two finite simplicial complexes with Whitehead torsion $\tau(\omega) = 0$ then there exists a homeomorphism $\omega' : X \times Q \to Y \times Q$ such that ω' and $\omega \times id_Q$ are homotopic.*
3. *(folklore) If ω is a homotopy equivalence between two finite dimensional complexes then $\omega \times id_{\mathbb{S}^1}$ has the Whitehead torsion $\tau(\omega \times id_{\mathbb{S}^1}) = 0$.*

Applications / Implications

- **Application to Topology:** establish relations (previously unknown to this author) between Homology, Novikov homology, $L-2$homology, Borel-Moore homology for some class of spaces.
- **Applications in computational topology:** New algorithms for the calculation of Betti numbers, Novikov Betti numbers, Alexander polynomial and Reidemeister torsion, as well as Refinements of Morse inequalities. (see references 2 and 3 for such algorithms.)

Applications /implications

- **Applications in data analysis:** Homological recognition of shapes which can be manifolds. Homological differentiations of shapes.
- **Applications in geometric analysis:** Refinement of Hodge - de Rham theorem on compact Riemannian maniflods, (Canonical base in the space of harmonic forms.)
- **Applications in dynamics:** For flows which admit an *action= a Lyapunov closed one form* detect presence of instants and closed trajectories

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Linear relations

- A **linear relation** $R : V_1 \rightsquigarrow V_2 \equiv R \subseteq V_1 \times V_2$.
- The concept Linear relation extends the concept of linear map, providing the morphisms in a category with objects vector spaces
- This category is abelian with reverse, i.e the reverse of $R : V_1 \rightsquigarrow V_2$ is $R^\dagger : V_2 \rightsquigarrow V_1$ s.t. $(R_1 \cdot R_2)^\dagger = R_2^\dagger \cdot R_1^\dagger$ and $R^{\dagger\dagger} = R$.

1. The **composition** of two liner relations $R_1 : V_1 \rightsquigarrow V_2$ and $R_2 : V_2 \rightsquigarrow V_2$ defined by $(v_1 (R_2 \cdot R_1) v_3$ iff $\exists v_2$ with $v_1 R_1 v_2$ and $v_2 R_2 v_3$.
2. The **identity** id_V is the diagonal $\Delta \subset V \times V$.
3. The reverse of $R : V_1 \rightsquigarrow V_2$ is $R^\dagger : V_2 \rightsquigarrow V_1$ $v_2 R^\dagger v_1$ iff $v_1 R v_2$.
4. Direct sum $R' \oplus R'' : V_1' \oplus V_1'' \rightsquigarrow V_2' \oplus V_2''$ of two relations $R' : V_1' \rightsquigarrow V_2'$ and $R'' : V_1'' \rightsquigarrow V_2''$ is defined by: $(v_1', v_1'')(R' \oplus R'')(v_2', v_2'')$ iff $(v_1' R' v_2')$ and $(v_1'' R'' v_2'')$.

Example :

1. $V_1 \xrightarrow{f} V_2 \ \Rightarrow R(f) : V_1 \rightsquigarrow V_2 = \{(v_1, f(v_1)) \in V_1 \times V_2\}$
2. $V_1 \xrightarrow{\alpha} W \xleftarrow[\beta]{} V_2 \ \Rightarrow R(\alpha, \beta) : V_1 \rightsquigarrow V_2 = \{(v_1, v_2 \mid \alpha(v_1) = \beta(v_2)\}$

Proposition

For $R : V \rightsquigarrow V$ there exists a linear isomorphism $T^R : V_{reg} \to V_{reg}$, s.t. :

1. *If $(R\colon V \rightsquigarrow V) \sim (R' : V' \rightsquigarrow V')$ are similar relations [a] then $T^R \sim T^{R'}$ are similar linear isomorphisms*
2. $(T^R)^{-1} = T^{R^\dagger}$
3. $T^{R' \oplus R''} = T^{R'} \oplus T^{R''}$
4. *Suppose $R_1 : V_1 \rightsquigarrow V_2, R_2 : V_2 \rightsquigarrow V_1$, then $T^{R_2 \cdot R_1} \sim T^{R_1 \cdot R_2}$.*

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[a] there exists an isomorphism of vector spaces $\omega : V \to V'$ such that $R' = R(\omega) \cdot R \cdot R(\omega^{-1})$

Denote by $\boxed{R_{reg} := R(T^R)}$: