LAPLACE TRANSFORM, FROM TOPOLOGY TO SPECTRAL GEOMETRY

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DYNAMICS:

X vector field on a closed manifold M

$\Psi_t : M \to M$ its flow
trajectory $\therefore$ $\theta : \mathbb{R} \to M$ s.t. $\theta(t) = \Psi_t(x)$

ELEMENTS OF DYNAMICS:

a) **Rest points** $\therefore \mathcal{X} = \{x \in M | X(x) = 0\}$

b) **Instantons** $\therefore \theta(t),\ \lim_{t \to \pm\infty} \theta(t) = x/y$, $x, y \in \mathcal{X}$

c) **Closed trajectories** $\therefore \tilde{\theta} \equiv (\theta, T)$ s.t. $\theta(t + T) = \theta(t)$.

Topology can be used to count a) b) c)

Riemannian geometry can be used to conveniently express the result

Laplace transform of Dirichlet series is the link between them

For $x \in \mathcal{X}$ the **stable/ unstable** set $W_x^{\pm}$

$$\boxed{W_x^{\pm} := \{y \in M | \lim_{t \to \pm\infty} = x\}}$$

(ND) NONDEGENERATE VECTOR FIELDS X

i) All rest points $x \in \mathcal{X}$ are **standard hyperbolic**, i.e. there exists coordinates x_i's about x so that

$X = -\sum_{i=1}^{k} x_i \frac{\partial}{\partial x_i} + \sum_{i=k+1}^{n} x_i \frac{\partial}{\partial x_i}$.

ii) All instantons are nondegenerate, i.e. $x, y \in \mathcal{X} \Rightarrow W_x^- \pitchfork W_y^+$

iii) All closed trajectories are nondegenerate, i.e.

$\theta(t) = \Psi_t(x),\ D_x(\Psi_t) : T_x(M) \to T_x(M)$
has 1 an eigenvalue with multiplicity one.

(MS) MORSE SMALE VECTOR FIELDS

Satisfy (i) and (ii)

Any closed trajectory $\tilde{\theta}$ has a sign,

$$\boxed{\operatorname{sign}(\tilde{\theta}) \in \{\pm 1\}}$$

and a period

$$\boxed{p(\tilde{\theta}) \in \mathbb{N}}.$$

Given X an ND vector field choose a collection of **orientations** $\mathcal{O} = \{\mathcal{O}_x,\ x \in \mathcal{X}\}$.

Any instanton θ from x to y with $ind(x) - ind(y) = 1$ has a sign,

$$\boxed{\operatorname{sign}(\theta) \in \{\pm 1\}}.$$

TOPOLOGY

$\xi \in H^1(M, \mathbb{R}) \ \Rightarrow \xi : H_1(M, \mathbb{Z}) \to \mathbb{R}$

$\boxed{\Gamma := H_1(M, \mathbb{Z}) / \ker \xi}$, $\ \xi : \Gamma \to \mathbb{R}$

For $x, y \in M$

$\hat{P}_{x,y}$ the set of equivalency classes $\hat{\alpha}$

of continuos paths α from x to y with

$\alpha_1 \equiv \alpha_2$ iff $\ \alpha_1^{-1} * \alpha_2 \in \ker \xi$.

$-\xi$ GRADIENT LIKE VECTOR FIELD

X is $-\xi$ gradient like vector field if there exists a closed one form ω representing ξ and a Riemannian metric g so that :

1) ω is a Morse form

2)$X = -\text{grad}_g(\omega)$

Proposition.

1) In the space of $-\xi$ gradient like vector fields those which are ND form a generic set.

2) If X is $-\xi$ gradient like and ND then:

i) $\mathcal{X}$ is finite and $x \in \mathcal{X}$ has a Morse index $ind(x)$.

ii) Isolated solitons from x to y exists only if $ind(x) - ind(y) = 1$. In each class $\hat{\alpha}$ there are finitely many solitons.

iii) In each class $\gamma \in \Gamma$ there are finitely many closed trajectories.

COUNTING FUNCTIONS:

1) For $x, \in \mathcal{X}_q$, $y, \in \mathcal{X}_{q-1}$, define $\mathbb{P}^{\mathcal{O}}_{x,y} : \hat{\mathcal{P}}_{x,y} \to \mathbb{Z}$ by

$$\boxed{\hat{\mathcal{P}}_{x,y}(\hat{\alpha}) = \sum_{\theta \in \hat{\alpha}} \operatorname{sign}(\theta)}$$

2)Define $\mathbb{Z}_{X,\xi} : \Gamma \to \mathbb{Q}$ by

$$\boxed{\mathbb{Z}_{X,\xi}(\gamma) := \sum_{\tilde{\alpha} \in \gamma} \frac{\operatorname{sign}(\tilde{\theta})}{p(\tilde{\theta})}}.$$

TOPOLOGICAL SOLUTION

Novikov has defined a ring Λ_ξ and a cochain complex of free Λ_ξ modules whose boundary homomorphisms are given in terms of $\mathbb{P}^{\mathcal{O}}_{x,y}$ interpreted as elements in Λ_ξ and cohomology is expressed in terms of the topology of M and ξ.

Hutchings - Lee and Pajinhof have interpreted $\mathbb{Z}_{X,\xi}$ as "torsion" associated to the Novikov complex.

SPECTRAL GEOMETRY SOLUTION

will be given in terms of **real valued functions** *which and determine the Novikov and Hutchings..... solutions.*

DIRICHLET SERIES:

$$f \equiv \begin{pmatrix} \lambda_1 & < & \lambda_2 & < & \cdots & \lambda_k & < & \lambda_{k+1} & \cdots \\ a_1 & & a_2 & & \cdots & a_k & & a_{k+1} & \cdots \end{pmatrix}$$

defines a measure with descrete support and Laplace transform

$$\boxed{L(f)(z) := \sum_i e^{-z\lambda_i} a_i}$$

with an abscissa of convergence $\rho(f) \leq \infty$ ($f(z)$ *convergent on* $\Re z > \rho(f)$ *and divergent on* $\Re z < \rho(f)$).

THE INVARIANT $\rho(X, \xi) \in [0, \infty]$.

X vector field with hyperbolic rest points, and , and $\xi \in H^1(M : \mathbb{R})$.

$$\boxed{\rho(X, \omega, g, x) := \inf\{\tau \in \mathbb{R} | \int_{W_x^-} e^{\tau h_x} Vol_{(i_x^-)^* g}\}}$$

$h_x : W_x^- :\to \mathbb{R}$ s.t. $dh_x = (i_x^-)^*(\omega),\ h_x(x) = 0$.

$\rho(\cdots)$ independent on g, and on $\omega \in \xi$.

$$\boxed{\rho(X, \xi) := \inf_{x \in \mathcal{X}} \rho(X, \omega, g, x)}$$

Proposition.

Let X be $-\xi$ gradient like which is ND.

1) The pairs $(\xi(\gamma), \mathbb{Z}_{X,\xi}(\gamma))$ with $\mathbb{Z}_{X,\xi}(\gamma)) \neq 0$ define a Dirichlet series with λ's given by $\xi(\hat{\alpha})$ and a's given by $\mathbb{Z}_{X,\xi}(\gamma)$.

2) Let ω be a closed one form representing ξ and $\mathcal{O}$ a collection of orientations. The pairs $(\omega(\hat{\alpha}), \mathbb{P}^{\mathcal{O}}(x,y)(\hat{\alpha}))$ with $\mathbb{P}^{\mathcal{O}}(x,y)(\hat{\alpha}) \neq 0$ define a Dirichlet series with λ's given by $\omega(\hat{\alpha})$ and a's given by $\mathbb{P}^{\mathcal{O}}(x,y)(\hat{\alpha})$. Changing ω and $\mathcal{O}$ one might change the sequence of $\lambda' s$ by sign and the sequence of $a' s$ by a factor.

3) If $\rho(X,\xi) < \infty$ the above series have a finite abscissa of convergence.

In particular the functions of one real variable t

$L(\mathbb{P}^{\mathcal{O}}_{x,y})(e^t)$ and $L(\mathbb{Z}_{X,\xi}(e^t)$ restricted (a, ∞)

determine by **analytic continuation** and

inverse Laplace transform the counting

functions $\mathbb{P}^{\mathcal{O}}_{x,y}$ and $\mathbb{Z}_{X,\xi}$.

SPECTRAL GEOMETRY:

(M, g) closed Riemannian manifold,

ω a Morse one form (*locally* $\omega = dh$, h *smooth function with all critical points nondegenerate*),

$t \in [0, \infty)$.

Consider

$$\boxed{(\Omega^*(M), d^*_\omega(t)),\ d^q_\omega(t) : \Omega^q(M) \to \Omega^{q+1}(M)}$$

with $d^q_\omega(t)(\alpha) := d\alpha + t\omega \wedge \alpha$.

Use g to define $(d^q_\omega(t))^\sharp : \Omega^{q+1}(M) \to \Omega^q(M)$

and define $\Delta^q_\omega(t) : \Omega^q(M) \to \Omega^q(M)$ by:

$$\Delta^q_\omega(t) := (d^q_\omega(t))^\sharp \cdot d^q_t + d^{q-1}_\omega(t) \cdot (d^{q-1}_\omega(t))^\sharp.$$

$$\boxed{\Delta^q_\omega(t) :== \Delta^q + t(L + L^\sharp) + t^2 ||\omega||^2 Id}$$

with L the Lie derivative along $-\text{grad}_g \omega$, $L^\sharp$ the adjoint of L, and $||\omega||^2$ the fiberwise norm of ω.

Theorem 1.

There exist $C_1, C_2, C_3, T > 0$ *so that for* $t > T$ *one has:*

i) Spect$\Delta^q_\omega(t) \cap [C_1 e^{-C_2 t}, C_3 t] = \emptyset$ *and* $1 \in (C_1 e^{-C_2 t}, C_3 t)$

ii)$\sharp\big($*Spect*$\Delta^q_\omega(t) \cap [0, C_1 e^{-C_2 t}]\big) = \sharp(\mathcal{X}_q)$.

iii) For all but finitely many t, $\dim(\ker \Delta_\omega(t))$ *is constant in* t.

Denote by :

$\Omega^*_{sm}(M)(t)$ the span of the eigenforms which correspond to eigenvalues smaller than 1.

$\Omega^*_{la}(M)(t)$ the span of the eigenforms which correspond to eigenvalues larger than 1.

Theorem 1 implies that for $t > T$

$$(\Omega^*(M), d_\omega(t)) =$$
$$\big(\Omega^*_{sm}(M)(t), d_\omega(t)\big) \oplus \big(\Omega^*_{la}(M)(t), d_\omega(t)\big)$$

and

$$\boxed{\Delta^q_\omega(t) = \Delta^q_{\omega,sm}(t) \oplus \Delta^q_{\omega,la}(t)}$$

with $\dim(\Omega^q(M)_{sm}(t)) = \sharp(\mathcal{X}_q)$ for any $t > T$.

Theorem2.

Suppose $X = -grad_g(\omega)$ *where* ω *is a closed one form representing* $\xi \in H^1(M:\mathbb{R})$, X *is MS,* $\rho(\xi, X) < \infty$ *and orientations* $\mathcal{O}$ *are given.*

There exists T, *and a canonical base of* $\Omega^q(M)_{sm}(t)$, $\{E_x^{\mathcal{O}}(t), x \in \mathcal{X}\}$, *so that for* $t > T$ *and* $y \in \mathcal{X}_q$

$$\boxed{d_\omega(t)E_y(t) = \sum_{x \in \mathcal{X}_{q+1}} I_{x,y}(t)E_y(t)} \text{ with}$$

$$\boxed{I_{y,x}(t) = L(\mathbb{P}_{x,y}^{\mathcal{O},\omega})(e^t)}.$$

Proposition.

Suppose X *is a* $-\xi$ *gradient like vector field* **with no rest points** *and all closed trajectories nondegenerate,* ω *a closed one form representing* $\xi \in H^1(M, \mathbb{R})$ *and* g *a Riemannian metric.*

Denote $\log T_{an}(t) := 1/2 \sum (-1)^{q+1} q \log \det(\Delta_\omega^q(t))$.

Then

$$\boxed{\log T_{an}(t) = (-1)^{n+1} t \int_M \omega \wedge X^*(\Psi(g))}$$

is the Laplace transform of the Dirichlet series $\mathbb{Z}_X$.

DROPPING THE HYPOTHESIS "NO REST POINTS"

ADDITIONAL SPECTRAL GEOMETRY

Denote by
$\log Vol(t) = \sum(-1)^q \log Vol\{E_x(t), x \in \mathcal{X}_q\}$

Define
$\log T_{an,la}(t) := 1/2 \sum(-1)^{q+1} q \log \det(\Delta^q_{\omega,la}(t)).$

ADDITIONAL RIEMANNIAN GEOMETRY
An invariant $\mathcal{R}(\omega, g, X)$ was introduced for

X with standard hyperbolic zeros,
ω closed one form,
g Riemannian metric

Theorem.

Suppose X is a $-\xi$ gradient like vector field which is ND and $\rho(\xi, X) < \infty$. Suppose (ω, g) is a pair with ω representing ξ and $X = -grad_g(\omega)$.

There exists a positive real number $R > \rho([\omega], X)$ so that for $t > R$ the function

$$\boxed{\log T_{an,la}(t) + \log Vol(t) + t\mathcal{R}(\omega, g, X)}$$

is the restriction of a holomorphic function on $\{z \in \mathbb{C} \mid \Re z > R\}$ which is inverse Laplace transform of the Dirichlet series $\mathbb{Z}_X$