LAPLACE TRANSFORM, FROM

TOPOLOGY TO SPECTRAL GEOMETRY

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DYNAMICS:

$X$ vector field on a closed manifold $M$

$\Psi_t : M \to M$ its flow

trajectory $\therefore \theta : \mathbb{R} \to M$ s.t. $\theta(t) = \Psi_t(x)$

ELEMENTS OF DYNAMICS:

a) Rest points $\therefore \mathcal{X} = \{x \in M | X(x) = 0\}$

b) Instantons $\therefore \theta(t), \lim_{t \to \pm \infty} \theta(t) = x/y, \ x, y \in \mathcal{X}$

c) Closed trajectories $\therefore \tilde{\theta} \equiv (\theta, T)$

s.t. $\theta(t + T) = \theta(t)$.

Topology can be used to count a) b) c)

Riemannian geometry can be used to conveniently express the result

Laplace transform of Dirichlet series is the link between them
For $x \in \mathcal{X}$ the \textbf{stable/ unstable} set $W_x^\pm$

\[
W_x^\pm := \{ y \in M \mid \lim_{t \to \pm \infty} = x \}
\]

(ND) NONDEGENERATE VECTOR FIELDS $X$

i) All rest points $x \in \mathcal{X}$ are \textbf{standard hyperbolic}, i.e. there exists coordinates $x_i$’s about $x$ so that

\[
X = -\sum_{i=1}^{k} x_i \frac{\partial}{\partial x_i} + \sum_{i=k+1}^{n} x_i \frac{\partial}{\partial x_i}.
\]

ii) All instantons are nondegenerate, i.e.

$x, y \in \mathcal{X} \Rightarrow W_x^- \cap W_y^+$

iii) All closed trajectories are nondegenerate, i.e.

$\theta(t) = \Psi_t(x), \ D_x(\Psi_t) : T_x(M) \to T_x(M)$

has 1 an eigenvalue with multiplicity one.

(MS) MORSE SMALE VECTOR FIELDS

Satisfy (i) and (ii)
Any closed trajectory $\tilde{\theta}$ has a sign,

$$\text{sign}(\tilde{\theta}) \in \{\pm 1\}$$

and a period

$$p(\tilde{\theta}) \in \mathbb{N}.$$

Given $X$ an ND vector field choose a collection of orientations $\mathcal{O} = \{\mathcal{O}_x, \ x \in \mathcal{X}\}$.

Any instanton $\theta$ from $x$ to $y$ with $\text{ind}(x) - \text{ind}(y) = 1$ has a sign,

$$\text{sign}(\theta) \in \{\pm 1\}.$$

**TOPOLOGY**

$$\xi \in H^1(M, \mathbb{R}) \Rightarrow \xi : H_1(M, \mathbb{Z}) \rightarrow \mathbb{R}$$

$$\Gamma := H_1(M, \mathbb{Z})/\ker \xi, \ \xi : \Gamma \rightarrow \mathbb{R}$$

For $x, y \in M$

$\hat{P}_{x,y}$ the set of equivalency classes $\hat{\alpha}$

of continuos paths $\alpha$ from $x$ to $y$ with

$\alpha_1 \equiv \alpha_2$ iff $\alpha_1^{-1} * \alpha_2 \in \ker \xi$. 
−ξ GRADIENT LIKE VECTOR FIELD

X is −ξ gradient like vector field if there exists a closed one form ω representing ξ and a Riemannian metric g so that:

1) ω is a Morse form
2) X = −grad\(_g\)(ω)

Proposition.

1) In the space of −ξ gradient like vector fields those which are ND form a generic set.

2) If X is −ξ gradient like and ND then:
   i) \( X \) is finite and \( x \in X \) has a Morse index \( \text{ind}(x) \).
   
   ii) Isolated solitons from \( x \) to \( y \) exists only if \( \text{ind}(x) − \text{ind}(y) = 1 \). In each class \( \hat{\alpha} \) there are finitely many solitons.
   
   iii) In each class \( \gamma \in \Gamma \) there are finitely many closed trajectories.
COUNTING FUNCTIONS:

1) For $x, \in \mathcal{X}_q$, $y, \in \mathcal{X}_{q-1}$, define $P^{\mathcal{O}}_{x,y} : \hat{\mathcal{P}}_{x,y} \to \mathbb{Z}$ by

$$\hat{\mathcal{P}}_{x,y}(\hat{\alpha}) = \sum_{\theta \in \hat{\alpha}} \text{sign}(\theta)$$

2) Define $Z_{X,\xi} : \Gamma \to \mathbb{Q}$ by

$$Z_{X,\xi}(\gamma) := \sum_{\hat{\alpha} \in \gamma} \frac{\text{sign}(\tilde{\theta})}{p(\tilde{\theta})}.$$ 

TOPOLOGICAL SOLUTION

Novikov has defined a ring $\Lambda_{\xi}$ and a cochain complex of free $\Lambda_{\xi}$ modules whose boundary homomorphisms are given in terms of $P^{\mathcal{O}}_{x,y}$ interpreted as elements in $\Lambda_{\xi}$ and cohomology is expressed in terms of the topology of $M$ and $\xi$.

Hutchings - Lee and Pajinhof have interpreted $Z_{X,\xi}$ as "torsion" associated to the Novikov complex.

SPECTRAL GEOMETRY SOLUTION

will be given in terms of real valued functions which and determine the Novikov and Hutchings..... solutions.
DIRICHLET SERIES:

\[ f \equiv \left( \begin{array}{cccc} \lambda_1 & < & \lambda_2 & < & \cdots & \lambda_k & < & \lambda_{k+1} & \cdots \\ a_1 & & a_2 & & \cdots & a_k & & a_{k+1} & \cdots \end{array} \right) \]

defines a measure with discrete support and Laplace transform

\[ L(f)(z) := \sum_i e^{-z\lambda_i} a_i \]

with an abscissa of convergence \( \rho(f) \leq \infty \)
\( (f(z) \text{ convergent on } \Re z > \rho(f) \text{ and divergent on } \Re z < \rho(f)). \)

THE INVARIANT \( \rho(X, \xi) \in [0, \infty]. \)

\( X \) vector field with hyperbolic rest points, and , and \( \xi \in H^1(M : \mathbb{R}). \)

\[ \rho(X, \omega, g, x) := \inf \{ \tau \in \mathbb{R} | \int_{W_x^-} e^{\tau h_x} \text{Vol}(i_x^-)_* g \} \]

\( h_x : W_x^- \to \mathbb{R} \) s.t. \( dh_x = (i_x^-)_* (\omega), \ h_x(x) = 0. \)

\( \rho(\cdots) \) independent on \( g, \) and on \( \omega \in \xi. \)

\[ \rho(X, \xi) := \inf_{x \in \mathcal{X}} \rho(X, \omega, g, x) \]
Proposition.

Let $X$ be $-\xi$ gradient like which is ND.

1) The pairs $(\xi(\gamma), Z_{X,\xi}(\gamma))$ with $Z_{X,\xi}(\gamma) \neq 0$ define a Dirichlet series with $\lambda$'s given by $\xi(\hat{\alpha})$ and $a$'s given by $Z_{X,\xi}(\gamma)$.

2) Let $\omega$ be a closed one form representing $\xi$ and $O$ a collection of orientations. The pairs $(\omega(\hat{\alpha}), P^O(x, y)(\hat{\alpha}))$ with $P^O(x, y)(\hat{\alpha}) \neq 0$ define a Dirichlet series with $\lambda$'s given by $\omega(\hat{\alpha})$ and $a$'s given by $P^O(x, y)(\hat{\alpha})$. Changing $\omega$ and $O$ one might change the sequence of $\lambda$'s by sign and the sequence of $a$'s by a factor.

3) If $\rho(X, \xi) < \infty$ the above series have a finite abscissa of convergence.

In particular the functions of one real variable $t L(P^O_{x,y}(e^t))$ and $L(Z_{X,\xi}(e^t))$ restricted $(a, \infty)$ determine by analytic continuation and inverse Laplace transform the counting functions $P^O_{x,y}$ and $Z_{X,\xi}$. 
SPECTRAL GEOMETRY:

$(M, g)$ closed Riemannian manifold,

$\omega$ a Morse one form (locally $\omega = dh$, $h$ smooth function with all critical points nondegenerate),

$t \in [0, \infty)$.

Consider

$$(\Omega^*(M), d^*(t)), \ d^q_\omega(t) : \Omega^q(M) \to \Omega^{q+1}(M)$$

with $d^q_\omega(t)(\alpha) := d\alpha + t\omega \wedge \alpha$.

Use $g$ to define $(d^q_\omega(t))^\# : \Omega^{q+1}(M) \to \Omega^q(M)$

and define $\Delta^q_\omega(t) : \Omega^q(M) \to \Omega^q(M)$ by:

$$\Delta^q_\omega(t) := (d^q_\omega(t))^\# \cdot d^q_t + d^{q-1}_\omega(t) \cdot (d^{q-1}_\omega(t))^\#.$$

$$\Delta^q_\omega(t) := \Delta^q + t(L + L^\#) + t^2||\omega||^2 Id$$

with $L$ the Lie derivative along $-\text{grad}_g \omega$, $L^\#$ the adjoint of $L$, and $||\omega||^2$ the fiberwise norm of $\omega$. 
Theorem 1.

There exist $C_1, C_2, C_3, T > 0$ so that for $t > T$ one has:

i) $\text{Spect} \Delta^q_\omega(t) \cap [C_1e^{-C_2t}, C_3t] = \emptyset$ and $1 \in (C_1e^{-C_2t}, C_3t)$

ii) $\#(\text{Spect} \Delta^q_\omega(t) \cap [0, C_1e^{-C_2t}]) = \#(X_q)$.

iii) For all but finitely many $t$, $\dim(\ker \Delta_\omega(t))$ is constant in $t$.

Denote by:

$\Omega^*_\text{sm}(M)(t)$ the span of the eigenforms which correspond to eigenvalues smaller than 1.

$\Omega^*_\text{la}(M)(t)$ the span of the eigenforms which correspond to eigenvalues larger than 1.

Theorem 1 implies that for $t > T$

$$(\Omega^*(M), d_\omega(t)) = (\Omega^*_{\text{sm}}(M)(t), d_\omega(t)) \oplus (\Omega^*_{\text{la}}(M)(t), d_\omega(t))$$

and

$$\Delta^q_\omega(t) = \Delta^q_{\omega,\text{sm}}(t) \oplus \Delta^q_{\omega,\text{la}}(t)$$

with $\dim(\Omega^q(M)_{\text{sm}}(t)) = \#(X_q)$ for any $t > T$. 
Theorem 2.

Suppose $X = -\grad_g(\omega)$ where $\omega$ is a closed one form representing $\xi \in H^1(M : \mathbb{R})$, $X$ is MS, $\rho(\xi, X) < \infty$ and orientations $\mathcal{O}$ are given.

There exists $T$, and a canonical base of $\Omega^q(M)_{sm}(t)$, $\{E^O_x(t), x \in \mathcal{X}\}$, so that for $t > T$ and $y \in \mathcal{X}_q$

$$d_\omega(t) E_y(t) = \sum_{x \in \mathcal{X}_{q+1}} I_{x,y}(t) E_y(t)$$

with

$$I_{y,x}(t) = L(\mathbb{P}^O_{x,y})(e^t).$$

Proposition.

Suppose $X$ is a $-\xi$ gradient like vector field with no rest points and all closed trajectories nondegenerate, $\omega$ a closed one form representing $\xi \in H^1(M, \mathbb{R})$ and $g$ a Riemannian metric.

Denote $\log T_{an}(t) := 1/2 \sum (-1)^{q+1} q \log \det(\Delta^q_\omega(t)).$

Then

$$\log T_{an}(t) = (-1)^{n+1} t \int_M \omega \wedge X^*(\Psi(g))$$

is the Laplace transform of the Dirichlet series $Z_X$. 
DROPPING THE HYPOTHESIS
”NO REST POINTS”
ADDITIONAL SPECTRAL GEOMETRY

Denote by
\[ \log \text{Vol}(t) = \sum (-1)^q \log \text{Vol}\{E_x(t), x \in X_q\} \]

Define
\[ \log T_{an,la}(t) := \frac{1}{2} \sum (-1)^{q+1} q \log \det(\Delta_{\omega,la}^q(t)). \]

ADDITIONAL RIEMANNIAN GEOMETRY
An invariant \( R(\omega, g, X) \) was introduced for

\( X \) with standard hyperbolic zeros,
\( \omega \) closed one form,
\( g \) Riemannian metric

Theorem.

Suppose \( X \) is a \( -\xi \) gradient like vector field which is ND and \( \rho(\xi, X) < \infty \). Suppose \( (\omega, g) \) is a pair with \( \omega \) representing \( \xi \) and \( X = -\text{grad}_g(\omega) \).

There exists a positive real number \( R > \rho([\omega], X) \) so that for \( t > R \) the function

\[ \log T_{an,la}(t) + \log \text{Vol}(t) + tR(\omega, g, X) \]

is the restriction of a holomorphic function on \( \{z \in \mathbb{C} \mid \Re z > R\} \) which is inverse Laplace transform of the Dirichlet series \( Z_X \).