WITTEN HELFFER SJÖSTRAND THEORY

NOTATIONS:

$\tau := (h, g)$

M^n	smooth manifold
$h : M^n \to \mathbb{R}$	smooth map
g	smooth, R-metric

$Cr(h)$ set of critical points

$X = -grad_g f$

$\gamma_y(t)$ trajectory of X, $\gamma_x(0) = x$.

$$W_x^{\pm} := \{y \in M^n \,|\, \lim_{t \to \pm\infty} \gamma_y(t) = x\}$$

THE MODEL: $\tau_{k,n-k,\alpha}$

$M^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$

$h(x, y) = \alpha - \frac{1}{2}\sum_{i=1}^{k} x_i^2 + \frac{1}{2}\sum_{i=k+1}^{n} x_i^2$

$g_{ij} = \delta_{ij}$

$Cr(h) = (0, 0), \qquad W_x^- = \mathbb{R}^k \times 0, \qquad W_x^+ = 0 \times \mathbb{R}^{n-k}$

$\tau = (h, g)$ A GENERALIZED TRIANGULATION if:

A1: For any $x \in Cr(h)$ there exists

$k,\ \epsilon > 0,\ \varphi : (U, x) \to (D^n(\epsilon), 0)$ s.t.

φ intertwines $\tau|_U$ with τ_k.

A2: For any $x, y \in Cr(h)$, $\quad W_x^- \pitchfork W_y^+$

ORIENTATIONS for $(\tau = (h, g)$,

$o = \{o_x, x \in Cr(h)\}, \qquad o_x$ orientation of W_x^-.

CONSEQUENCES:

C0: $\sharp Cr(h)$ finite,

C1: $M = \bigcup_{x\in Cr(h)} W_x^-$ with $W_x^- \equiv \mathbb{R}^{indexx}$

C2: $\mathcal{M}(x,y) = W_x^- \cap W_y^+/\mathbb{R}$ is a smooth manifold of dimension $i(x) - i(y) - 1$

Theorem A. *(Floer, A-B). There exists:*

a canonical compactification of $\hat{W}_x^-$ *of* W_x^-

the extension $\hat{in}_x : \hat{W}_x^- \to M^n$ *of* $in_x : W_x^- \to M^n$,

$\hat{W}_x^-$ *a smooth manifold with corners so that:*

(1): $\partial_1 W_x^- = \bigsqcup_{i(x)>i(y)} \mathcal{M}(x,y) \times W_y^-$

(2): $\hat{in}_x|_{\mathcal{M}(x,y)\times W_y^-} = in_y \cdot pr_{W_y^-}$

In particular

C3: The partition $M = \bigcup_{x\in Cr(h)} W_x^-$ is a smooth CW-complex

$\tilde{\tau} = (\tau, o)$

THE GEOMETRIC COMPLEX $(C^*(M,\tau), \partial^*_{\tau,o})$

$C^q(M,\tau) = \mathrm{Maps}(Cr_q(h), \mathbb{R})$

$\delta^q : C^q(M,\tau) \to C^{q+1}(M,\tau)$

NOTE: It is convenient to write

$C^*(M,\tau) = \oplus_{x \in Cr(h)} \Omega^{*-i(x)}(x)$

$\partial^* = d^* + \delta^*$, d^* exterior differential,

in this case $= 0$.

THE INTEGRATION MORPHISM

$Int^* : (\Omega^*(M), d^*) \to (C^*(M,\tau), \partial^*)$

WITTEN DEFORMATION $(\Omega(M), d^*(t))$

$d^*(t) = e^{-th} \cdot d^* \cdot e^{th}$

$\Delta_q^\tau(t) = \Delta_q + t(L_X + L_X^*) + t^2||\mathrm{grad}_g h||^2$

In coordinates, near critical ponts

$\Delta_q^\tau(t) = -\sum_i \frac{\partial^2}{\partial x_i^2} + t\epsilon_q + t^2 \sum_i x_i^2,$

with

$\epsilon_q(\sum_I a_I(x_1, \cdots, x_n) dx_I) = \sum \epsilon_{q,I} a_I(x_1, \cdots, x_n) dx_I$

THE DIAGRAM

$$\begin{array}{ccc} (\Omega^*(M), \overline{d}^*(t)) & \xrightarrow{e^{th}} & (\Omega(M)^*, d^*) \\ {\scriptstyle H^*(t)}\downarrow & & \downarrow{\scriptstyle Int^*} \\ (C^*(M,\tau), \partial^*)) & \xleftarrow[S^*(t)]{} & (C^*(M,\tau), \partial^*) \end{array}$$

$S^q(t)(x) = (\pi/t)^{(n-2k)/4} e^{-tf(x)}$

Theorem B. *(Witten) There exists positive constants* C, C', C''*, and* t_0 *so that for* $t \geq t_0$*:*

(1): $\operatorname{spec}(\Delta_q(t)) \subset [0, Ce^{-tC'}) \cup (C''t, \infty)$,

(2): $\sharp(\operatorname{spec}(\Delta_q(t)) \cap [0, Ce^{-tC'}) = \sharp(Cr_q(h)$

THE SMALL COMPLEX

$(\Omega(M)(t)_{sm}, \overline{d}^*(t))$

spanned by the eigenforms coresponding

to eigenvalues in $[0, Ce^{-tC'})$

Theorem C. *(Helffer Sjöstrandt) There exists* $T \in \mathbb{R}_+$ *and a smooth family of isometries* $J^q(t) : C^q(M, \tau) \to \Omega^q(M)(t)_{sm}$ *so that*

$H^q(t)_{sm} \cdot J^q(t) = Id + O(1/t)$

i.e $(J^q(t))^{-1} \cdot \overline{d}^*(t) \cdot J^q(t) = \partial^* + O(1/t)$

THE MAIN RESULTS OF THE WHS-THEORY:

Given a generalized triangulation $\tau = (h, g)$

(1) one has a cannonical smooth cell complex structure on M, therefore a chain complex of finite dimensional vector spaces $(C^*(M,\tau), \partial^*)$.
(Theorem A)

(2) $\mathrm{spec}(\Delta_q(t)) \subset [0, Ce^{-tC'}) \cup (C"t, \infty)$
(Theorem B)

3) the subcomplex $(\Omega(M)_{sm}(t), \overline{d}^*(t))$ is asymptotically isometric to $(C^*(M,\tau), \partial^*)$. (Theorem C)

G-WITTEN HELFFER-SJÖSTRAND THEORY

NOTATIONS:

G compact Lie group, $\mathfrak{g}$ its Lie algebra

ξ irreducible representation

V G-vector space $V = \tilde{\oplus} V_\xi$

M G manifold

h G-invariant smooth function

g G-invariant R-metric

$W^\pm_\Sigma = \bigcup_{x\in\Sigma} W^\pm_x$, Σ critical orbit

$\Delta_q(t) = \oplus \Delta_q(t)_\xi$

THE MODEL $\tau_{(\mathcal{G},\ <,>_{\mathfrak{g}},\ \alpha)}$.

$\mathcal{G} = (G, H \subset G, \rho_\pm \to O(V_\pm))$

$(V_\pm, < \ , \ >_\pm)$ Euclidean spaces

$<,>_{\mathfrak{g}}$ an ad$G-$ invariant scalar product on $\mathfrak{g}$.

$M = G \times_H V$,

$\pi : G \times_H V \to G/H, \qquad \pi^\pm : G \times_H V^\pm \to G/H$

$G-$vector bundles

$h(g, v_-, v_+) = \alpha - |v_-|^2 + |v_+|^2$

g induced by $<,>_{\mathfrak{g}}$ and $< \ , \ >_\pm$

$Cr(h) = \Sigma, \quad W^-_x \sim V_- \qquad W^-_\Sigma = G \times_H V_-$

$\tau = (h, g)$ A G-TRIANGULATION (GOOD)

A1: For any Σ critical orbit, there exists $(\mathcal{G}, <,>_{\mathfrak{g}}, \alpha)$,
$\epsilon > 0$, $\varphi : (U, x) \to (G \times_H D^n(\epsilon), 0)$ $G-$ equivariant,
s.t. φ intertwines $\tau|_U$ and $\tau_{(\mathcal{G}, <,>_{\mathfrak{g}}, \alpha)}$.

A2: For any two critical orbits Σ, Σ' and $x \in \Sigma$.
$W^-_x \pitchfork W^+_{\Sigma'}$ (then $W^-_{\Sigma} \pitchfork W^+_{\Sigma'}$).

(A.3:) ρ_- is trivial for any critical orbit.

ORIENTATIONS $o = \{x_\Sigma, o_\Sigma\}$
o_Σ an orientation in $W^-_{x_\Sigma}$

CONSEQUENCES:
C0:Finitely many critical orbits.
C1: $M = \bigcup_\Sigma W^-_\Sigma$
C3: $\mathcal{M}(\Sigma, \Sigma') = W^-_\Sigma \cap W^+_{\Sigma'}/\mathbb{R}$ manifold of dimension $i(\Sigma) - i(\Sigma') - 1 + \dim \Sigma$, $u^- : \mathcal{M}(\Sigma, \Sigma') \to \Sigma'$, map $u : W^-_\Sigma \to \Sigma$, $u^+ : \mathcal{M}(\Sigma, \Sigma') \to \Sigma$ bundles .

Theorem A. *There exists:*

a)canonical compactification $\hat{W}^-_\Sigma$ *of* W^-_Σ*,and*

b)the extension $\hat{in}_\Sigma : \hat{W}^-_\Sigma \to M^n$ *of* $in_\Sigma : W^-_\Sigma \to M^n$ *s.t.:*

(1) $\partial_k \hat{W}^-_\Sigma = \bigsqcup_{i(\Sigma)>\cdots>i(\Sigma_k)} W^-_{\Sigma,\Sigma_1\cdots,\Sigma_k}$

$W^-_{\Sigma,\Sigma_1\cdots,\Sigma_k} = \mathcal{M}(\Sigma,\Sigma_1)\times_{\Sigma_1}\cdots\mathcal{M}(\Sigma_{k-1},\Sigma_k)\times_{\Sigma_k} W^-_{\Sigma_k}$

(2) $\hat{in}_\Sigma | W^-_{\Sigma,\Sigma_1\cdots,\Sigma_k} = in_{\Sigma_k}\cdot pr_{W^-_{\Sigma_k}}$

In particular

C3. (1)$\partial_1 W^-_\Sigma = \bigsqcup_{i(\Sigma)>i(\Sigma')} \mathcal{M}(\Sigma,\Sigma')\times_{\Sigma'} W^-_{\Sigma'}$

C3. (2) $\hat{in}_\Sigma, \quad \pi^-_\Sigma : W^- \to \Sigma \;\Rightarrow$

$Int^* : \Omega^*(M) \to \Omega^{*-i(\Sigma)}(\Sigma, o(\pi^-_\Sigma))$

C3. (3):$u^\pm_{\Sigma,\Sigma'} \qquad o_\Sigma,\; o_\Sigma \;\Rightarrow$

$\delta^*_{\Sigma,\Sigma'} : \Omega^*(\Sigma'; o(\pi^-_{\Sigma'})) \to \Omega^{*-i(\Sigma)+i(\Sigma')+1}(\Sigma; o(\pi^-_\Sigma))$

when $i(\Sigma)\le i(\Sigma')$, $\delta^*_{\Sigma,\Sigma'} = 0$ when $i(\Sigma)\le i(\Sigma')$.

$\tilde{\tau} = (\tau, o)$.

THE GEOMETRIC COMPLEX $(C^*(M,\tau), \partial^*_{\tau,o})$

$C^q(M,\tau) = \sum_{r+i(\Sigma)=k} \Omega^r(\Sigma; o(\pi^-_\Sigma))$

$\delta^q : C^q(M,\tau) \to C^{q+1}(M,\tau)$ given by $\delta^*_{\Sigma,\Sigma'}$, when $i(\Sigma) \leq i(\Sigma')$ and by d^* when $i(\Sigma) = i(\Sigma')$.

THE ORBIT COMPLEX

$(C^*(M,\tau), d^*) = \oplus_\Sigma (\Omega^{*-i(\Sigma)}(\Sigma; o(\pi^-_\Sigma)), d^{*-i(\Sigma)})$

THE INTEGRATION MORPHISM

$Int^* : (\Omega^*(M), d^*) \to (C^*(M,\tau), \partial^*)$

induced by integration along the fiber (C.2)

ξ irreducible $\Rightarrow C^*(M,\tau)_\xi$ finite dimensional graded vector space. For q and ξ given, let $\alpha_1, \alpha_2, \cdots, \alpha_N$ the eigenvalues of the the $q-$Laplacian in the finite dimensional complex $\oplus_\Sigma (\Omega^{*-i(\Sigma)}(\Sigma; o(\pi^-_\Sigma))_\xi, d_\xi^{*-i(\Sigma)})$.

The G-version of Theorem A leads to

the GEOMETRIC COMPLEX $(C^*(M,\tau), \partial^*_{\tau,o})$

and the INTEGRATION MORPHISM

$Int^* : (\Omega^*(M), d^*) \to (C^*(M,\tau), \partial^*)$
which decomposes as a completed direct sum of
$Int^*_\xi : (\Omega^*(M)_\xi, d^*_\xi) \to (C^*(M,\tau)_\xi, \partial^*_\xi)$

UNFORTUNATELY, the target of Int^* is not finite dimensional and spec$(\Delta_q(t)_\xi)$ is not convergent to 0 and ∞,

FORTUNATELY, for an irreducible ξ the target of Int^*_ξ is finite dimensional and one has the G-version of Theorem B.

Theorem B. *For any irreducible representation* ξ *and integer* q *there exists the positive constants* C, C', C'', t_0 *and the colection of nonegative real numbers* $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_N$ *depending only on the* $G-$*manifold* M^n, *so that for* $t \geq t_0$*:*

(1): $\mathrm{spec}(\Delta_q(t)_\xi)$ *is contained in* $\bigcup_{i=1\cdots N}(\alpha_i - Ce^{-tC'}, \alpha_i + Ce^{-tC'}) \cup (C''t, \infty)$,

(2): $\sharp(\mathrm{spec}(\Delta_q(t)_\xi) \cap (\alpha_i - Ce^{-tC'}, \alpha_i + Ce^{-tC'}) = 1$ *for* t *large enough.*

THE SMALL SUBCOMPLEX

$(\Omega(M)(t)_{sm})_\xi, \overline{d}^*(t)_\xi$

the span of the eigenforms coresponding to the eigenvalues in $\bigcup_{i=1\cdots N}(\alpha_i - Ce^{-tC'}, \alpha_i + Ce^{-tC'})$

Theorem C. *Given an irreducible representation ξ there exists $T_\xi \in \mathbb{R}_+$ and a smooth family of isometries $J^q(t)_\xi : C^q(M,\tau)_\xi \to \Omega^q(M)_{sm}(t)$ so that for $t \geq T(\xi)$*

$$H^q(t)_\xi \cdot J^q(t)_\xi = Id + O(1/t).$$