DIRICHLET SERIES,

NOVIKOV THEORY AND

SPECTRAL GEOMETRY

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$$(M, g, \omega)$$

M closed manifold

ω closed one form i.e. $d(\omega) = 0$

g Riemmanian metric.

PURPOUSE

Provide analytic interpretation of:

a)Novikov incidences (of critical points of ω,)

b) the counting function (for closed trajectories of $-grad_g\omega$)

in terms of the spectral theory of $\Delta_\omega(t)$, the

Witten Laplacians associated to g and ω.

This is done with the help of Dirichlet series.

DIRICHLET SERIES

A Dirichlet series f is given by:

$$\begin{Bmatrix} \lambda_1 < \lambda_2 < \lambda_3 \cdots < \lambda_k < \lambda_{k+1} \cdots \\ a_1 \quad a_2 \quad a_3 \cdots \quad a_k \quad a_{k+1} \end{Bmatrix}$$

with $\lambda_k \to \infty$ if the sequences are infinite

The associated series is

$$f(z) = \sum_i e^{-z\lambda_i} a_i$$

and has an

Abscissa of Convergence,

$$\rho(f) \leq \infty,$$

a) If $\Re z > \rho(f) \Rightarrow f(z)$ is convergent and defines a holomorphic function

b) If $\Re z < \rho(f) \Rightarrow f(z)$ is divergent.

NOVIKOV RINGS

$[\omega] : \Gamma \to \mathbb{R}$ injective group homomorphism

$$\Lambda_{[\omega]} := \{f : \Gamma \to \mathbb{C} | f \text{has property N}\}$$

Property N: f has propery N if for any $R \in \mathbb{R}$

$\sharp\{\text{Supp f}\} \cap \{\gamma \in \Gamma | \omega(\gamma) < R\} < \infty$

$$f \in \Lambda_\omega \Rightarrow \left\{ \begin{array}{l} \lambda' s \equiv \omega(\gamma), f(\gamma) \neq 0 \\ a' s \equiv f(\gamma) \end{array} \right\}$$

$$\Lambda_{[\omega],\rho} := \{f \in \Lambda_{[\omega]}, \rho(f) < \rho\}$$

Observation. *With respect to convolution*

a) $\Lambda_{[\omega]}$ *is a field,*

b) $\Lambda_{[\omega],\rho}$ *is a subring of* $\Lambda_{[\omega]}$,

The evaluation at $z, \Re z > \rho, \quad ev_z : \Lambda_{[\omega],\rho} \to \mathbb{C}$ *is a ring homomorphism and the Laplace transform*

$$L : \Lambda_{[\omega],\rho} \to C^\omega(\rho, \infty)$$

is an injective ring homomorphism.

TOPOLOGY, NOVIKOV THEORY

Consider (M, g, ω)

M closed manifold

ω closed one form, i.e. $d(\omega) = 0$

g Riemmanian metric.

Denote by:

$X = -grad_g \omega$

$\Psi_t : M \to M$ the flow of X.

$\mathcal{P}_{x,y}$:= homotopy classes α of paths from x to y

Define:

$\omega \Rightarrow [\omega] : H_1(M)/\ker([\omega]) \equiv \Gamma \to \mathbb{R}$

$\hat{\mathcal{P}}_{x,y} := \mathcal{P}_{x,y}/\sim$, $(\alpha \sim \alpha'$ iff $\alpha^{-1} \circ \alpha' \in \ker([\omega]))$

The group Γ acts freely and transitively on $\hat{\mathcal{P}}_{x,y}$

$\pi : \tilde{M} \to M$ the associated Γ -principal covering.

H1: ω is a Morse form

i.e. $\omega = dh$, $h(x_1, x_2, \cdots, x_n) = x_1$ or

$h(x_1, \cdots, x_n) = -1/2 \sum_1^k x_i^2 + 1/2 \sum_{k+1}^n x_i^2$

Critical points:

$\mathcal{X} := \{x \in M | \omega(x) = 0\}$

$\mathcal{X}_q := \{x \in \mathcal{X} | ind(x) = q\}$

Stable/unstable sets:

$W_x^\pm := \{y \in M | \lim_{t \to \pm\infty} \Psi_t(y) = x\}$, $x \in \mathcal{X}$.

Observation 1. *$\mathcal{X}$ is finite*

H2: (ω, g) is Morse Smale

i.e. $x, y \in \mathcal{X} \quad \Rightarrow W_x^- \pitchfork W_y^+$

Observation 2. *Suppose H1 and H2 are satisfied. If* $x \in \mathcal{X}_q,\ y \in \mathcal{X}_{q-1},\ \alpha \in \hat{\mathcal{P}}_{x,y}$ *then the set of trajectories from* x *to* y *in the equivalence class* α *is finite*

Novikov incidences.

Choose:

$\mathcal{O} := \{\mathcal{O}_x, x \in \mathcal{X}\}$, $\mathcal{O}_x$ orientation of W_x^-,

Denote by:

$I_{\mathcal{O}}(x, y, \alpha) \in \mathbb{Z}$, the algebraic cardinality of trajectories from x to y in $\alpha \in \hat{\mathcal{P}}_{x,y}$.

Lifts:

$$\mathcal{L} := \{s : \mathcal{X} \to \tilde{M} | \pi \cdot s = id\}$$

Choose a lift $s \in \mathcal{L}$.

One can better organize the integers $I_{\mathcal{O}}(x, y, \alpha)$ as follows:

Define $I_{\mathcal{O},s}(x, y) := I_{\mathcal{O}}(x, y, \alpha_s)$, α_s the image by π

of a path between $s(x)$ and $s(y)$

Define

$$\mathbb{I}_{\mathcal{O},s}(x, y) : \Gamma \to \mathbb{Z} \subset \mathbb{C}$$

by $\mathbb{I}_{O,s}(x, y)(\gamma) := I_{\mathcal{O}}(x, y, \gamma\alpha_s)$, A trajectory $\theta(t) := \Psi_t(x)$ is closed

iff there exists $T \in \mathbb{R}_+$ so that $\theta(t+T) = \theta(t)$.

$\Rightarrow \theta : S^1 = \mathbb{R}/T\mathbb{Z} \to M$

A closed trajectory θ is nondegenerate

iff $D_x(\Psi_T) : T_xM \to T_xM$ has only one no nonzero eigenvector with eigevalue 1.

H3: All closed trajectories of X are nondegenerate.

Observation 3. *If H3 is satisfied then for any $\gamma \in \Gamma$,*

the set of closed trajectories in the class γ is finite

For $\theta(t)$ a nondegenerate closed trajectory

$\epsilon(\theta) := \operatorname{sign} \det(D_x(\Psi_T))$

$p(\theta)$ the period

$\zeta(\gamma) := \sum_{\theta \in \gamma} \frac{(-1)^{\epsilon(\theta)}}{p(\theta)} \in \mathbb{Q}.$

$$\mathcal{Z}_{\omega,g} : \Gamma \to \mathbb{Q} \subset \mathbb{C},$$

$\mathcal{Z}_{\omega,g}(\gamma) = \zeta(\gamma)$

Novikov Theory. *(Novikov, Huchings, Patzhitnov)*

Suppose (M, g, ω) *satisfies H1-H3. Choose* $\mathcal{O}$ *and* $s \in \mathcal{L}$.

1. For $x \in \mathcal{X}_q$ *and* $y \in \mathcal{X}_{q-1}$*, the maps* $\mathbb{I}_{\mathcal{O},s}(x, y)$ *and* $\mathcal{Z}_{\omega,g}$ *are in* $\Lambda_{[\omega]}$.

2. The free modules $C^q := Maps(\mathcal{X}_q, \Lambda_{[\omega]})$ *and the* $\Lambda_{[\omega]}-$*linear maps* $\partial^q : C^{q-1} \to C^q$*, defined by*

$$\partial(E_y) = \sum_{x \in \mathcal{X}_q} \mathbb{I}_{\mathcal{O},s}(x, y) E_x,$$

E_x *the characteristic map of* $x \in \mathcal{X}_q$ *define a cochain complex (the Novikov complex).*

3. The function $\mathcal{Z}_{\omega,g}$ *has interpretation as a torsion element in* $Wh(\Lambda_\omega)$

Note: Both functions $\mathbb{I}_{\mathcal{O},s}(x, y)$ and $\mathcal{Z}_{\omega,g}$ can be defined without the hypotheses H2 and H3.

ANALYSYS, SPECTRAL GEOMETRY

Witten deformation:

$$(\Omega^*(M), d_\omega(t)), \quad d_\omega(t) := d + t\omega \wedge \cdot$$

g induce the **Witten Laplacians** $\Delta^q(t) := \Delta^q_\omega(t)$

$\Delta^q_\omega(t) := d_\omega(t) \cdot d^\sharp_\omega(t) + d^\sharp_\omega(t) \cdot d_\omega(t)$

$\Delta^q_\omega(t) = \Delta^q + t(L_X - L^\sharp_X) + t^2||X||^2 Id$

Theorem (Witten). *If ω is a Morse form then there exists positive constants C_1, C_2, C_3, T so that for $t > T$:*

1) $Spec\Delta^q(t) \cap [C_1 e^{-C_2 t}, C_3 t] = \emptyset, 1 \in [C_1 e^{-C_2 t}, C_3 t],$

2) $\sharp(Spect\Delta^q(t) \cap [0, 1] = \sharp(\mathcal{X}_q)$

Witten Theorem implies that for $t > T$

$$(\Omega(M), d_\omega(t)) = (\Omega(M,)_{sm}(t), d_\omega(t)) \oplus (\Omega(M,)_{la}(t), d_\omega(t))$$

$$\Delta^q(t) = \Delta^q_{sm}(t) \oplus \Delta^q_{la}(t)$$

Theorem 1.

Suppose (M, g, ω) *satisfies H1 and H2. Choose* $\mathcal{O}$ *and* $s \in \mathcal{L}$. *There exist* $\rho(\omega, g) > 0$ *and a canonical base* $\{E_x(t), x \in \mathcal{X},\}$ *of* $\Omega^*(M)_{sm}(t)$ *for* $t > \rho(\omega, g)$ *so that the inverse Laplace transform of* $(\Omega^*(M)_{sm}(t), d^*(t), E_x(t))$ *is exactely the Novikov complex associated to* $(\omega, g), \mathcal{O}$ *and* $s \in \mathcal{L}$.

Precisely, $\mathbb{I}_{\mathcal{O},s}(x, y) \in \Lambda_{[\omega],\rho}$ *and with respect to the canonical base*

$d_\omega(t)(E_y(t)) = \sum_{x\in\mathcal{X}_q} I(x, y)(t)E_x(t), \;\; y \in \mathcal{X}_{q-1}$

with $I_{x,y}(t) = L(\mathbb{I}_{\mathcal{O},s}(x, y))$.

Introduce the functions:

a) $\log V_s(t) := \sum(-1)^q \log \mathrm{Vol}(E_x(t), x \in \mathcal{X}_q))$,

b) $\log T_{an,la}(t) := 1/2 \sum_q (-1)^{q+1} q \log Det\Delta^q_{la}(t)$

The metric g and the vector field X define

c) $\Phi(g, X) \in \Omega^{n-1}(M \setminus \mathcal{X})$

$$\left\{ \begin{array}{l} \Phi(g) \in \Omega^{n-1}(TM \setminus M) \\ X : M \to TM \end{array} \right\}$$

Theorem 2. *If (ω, g) satisfies H1-H3 then*

1. The integral $A = \int_{M\setminus\mathcal{X}} \omega \wedge \Phi$ is convergent

Consider $\log \mathbb{T}_{an,la,s}(t) := \log T_{an,la}(t) + \log V_s(t) + tA$

the corrected large torsion. Then:

2. $\mathcal{Z}_{\omega,g} \in \Lambda_{[\omega],\rho}$ and

$$\frac{L(\mathcal{Z}_{\omega,g}) - \log \mathbb{T}_{an,la,s}(t)}{t} = C \in [\omega](\Gamma) \subset \mathbb{R}$$

with C a computable costant depending on ω, g, s.

ABOUT THE PROOF

TOPOLOGY (Compactification Theorem)

ANALYSIS (Integration Theory)

SPECTRAL GEOMETRY (Relative Torsion)

Denote by:

$\tilde{\mathcal{X}}_q = \pi^{-1}(\mathcal{X}_q)$

$h_{\tilde{x}} : \tilde{M} \to \mathbb{R}$, s.t. $d(h_{\tilde{x}}) = \pi^*(\omega)$, $h_{\tilde{x}}(\tilde{x}) = 0$

For $\tilde{x} \in \tilde{\mathcal{X}}$, $W^{\pm}_{\tilde{x}}$

$\tilde{i}^{\pm}_{\tilde{x}} : W^{\pm}_{\tilde{x}} \to \tilde{M}$ smooth embedding,

$\pi \cdot \tilde{i}^{\pm}_{\tilde{x}} : W^{\pm}_{\tilde{x}} \to W^{\pm}_{\pi(\tilde{x})} \subset \tilde{M}$ injective smooth immersion

$\mathcal{M}(\tilde{x}, \tilde{y}) = W^{-}_{\tilde{x}} \cap W^{+}_{\tilde{y}}$, $\mathcal{T}(\tilde{x}, \tilde{y}) = \mathcal{M}(\tilde{x}, \tilde{y})/\mathbb{R}$

Broken trajectories:

$$\mathcal{B}(\tilde{x},\tilde{y}) = \bigcup_{\substack{k\geq 0,\ \tilde{y}_0,\ldots,\tilde{y}_{k+1}\in\tilde{\mathcal{X}} \\ \tilde{y}_0=\tilde{x},\ \tilde{y}_{k+1}=\tilde{y} \\ ind(\tilde{y}_i)>ind(\tilde{y}_{i+1})}} \mathcal{T}(\tilde{y}_0,\tilde{y}_1)\times\cdots\times\mathcal{T}(\tilde{y}_k,\tilde{y}_{k+1}).$$

Completed unstable sets:

$$\hat{W}^-_{\tilde{x}} = \bigcup_{\substack{k\geq 0,\ \tilde{y}_0,\ldots,\tilde{y}_k\in\tilde{\mathcal{X}} \\ \tilde{y}_0=\tilde{x} \\ ind(\tilde{y}_i)>ind(\tilde{y}_{i+1})}} \mathcal{T}(\tilde{y}_0,\tilde{y}_1)\times\cdots\times\mathcal{T}(\tilde{y}_{k-1},\tilde{y}_k)\times W^-_{\tilde{y}_k}.$$

Let

$i^-_{\tilde{x}} : \hat{W}^-_{\tilde{x}} \to \tilde{M}$

$\hat{i}_{\tilde{x}} = h_x \cdot i^-_{\tilde{x}}$

defined in the obvious way

Compactification Theorem.
Let (ω, g) be a Morse-Smale pair.

1.For any $\tilde{x}, \tilde{y} \in \tilde{\mathcal{X}}$, $\mathcal{B}(\tilde{x}, \tilde{y})$ is compact, in the topology induced from $C^0\big([h(\tilde{y}), h(\tilde{x})], \tilde{M}\big)$.

The smooth manifold $\mathcal{T}(\tilde{x}, \tilde{y})$ has $\mathcal{B}(\tilde{x}, \tilde{y})$ as a canonical compactification.

Moreover, $\mathcal{B}(\tilde{x}, \tilde{y})$ has the structure of a compact smooth manifold with corners, with $\mathcal{B}(\tilde{x}, \tilde{y})_k$ being

$$= \bigcup_{\substack{\tilde{y}_0, \ldots, \tilde{y}_{k+1} \in \tilde{\mathcal{X}} \\ \tilde{y}_0 = \tilde{x},\ \tilde{y}_{k+1} = \tilde{y} \\ ind(\tilde{y}_i) > ind(\tilde{y}_{i+1})}} \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_k, \tilde{y}_{k+1})$$

and $\mathcal{B}(\tilde{x}, \tilde{y})_0 = \mathcal{T}(\tilde{x}, \tilde{y})$.

2. For any critical point $\tilde{x} \in \tilde{\mathcal{X}}$, $\hat{W}^-_{\tilde{x}}$ has a structure of a smooth manifold with corners, with $(\hat{W}^-_{\tilde{x}})_k$ being

$$= \bigcup_{\substack{\tilde{y}_0, \ldots, \tilde{y}_k \in \tilde{\mathcal{X}} \\ \tilde{y}_0 = \tilde{x} \\ ind(\tilde{y}_i) > ind(\tilde{y}_{i+1})}} \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_{k-1}, \tilde{y}_k) \times W^-_{\tilde{y}_k}$$

and $(\hat{W}^-_{\tilde{x}})_0 = W^-_{\tilde{x}}$.

Moreover $\hat{i}_{\tilde{x}}$ and $\hat{h}_{\tilde{x}}$ are smooth and proper maps, and $\hat{i}_{\tilde{x}}$ is a closed map.

3.If X has no closed trajectories then W_x^-'s are smooth submanifolds providing a partition of M is open cells (of a CW complex).

INTEGRATION THEORY

$$Int_t : \Omega^q(M)_{sm}(t) \to Maps(\mathcal{X}_q, \mathbb{C})$$

$$Int_t(a) := \int_{W^-_{s(x)}} i^*_{s(x)}(e^{h_{s(x)}t} \cdot \pi^*(a))$$

a)Convergence

b)Int_t is isomorphism for $t > \rho(\omega, g)$