A NEW (computer friendly) APPROACH TO MORSE-NOVIKOV THEORY

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Classical finite dimensional MN theory

- **Considers**

1. Riemannian manifold (M, g)
2. Smooth map $f : M \to \mathbb{S}^1$

s.t (g, f) is Morse pair.

$$\boxed{f : M \to \mathbb{S}^1, \quad \mathrm{grad_g f}}$$

- **Relates**

1. Critical points of f = Rest points of $\mathrm{grad_g f}$
2. Instantons= isolated trajectories between rest points
3. Closed trajectories (Poincaré return maps of trajectories)

with

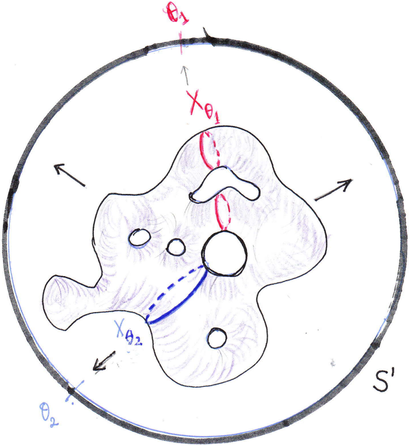
- **Topology of** (M, ξ_f) $\xi_f \in H^1(X; \mathbb{Z})$

1. Betti numbers, Novikov-Betti numbers
2. Monodromy, Reidemeister Torsion

Note : The above dynamical elements remain the same for any vector field which has f as a Lyapunov map which happens for any vector field which "minimizes an action". The relationship can be used both ways : Topology $\Leftrightarrow$ Dynamics

- **Draw backs** for use outside mathematics

1. few shapes / maps are smooth manifolds / smooth maps
2. elements hard or impossible to compute
3. lack of stability at small perturbations


θ_1
X_{θ_1}
X_{θ_2}
θ_2
S^1

Tame angle valued maps

$\boxed{f : X \to \mathbb{S}^1}$ is tame if:

1. X a compact ANR and $f : X \to \mathbb{S}^1$ continuous
2. For any $\theta \in \mathbb{S}^1$, $f^{-1}(\theta)$ open neighborhood retract
3. $\Sigma \subset \mathbb{S}^1$ finite s.t $f : X \setminus f^{-1}(\Sigma) \to \mathbb{S}^1 \setminus \Sigma$ fibration

Note :

- Morse maps defined on compact smooth manifold are tame,
- Simplicial maps defined on finite simplicial complex are tame,
- Tame maps are generic - the subspace of tame maps is homotopy equivalent to the space of continuous maps.

Elements associated with f

1. Critical values = angles = elements in Σ
2. $r-$Bar codes **denoted by** $\mathcal{B}_r(f)$:
   - closed $[a,b]$ $\therefore$ **denoted by** $\mathcal{B}_r^c(f)$
   - open (a,b) $\therefore$ **denoted by** $\mathcal{B}_r^o(f)$
   - open-closed $(a,b]$ $\therefore$ **denoted by** $\mathcal{B}_r^{oc}(f)$
   - closed-open $[a,b)$ $\therefore$ **denoted by** $\mathcal{B}_r^{co}(f)$

   $a \leq b, b = b' + 2\pi k,\ a, b' \in \Sigma,\ k \in \mathcal{Z} \geq 0$
3. $r-$Jordan blocks $J = (V, T) \therefore$ **denoted by** $\mathcal{J}_r(f)$

   $$J = \begin{cases} V & \text{vector space,} \\ T : V \to V & \text{isomorphism,} \\ (V, T) & \text{indecomposable} \end{cases}$$

   Jordan $\lambda-$blocks $\mathcal{J}_{r,\lambda}(f) := \{J \in \mathcal{J}_r(f) \mid \lambda \in \mathrm{spec\ T}\}$

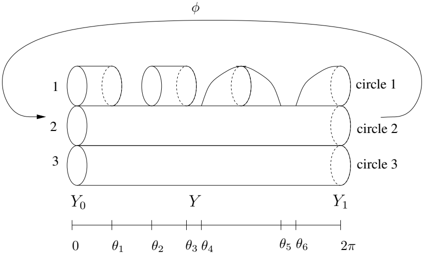
Meaning of barcodes and Jordan blocks

Bar codes describe

- **Death** right and left for elements in $H_r(X_\theta)$
- **Observability** right and left for elements in $H_r(X_\theta)$

Jordan blocks describe

- **Return** after a $2\pi-$ period for elements in $H_r(X_\theta) = H_r(X_{\theta+2\pi})$



map ϕ	r-invariants		
	dimension	bar codes	Jordan cells
circle 1: 1 time around circle 1 -3 times around 2, - 2 times around 3 circle 2: 1 time around circle 1, 4 times around 2, 1 time around 3 circle 3: 2 time around 1, 2 times around 2, 2 times around 3	0	(θ_2, θ_3)	$(1, 1)$
	1	$(\theta_6, \theta_1 + 2\pi]$ $[\theta_2, \theta_3]$ (θ_4, θ_5)	$(3, 2)$

Figure: Example of r-invariants for a circle valued map

Note : - If one add a cord from the $\theta_2 =$level to θ_3- level one introduces a $0-$open bar code (θ_2, θ_3).

Algebraic topology of $f : X \to \mathbb{S}^1$

- $f : X \to \mathbb{S}^1 \Rightarrow \xi_f \in H^1(X;\mathbb{Z})$
- $\xi \in H^1(X;\mathbb{Z}) \Rightarrow \tilde{X} \to X$, infinite cyclic cover with
- $\tau : \tilde{X} \to \tilde{X}$ deck transformation

- Betti numbers of X $\therefore$ $\beta_r(X)$
- Novikov–Betti numbers of (X,ξ) $\therefore$ $\beta_r^N(X,\xi)$
- $r-$ Monodromies of (X,ξ) $\therefore$ $(V_r, T_r : V_r \to V_r)$

Topological invariants recovered

$$f : X \to \mathbb{S}^1 \Rightarrow \begin{cases} \mathcal{B}^c_r(f),\ \mathcal{B}^o_r(f), \quad \mathcal{B}^{co}_r(f),\ \mathcal{B}^{oc}_r(f) \\ \mathcal{J}_r(f); \quad \boxed{\mathcal{J}_{r,1}(f)} \subseteq \mathcal{J}_r(f) \end{cases}$$

Theorem

(*Homotopy invariance*) *If $f : X \to \mathbb{S}^1$ is a tame map and κ a field then:*

1. $\sharp\mathcal{B}^c_r(f) + \sharp\mathcal{B}^o_{r-1}(f)$ is a homotopy invariant of the pair (X, ξ_f)
= to the Novikov Betti number $\beta^N_r(X, \xi_f)$,

2 The collection $\mathcal{J}_r(f)$ is a homotopy invariant of the pair (X, ξ_f).
= the monodromy $(V_r(\xi_f), T_r(\xi_f)) = \bigoplus_{J \in \mathcal{J}_r}(V(J), T(J))$,

3. $\sharp\mathcal{B}^c_r(f) + \sharp\mathcal{B}^o_{r-1}(f) + \sharp\mathcal{J}_{r,1}(f) + \sharp\mathcal{J}_{r-1,1}(f)$ is a homotopy invariant of X
= the Betti number $\beta_r(X)$.

Consider

- The space : $\boxed{\mathbb{T} \equiv \mathbb{R}^2/\mathbb{Z} \equiv \mathbb{C} \setminus 0}$

  The action : $\mu : \mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}^2$, $\mu(n, (a, b)) = (a + 2\pi n, b + 2\pi n)$
  The map : $[[x, y]] \in \mathbb{T} \quad \rightsquigarrow \quad e^{(y-x)+ix} \in \mathbb{C} \setminus 0$

- The space: $\mathbb{T} \setminus \Delta_T \equiv (\mathbb{R}^2 \setminus \Delta)/\mathbb{Z} \equiv \mathbb{C} \setminus \{S^1 \cup 0\}$

- The configuration $C_r(f)$: $\boxed{\{\mathcal{B}_r^c, \mathcal{B}_{r-1}^o\} \Longrightarrow C_r(f)}$

  - $[a, b] \rightsquigarrow [[a, b]] \in \mathbb{R}^2/\mathbb{Z} \rightsquigarrow e^{(b-a)+ia} \in \mathbb{C} \setminus 0$
  - $(a, b) \rightsquigarrow [[b, a]] \in \mathbb{R}^2/\mathbb{Z} \rightsquigarrow e^{(a-b)+ib} \in \mathbb{C} \setminus 0$

- The configuration $C_r^m(f)$: $\boxed{\{\mathcal{B}_r^{c,o}, \mathcal{B}_{r-1}^{o,c}\} \Longrightarrow C_r^m(f)}$

  - $[a, b) \rightsquigarrow [[a, b]] \in \mathbb{R}^2/\mathbb{Z} \rightsquigarrow e^{(b-a)+ia} \in \mathbb{C} \setminus \{S^1 \cup 0\}$
  - $(a, b] \rightsquigarrow [[b, a]] \in \mathbb{R}^2/\mathbb{Z} \rightsquigarrow e^{(a-b)+ib} \in \mathbb{C} \setminus \{S^1 \cup 0\}$

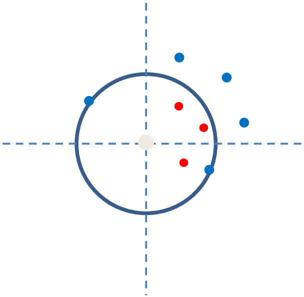


Figure: Configuration $C_r(f)$

WHAT IS THIS GOOD FOR ?

- $\mathrm{Conf}_n(\mathbb{T}) = S^n(\mathbb{T}) \equiv \mathbb{C}^{n-1} \times (\mathbb{C} \setminus 0)$ (complete metric space)

$S^n(\mathbb{T})$ = degree n monic polynomials with nonzero free coefficient

- $C_\xi(X; \mathbb{S}^1)$ = continuous maps f with $\xi_f = \xi$ with the compact-open topology. (complete metric space)

Theorem

(Stability) When restricted to tame maps the assignment $f \rightsquigarrow C_r(f)$ is continuous. The assignment extends to a continuous map $C_\xi(X; \mathbb{S}^1) \rightsquigarrow S^n(\mathbb{T})$, $n = \beta_r^N(X; \xi)$.

- The monic polynomial $P_r(f)(z)$ provides a refinement of the Novikov Betti number $\beta_r^N(\xi_f)$.
- If f real valued and $H_r(X)$ equipped with a scalar product (e.g. X a Riemannian manifold, $C_r(f)$ provides a canonical decomposition

$$H_r(X) = \oplus_{I \in \mathcal{B}^c_r(f)} H_{r,I}(X) \bigoplus \oplus_{I \in \mathcal{B}^o_{r-1}(f)} H_{r,I}(X)$$

- For an open and dense set of $f \in C(X; \mathbb{R})$ one has $\dim H_{r,I}(X) = 1$.

Theorem

(*Poincaré Duality*) *If* M^n *is a closed* $\kappa-$*orientable topological manifold with* $f : M \to \mathbb{S}^1 \subset \mathbb{C}$ *a continuous map then*

$$C_r(f)(z) = C_{n-r}(\bar{f})(z^{-1})$$

Definition (1) of Bar codes and Jordan blocks.

Based on representations of the oriented graph G_{2m} with $2m$ vertices

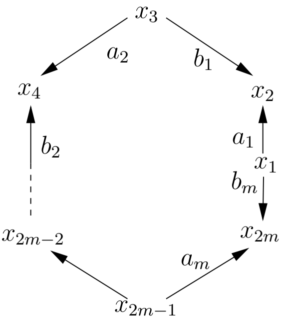


Figure: Graph G_{2m}

Representation

$$\rho = \begin{cases} V_r, & 1 \leq r \leq 2m \\ \alpha_i : V_{2i-1} \to V_{2i}, & 1 \leq i \leq \;\; m \\ \beta_i : V_{2i+1} \to V_{2i}, & 1 \leq i \leq \;\; m \end{cases}$$

Indecomposable representations

Indexed by

$$\begin{cases} \rho^I(< r, s >), & r \leq s \text{ integers } \; 1 \leq r \leq 2m, r \leq s \\ \rho^{II}(\lambda, k), & \lambda \in \overline{\kappa} \setminus 0, \; k \; \text{ positive } \textit{integer} \end{cases}$$

$\rho^I(\cdots)$s give bar codes $\rho^{II}(\cdots)$s give Jordan Blocks

$\rho^I(< r, s >)$, $s = r' + 2mk$ described by a spiral

from vertex x_r to $x_{r'}$ going counter-clockwise k times around origin,

$$\rho^{II}(\lambda; k) = \begin{cases} V_i = \kappa^k \\ \alpha_i = \beta_i = \beta_1 = id, 2 \leq i \leq m \\ \alpha_1 = T(\lambda; k) \end{cases}$$

$$T(\lambda, k) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{pmatrix}.$$

Construction of $\rho_r(f)$

- Consider $0 < \theta_1 < \theta_2 < \cdots < \theta_m \leq 2\pi$ critical angles for f
- Choose $0 < t_1 < t_2 < \cdots t_m < 2\pi$ regular angles s.t. $0 < t_1 < \theta_1 < t_2 < \theta_2 < \cdots t_m < \theta_m$
- Use tameness to define $a_i : f^{-1}(t_i) \to f^{-1}(\theta_i)$ and $b_i : f^{-1}(t_{i+1}) \to f^{-1}(\theta_i)$.
- Define

$$\rho_r(f) = \begin{cases} V_{2r} = H_r(X_{\theta_r}), V_{2r+1} = H_r(X_{t_r}) \\ \alpha_r,\ \beta_r \text{ induced in homology by } a_r, b_r. \end{cases}$$

and assign

- Bar codes of $\rho_r(f) \Rightarrow \rho_r(f)^I(\cdots)$,

$$\begin{cases} \rho^I(< 2i, 2j + 2mk >) & \Rightarrow [\theta_i, \theta_j + 2\pi k] \\ \rho^I(< 2i+1, 2j - 1 + 2mk >) & \Rightarrow (\theta_i, \theta_j + 2\pi k) \\ \rho^I(< 2i, 2j - 1 + 2mk >) & \Rightarrow [\theta_i, \theta_j + 2\pi k) \\ \rho^I(< 2i+1, 2j + 2mk >) & \Rightarrow (\theta_i, \theta_j + 2\pi k] \end{cases}$$

- Jordan blocks of $\rho_r(f) \Rightarrow \rho_r^{II}(J)$

$$\rho^{II}(J) \Rightarrow J = (V, T) \Rightarrow \{(\lambda, k)\}$$

Alternative definition

- Consider $\tilde{f} : \tilde{X} \to \mathbb{R}$
- Denote $\tilde{X}_a = f^{-1}(-\infty, a], \quad \tilde{X}^a = f^{-1}[a, \infty)$
- Consider

$$\mathbb{I}_a(r) := img(H_r(\tilde{X}_a) \to H_r(\tilde{X}))$$
$$\mathbb{I}^a(r) := img(H_r(\tilde{X}^a) \to H_r(\tilde{X}))$$

- Denote $\boxed{F_r(a, b) := \dim(\mathbb{I}_a(r) \cap \mathbb{I}^b(r))}$

- Define

$$\boxed{\underline{\delta}_r(a,b) := \lim_{\epsilon\longrightarrow 0}\left\{\begin{array}{ll} -F_r(a-\epsilon,b-\epsilon)- & F_r(a+\epsilon,b+\epsilon) \\ +F_r(a-\epsilon,b+\epsilon)+ & F_r(a+\epsilon,b-\epsilon) \end{array}\right.}$$

- Note that $\delta_r(a,b) = \delta_r(a+2\pi, b+2\pi)$
- Define

$$\delta_r(e^{\alpha+i\beta}) = \underline{\delta}_r(\alpha,\alpha+\beta)$$

Proposition

$$\delta_r = C_r(f)$$

Computation for $f : X \to S^1$, X finite simplicial complex, f linear on each simplex

Choose a total ordering of the simplices of X s.t

1. $\dim \tau < \dim \sigma \Rightarrow \tau < \sigma$

Note that the total order for vertices of X induces an orientation on each simplex hence an incidence number $I(\tau, \sigma) = \pm 1$ if τ face of σ and $I(\tau, \sigma) = 0$ otherwise.

INPUT:

1. A square $N \times N$ matrix, N the total number of simplices of X, with entries the numbers $I(\tau, \sigma)$.
2. A column ($1 \times K$ matrix, K the number of vertices) with entries the values of f on vertices.

OUTPUT:

A table with two columns and $\dim X$ rows.

The $r-$th row contains:

1. the collection of $r-$barcodes; pairs $(\pm a, \pm b)$ with a resp. b the left resp. right end and with the sign $+/-$ indicating that the end is closed / open.
2. the collection of $r-$Jordan cells, pairs (λ, k).

Morse–Novikov theory

- (M, g) Riemannian manifold, $f : M \to \mathbb{S}^1$ Morse function, $\text{grad}_g f$ associated vector field.

- Consider critical points $x_1, x_2, \cdots, x_N$ of index $i_1, i_2, \cdots, , i_N$

- Consider the **isolated trajectories** $\gamma \in T(r, s)$ between critical points x_r and x_s.

These sets might be infinite, however each trajectory has a sign $\epsilon(\gamma) = \pm$ and a winding number $n(\gamma) \in \mathbb{Z}$. Consider the sets $T(r, s; k) := \{\gamma \in T(r, s) \mid n(\gamma) = k\}$. These sets are finite.

• Define

$$\delta x_r = \sum_{s=1,2,\cdots N} (\sum_{k\in\mathbb{Z}} t^k \sum_{\gamma\in T(r,s;k)} \epsilon(\gamma))x_s$$

The key observations due to Novikov are :

1. $\sum_{k\in\mathbb{Z}}(\sum_{\gamma\in T(r,s;k)})t^k \in \kappa[t^{-1}, t]]$
2. $\delta^2 = 0$

• Form the Novikov complex whose $k-$components are the $\kappa[t^{-1}, t]]-$ vector spaces generated by the critical points of index k.

Morse theory

- (M, g) Riemannian manifold,
$f : M \to \mathbb{R}$ Morse function, $\Rightarrow$ $\text{grad}_g\, f$
- Consider critical points $x_1, x_2, \cdots, x_N$ of index $i_1, i_2, \cdots, , i_N$
- Count isolated trajectories between critical points $n(x_r, x_s)$
- Define Morse complex ($C_* = \kappa[\{x_i\}]$ and
$\partial(x_r) = \sum_s n(x_r, x_s)x_s$.)
- Calculate dim $H_r(C_*, \partial_*) = \beta_r(M)$ (Betti numbers).

Morse–Novikov theory

• (M, g) Riemannian manifold, $f : M \to \mathbb{S}^1$ Morse function, $\mathrm{grad}_g\, f$

• Consider critical points $x_1, x_2, \cdots, x_N$ of index $i_1, i_2, \cdots, , i_N$

• Count **isolated trajectories** between critical points x_r and x_s, however these sets might be infinite.
If so decompose in a countable union of finitely many as follows:

1. consider $\tilde{f} : \tilde{M} \to \mathbb{R}$ the infinite cyclic covering of f and $\tau : \tilde{M} \to \tilde{M}$ the deck transformation,
2. choose $\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_N$ lifts of critical points
3. count isolated trajectories from $\tilde{x}_r$ to $\tau^k(\tilde{x}_s)$ $\therefore$ $n(\tilde{x}_r, \tilde{x}_s; k)$

- Define Novikov complex:

$$C_* = \kappa[T^{-1}, T]][\{\tilde{x}_i\}],$$
$$\partial(\tilde{x}_r) = \sum_{\tilde{x}_s, k} n(\tilde{x}_r, \tilde{x}_s; k)\tilde{x}_s.$$

- $(C_*, \partial_*) \Rightarrow \beta_r^N(M; \xi_f)$ Novikov–Betti numbers.