G-MORSE THEORY REVISITED

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ELEMENTARY MORSE THEORY .

M closed manifold,

$h : M \to \mathbb{R}$ smooth function

$X : M \to TM$ smooth vector field.

h **Morse function**

$dh_x = o \Rightarrow$

$$(h, M, x) \sim \left\{ \begin{array}{l} h := h(x) - 1/2 \sum_{i=1}^{q} x_i^2 + 1/2 \sum_{i=q+1}^{n} x_i^2 \\ \mathbb{R}^q \times \mathbb{R}^{n-q} \\ \quad 0 \end{array} \right\}$$

$Ind(x) = q$

$\boxed{Cr(h) = \{x \in M, dh_x = 0\}}$, $Cr(h) = \sqcup_q Cr(h)_q$

$X \quad -h$ **gradient like vector field**

a)$X(h)(x) < 0, x \in M \setminus Cr(h)$

b)$(X, M, x) \sim (-grad_g h, M, x)$ for some Riemannian metric $g \Rightarrow \ Cr(h) = \mathcal{X}$

Stable/Unstable set

x critical point i.e $x \in Cr(h)$

$$\boxed{W_x^{\pm} = \{y \in M | \lim_{t \to \pm\infty} \gamma_y(t) = x\}} \quad \boxed{i_x^{\pm} : W_x^{\pm} \to M}$$

$\Rightarrow W_x^{\pm} \sim \mathbb{R}^{Ind(x)/n-Ind(x)}$

(X, h) **Morse pair**

a) h Morse function,

b)X $(-h)$-gradient like vector field

(X, h) **Morse Smale pair**

a) (X, h) Morse pair

b)$x, y \in Cr(h) \Rightarrow W_x^- \pitchfork W_y^+$

Proposition 1. *1. Given f smooth function there exists h a Morse function arbitrary close from f in C^r topology .*

2. Given h a Morse function there exists X $(-h)$-gradient like vector field.

3. Given (X, h) a Morse pair there exists X' arbitray close from X in C^0 topology so that (X', h) is a Morse Smale pair. One can actually choose X' to be equal to X away from an arbitrary neighborhood of $Cr(h)$

Let (X, h) be a Morse-Smale pair

$x, y \in Cr(h) \Rightarrow \boxed{\mathcal{T}(x, y) := W_x^- \cap W_y^+ / \mathbb{R}}$

smooth manifold of dimension $ind(x) - ind(y) - 1$

Consider $\hat{\mathcal{T}}(x, z), \;\; \hat{W}_x^- \; and \; \hat{i}_x^- : \hat{W}_x^- \to M$ defined by

$$\boxed{\hat{\mathcal{T}}(x, z) := \sqcup_r \bigsqcup_{y_1, \cdots, y_r} \mathcal{T}(x, y_1) \times \cdots \mathcal{T}(y_{r-1}, y_r) \times \mathcal{T}(y_r, z)}$$

$$\boxed{\hat{W}_x^- := \sqcup_r \bigsqcup_{y_1, \cdots, y_r} \mathcal{T}(x, y_1) \cdots \mathcal{T}(y_{r-1}, y_r) \times W_{y_r}^-}$$

$$\boxed{\hat{i}_x^- |_{\bigsqcup_{y_1, \cdots, y_r} \mathcal{T}(x, y_1) \cdots \mathcal{T}(y_{r-1}, y_r) \times W_{y_r}^-} := i_{y_r}^- \cdot Pr_{W_{y_r}^-}}.$$

Theorem 1. *1. $\hat{\mathcal{T}}(x, z)$ is a smooth compact manifold with corners whose $k-$ corner is*

$\hat{\mathcal{T}}(x, z)_r = \bigsqcup_{y_1, \cdots, y_r} \mathcal{T}(x, y_1) \times \cdots \mathcal{T}(y_{r-1}, y_r) \times \mathcal{T}(y_r, z).$

2. $\hat{W}_x^-$ is a smooth compact manifold with corners whose $k-$ corner is

$(\hat{W}_x^-)_r = \bigsqcup_{y_1, \cdots, y_r} \mathcal{T}(x, y_1) \cdots \mathcal{T}(y_{r-1}, y_r) \times W_{y_r}^-$

3. $\hat{i}_x^-$ is a smooth map.

APPLICATIONS

1) Smooth cell structure of M

Corollary 1. *1. The partition of M into $\mathcal{W}_x^-$, $x \in Cr(h)$, is a smooth CW-complex whose closed cells are given by*
$\hat{i}_x^- : \hat{W}_x^- \to M$.
2. There exists compatible smooth triangulations of M, $\hat{\mathcal{T}}(x, y)$, $\hat{W}_x^-$. In particular the open simplexes of W_x^- define the collection of open simplexes of M

2)The geometric complex

Fix orientations $\mathcal{O} := \{o_x, x \in Cr(h)\}$ of $T_x(W_x^-)$
Define the complex $(C^*(X), \delta^*_{h,\mathcal{O}}(t))$ by:

$$\boxed{C^q(X) := Maps(Cr(h)_q, \mathbb{R})}$$

$$\boxed{((\delta^q_{h,\mathcal{O}}(t))(f))(y) := \sum_{x \in Cr(h)_q} e^{-t(h(x)-h(y))}\delta(x,y)f(x)}$$

$y \in Cr(h)_{q+1}$, $\delta(x, y) = \sharp(\mathcal{T}(y, x))$, $f \in C^q(X)$

3)Integration Theory

$$Int_h^q(t)(\omega) := \int \hat{W}_x^-(i_x^-)^*(e^{(-t(h-h(x))}\omega)$$

$$Int_h^*(t) : (\Omega^*(M), d_h^*(t) = d^* + tdh\wedge) \to (C^*(X), \delta_{h,\mathcal{O}}^*(t))$$

4)WHS-theorem

Theorem 2. *Given a Riemannian metric g (which is admissible for (X, h)) there exists $T > 0$ and a canonical decomposition*

$$(\Omega^*(M), d_h^*(t)) = (\Omega_{sm}^*(M), d_h^*(t)) \oplus (\Omega_{la}^*(M), d_h^*(t))$$

so that

$$Int_h(t) : (\Omega_{sm}^*(M), d_h^*(t)) \to (C^*(X, \mathcal{O}), \delta_{h,\mathcal{O}}^*(t))$$

is an $O(1/t)$ isometry provided $t > T$.

ELEMENTARY G-MORSE THEORY .

$\tilde{M} = (M, \mu : G \times M \to M)$ closed $G-$ manifold,

Let $x \in M$.

$H := G_x = \{g \in G, \mu(g, x) = x\}$

$\Sigma(x) := \mu(G, x) = G/H$ orbit of $x \in M$

$\rho^x : H \to Gl(V = T_x(M)/T_x(\Sigma)$ isotropy repr.

$$(U, \Sigma) \sim (G \times_H V, G/H)$$

$$G \times_H V \leftrightarrows G/H$$

$h : M \to \mathbb{R}$ a smooth $G-$invariant function

$X : M \to TM$ smooth $G-$ equivariant vector field.

$x \in Cr(h) \Rightarrow \Sigma(x) \subset Cr(h)$

$x', x" \in \Sigma \Rightarrow \rho_{x'} \equiv \rho_{x"}$

$G-$ **Morse function** h satisfies:

a) $\rho^x = \rho^x_- \oplus \rho^x_+, \ \rho^x_\pm : H \to O(V_\pm), \ V = V_- \oplus V_+$

$x \in Cr(h) \Rightarrow \Sigma(x) \subset Cr(h)$

b) $\Sigma \subset Cr(h) \Rightarrow$

$$(h, M, \Sigma) \sim \left\{ \begin{array}{l} h := h(x) - 1/2||v_|||^2 + 1/2||v_|||^2 \\ E(\rho_- \oplus \rho_+) \\ \ \Sigma \end{array} \right\}$$

$Ind(\Sigma) := dim(V_-)$

$G-$ normal Morse function

h satisfies

a) h is a $G-$ Morse function,

b) ρ_x^- trivial for any $x \in Cr(h)$

X $-h$ gradient like vector field and $g-$ equivariant

$\boxed{\mathcal{X} := \{x \in M | X(x) = 0\}} \Rightarrow \mathcal{X} = Cr(h)$

$\Sigma \subset Cr(h)$

$$\boxed{W_\Sigma^\pm = \{y \in M | \lim_{t \to \pm\infty} \gamma_y(t) \in \Sigma\} = \bigcup_{x \in \Sigma} W_x^\pm}$$

$\boxed{i_\Sigma^\pm : W_\Sigma^\pm \to M}$ $\boxed{p_\Sigma^\pm : W_\Sigma^\pm \to \Sigma}$

$p_\Sigma^\pm$ is a $G-$ bundle eqv. to

$G \times_{G_x} W_x^\pm \to \Sigma(x)$ equivalent to $G \times_{G_x} V_\pm \to G/G_x$, for some (and then any) $x \in \Sigma$.

(X, h) **G-Morse pair**

$\ldots$

(X, h) **G- normal Morse pair**

...

G-Morse Smale pair (normal Morse Smale pair)

(X, h) satisfies

a) (X, h) a G-Morse pair (G- normal Morse pair)

b) For any $\Sigma, \Sigma', x \in \Sigma \Rightarrow W_x^- \pitchfork W_{\Sigma'}^+$ It suffices to check for one x.

Let (X, h) be a $G-$Morse-Smale pair

$\Sigma, \Sigma' \subset Cr(h) \Rightarrow$

$$\boxed{\mathcal{T}(\Sigma, \Sigma') := W_\Sigma^- \cap W_{\Sigma'}^+ / \mathbb{R}}$$

smooth manifold of dimension $dim(\Sigma) + ind(\Sigma) - ind(\Sigma') - 1$

$\Sigma, \Sigma' \subset Cr(h), x \in \Sigma \Rightarrow$

$$\boxed{\mathcal{T}(x, \Sigma') := W_\Sigma^- \cap W_{\Sigma'}^+ / \mathbb{R}}$$

smooth manifold of dimension $ind(\Sigma) - ind(\Sigma') - 1$ on which G_x acts.

$$\boxed{u : \mathcal{T}(\Sigma, \Sigma') \to \Sigma, \ l : \mathcal{T}(\Sigma, \Sigma') \to \Sigma'}$$

smooth maps, u a smooth $G-$ bundle isomorphic to $G \times_{G_x} \mathcal{T}(x, \Sigma') \to G/G_x$

ρ_x^- trivial $\Rightarrow$ p_Σ^- and u are trival bundles.

Define:

$\hat{\mathcal{T}}(\Sigma, \Sigma'), \hat{W}_\Sigma^-$ G-spaces,

$\hat{u} : \hat{\mathcal{T}}(\Sigma, \Sigma') \to \Sigma,\ l : \hat{\mathcal{T}}(\Sigma, \Sigma') \to \Sigma'$ G-equiv. maps,

$\hat{p}_\Sigma : \hat{W}_\Sigma^- \to \Sigma, : \hat{i}_\Sigma^- : \hat{W}_\Sigma^- \to M$ G- equiv.maps.

$$\boxed{\hat{\mathcal{T}}(\Sigma, \Sigma') := \sqcup_r \bigsqcup_{\Sigma_1, \cdots, \Sigma_r} \mathcal{T}(\Sigma, \Sigma_1) \times_{\Sigma_1} \cdots \mathcal{T}(\Sigma_{r-1}, \Sigma_r) \times_{\Sigma_r} \mathcal{T}(\Sigma_r, \Sigma')}$$

$$\boxed{\hat{W}_\Sigma^- := \sqcup_r \bigsqcup_{\Sigma_1, \cdots, \Sigma_r} \mathcal{T}(\Sigma, \Sigma_1)_{\Sigma_1} \cdots_{\Sigma_{r-1}} \mathcal{T}(\Sigma_{r-1}, \Sigma_r) \times_{\Sigma_r} W_{\Sigma_r}^-}$$

$$\boxed{\hat{i}_\Sigma^-|_{\bigsqcup_{\Sigma_1, \cdots, \Sigma_r} \mathcal{T}(\Sigma, \Sigma_1)_{\Sigma_1} \cdots_{\Sigma_{r-1}} \mathcal{T}(\Sigma_{r-1}, \Sigma_r) \times_{\Sigma_r} W_{\Sigma_r}^-} := i_{\Sigma_r}^- \cdot Pr_{W_{\Sigma_r}^-}}.$$

Theorem 3. *1. $\hat{\mathcal{T}}(\Sigma, \Sigma')$ is a smooth compact manifold with corners whose $k-$ corner is*

$$\hat{\mathcal{T}}(\Sigma, \Sigma')_r = \bigsqcup_{y_1, \cdots, y_r} \mathcal{T}(\Sigma, \Sigma_1)_{\Sigma_1} \cdots_{\Sigma_{r-1}} \mathcal{T}(\Sigma_{r-1}, \Sigma_r) \times_{\Sigma_r} \mathcal{T}(\Sigma_r, \Sigma').$$

The maps $\hat{\mathrm{u}}$ and $\hat{l}$ are smooth with $\hat{\mathrm{u}}$ a $G-$ bundle with fiber

$$T_x := \sqcup_r \bigsqcup_{\Sigma_1, \cdots, \Sigma_r} \mathcal{T}(x, \Sigma_1) \times_{\Sigma_1} \cdots_{\Sigma_{r-1}} \mathcal{T}(\Sigma_{r-1}, \Sigma_r) \times_{\Sigma_r} \mathcal{T}(\Sigma_r, \Sigma').$$

This bundle is isomorphic to $G \times_{G_x} T_x \to G/G_x$ and is trivial if ρ_- is trivial.

2. $\hat{W}^-_\Sigma$ is a smooth compact manifold with corners whose $k-$ corner is

$$(\hat{W}^-_\Sigma)_r = \bigsqcup_{y_1, \cdots, y_r} \mathcal{T}(\Sigma, \Sigma_1)_{\Sigma_1} \cdots_{\Sigma_{r-1}} \mathcal{T}(\Sigma_{r-1}, \Sigma_r) \times_{\Sigma_r} W^-_{\Sigma_r}.$$

The map $\hat{p}^-_\Sigma$ a smooth $G-$ bundle with fiber

$$F_x = \bigsqcup_{y_1, \cdots, y_r} \mathcal{T}(x, \Sigma_1)_{\Sigma_1} \cdots_{\Sigma_{r-1}} \mathcal{T}(\Sigma_{r-1}, \Sigma_r) \times_{\Sigma_r} W^-_{\Sigma_r}.$$

This bundle is isomorphic to $G \times_{G_x} F_x \to G/G_x$ and is trivial if ρ_- is trivial.

3. $\hat{i}^-_\Sigma$ is a smooth map.

Suppose (X, h) is G-normal Morse Smale. Choose for any $\Sigma \subset Cr(h)$ a point $x \in \Sigma$

By "passing to the orbit spaces" $i^-_\Sigma : W^-_\Sigma \to M \Rightarrow$ $\boxed{i^-_x : W^-_x \to M/G}$

Corollary 2. *The manifolds W^-_x have canonical compactifications to smooth manifolds with corners F_x and the maps i^-_x have (smooth) completions $\boxed{\hat{i}^-_x : F_x \to M/G}$ providing a (smooth) cell complex structure for M/G.*

Theorem 4. *1. (K.H.Meyer) Given a $G-$ invariant smooth function f there exists h a normal G-Morse function arbitrary close to f in C^0 topology.*

2. Given h a $G-$ Morse function there exists $(-h)$-gradient like vector fields.

3. ([B])Given (X, h) a normal $G-$ Morse pair, there exists X' arbitray close to X in the C^0 topology so that (X', h) is a G-normal Morse Smale pair. One can choose X' to be equal to X away from an arbitrary G-invariant neighborhood of $Cr(h)$.

APPLICATIONS:

1) **Smooth cell structure**

Corollary 3. *Suppose (X, h) is a G- Morse Smale pair.*

1. The partition of M into W^-_Σ is a smooth $G-$ handle body whose G- handles are $\hat{i}^-_\Sigma : \hat{W}^-_\Sigma \to M$ where $\hat{W}^-_\Sigma$ are the total spaces of the bundles $\hat{p}_\Sigma : \hat{W}^-_\Sigma \to \Sigma$.

2. If the pair is normal this partition induces a smooth cell structure for M/G and compatible smooth triangulations of F_x, T_x lead to compatible smooth triangulations of M/G; equivalently to a smooth G-triangulation of $\tilde{M}$. The open simplexes of M/G are open simplexes of W^-_x.

2)**The geometric complex**

The smooth maps i^-_Σ combined with integration along the fibers ($\int^F$) provide for any critical orbit Σ the linear maps

$$\boxed{Int^*_\Sigma(t) : \Omega^*(M) \to \Omega^{*-Ind(\Sigma)}(\Sigma, o_\Sigma)}$$

by $\boxed{Int^*_\Sigma(t)(\omega) := \int^F (\hat{i}^-_x)^*(e^{-t(h-h(\Sigma))}\omega)}$

Choose orientations $\mathcal{O} : \{O_x orientation\ in T_x(W_x^-)$.

$\mathcal{O} \Rightarrow$ "orientations" o_Σ of the bundles $\hat{p}_\Sigma : \mathcal{W}_\Sigma^- \to \Sigma$.

The integration along the fibers induces the linear maps

$$\boxed{\partial_{\Sigma,\Sigma'} : \Omega^*(\Sigma', o'_\Sigma) \to \Omega^{*-Ind(\Sigma)+Ind(\Sigma')+1}(\Sigma', o'_\Sigma)}.$$

$$\boxed{\partial_{\Sigma,\Sigma} : \Omega^*(\Sigma, o_\Sigma) \to \Omega^{*+1}(\Sigma, o_\Sigma) = d^*|_\Sigma}$$

Define:

$$\boxed{C^q(X) = \sum_\Sigma \Omega^{q-Ind(\Sigma)}(\Sigma, o_\Sigma)}$$

and

$$\boxed{\delta^q_{h,\mathcal{O}}(t)(\omega) := \oplus_{\Sigma'} e^{-t(h(x')-h(x))}\delta_{\Sigma,\Sigma'}(\omega)}$$

for $\omega \in \Omega^q(\Sigma, o_\Sigma)$.

4 Integration Theory

$$\boxed{Int_h^*(t) : (\Omega^*(M), d_h^*(t) = d^* + tdh\wedge) \to (C^*(X, \mathcal{O}), \delta^*_{h,\mathcal{O}}(t))}$$

$$\boxed{Int_h(t)(\omega) = \oplus_\Sigma Int_\Sigma^*(t)(i_x^-)^*\omega}$$

$(\Omega^*(M), d_h^*(t))$ and $(C^*(X), \delta^*_{h,\mathcal{O}}(t))$ are complexes of G- representations and $Int_h^*(t)$ is $G-$ equivariant.

$\Rightarrow$

$$\boxed{(Int_h^*(t)) = \oplus (Int_h^*(t))_\xi}$$

$$\boxed{(Int_h^*(t))_\xi : (\Omega^*(M)_\xi, d_h^*(t)) \to (C^*(X,)_\xi, \delta_h(t))}$$

with $(C^*(X)_\xi, \delta^*_{h,\mathcal{O}}(t))$ finite dimensional cochain complex.

4)WHS-theorem

Theorem 5. *Given a Riemannian metric g admissible for (X, h) and ξ an irreducible representation there exists $T_\xi > 0$ and a canonical decomposition*

$$\boxed{(\Omega^*(M)_\xi, d_h^*(t)) = (\Omega^*_{\xi,sm}(M), d_h^*(t)) \oplus (\Omega^*_{\xi,la}(M), d_h^*(t))}$$

so that

$$(Int_h(t))_\xi : (\Omega^*_{\xi,sm}(M), d_h^*(t)) \to (C^*(X)_\xi, \delta^*_{h,\mathcal{O}}(t))$$

is an $O(1/t)$ isometry provided $t > T_\xi$.

PROOF OF THEOREMS 3,4)

$K := W_\Sigma^- \cap M^{c-2\epsilon}$, $K \sim G \times_H K_x, \quad K_x = W_x^- \cap M^{c-2\epsilon}$

$Z = (\bigcup_{h(\Sigma'))\leq c} W_{\Sigma'}^+ \cap M^{c-2\epsilon}, \quad Z \subset M^{c-2\epsilon}$ Whitney stratif.

$$Maps_G(K\ , M\ \ |K_x \pitchfork Z\ \ ) \subset Maps_G(K\ \ , M\ \ )$$
$$Maps_H(K_x, M\ \ |K_x \pitchfork Z\ \ ) \subset Maps_H(K_x, M\ \ )$$
$$Maps\ \ (K_x, M^H|K_x \pitchfork Z^H) \subset Maps\ \ (K_x, M^H)$$

MANIFOLDS WITH CORNERS $\mathcal{P}$:

$\mathcal{P}$ is locally diffeomorphic $\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) \mid x_i \geq 0\}$.

$\mathcal{P}_k$ points of $\mathcal{P}$ which in some chart have exactly k vanishing coordinates. $\mathcal{P}_k$ is a smooth manifold of dimension $(n-k)$.

$\partial\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \cup \mathcal{P}_n$ is closed in $\mathcal{P}$ and a topological manifold. $(\mathcal{P}, \partial\mathcal{P})$ is a topological manifold with boundary $\partial\mathcal{P}$.

A product of manifolds with boundary is a manifold with corners.

Remark 1. *If $\mathcal{P}$ is a smooth manifold with corners, $\mathcal{O}, \mathcal{S}$ smooth manifolds, $p : \mathcal{P} \to \mathcal{O}$ and $s : \mathcal{S} \to \mathcal{O}$ smooth maps so that p and s are transversal (p is transversal to s if its restriction to each k-boundary $\mathcal{P}_k$ is transversal to s), then $p^{-1}(s(\mathcal{S}))$ is a smooth submanifold with corners of $\mathcal{P}$.*

PROOF OF THEOREM 3 1)

$c_N > \cdots > c_i > c_{i-1} > \cdots > c_1$, all critical values of h.

$\epsilon > 0$ small so that $c_i - \epsilon > c_{i-1} + \epsilon$ Denote:

$$M_i := h^{-1}(c_i),$$
$$M_i^{\pm} := h^{-1}(c_i \pm \epsilon_i)$$
$$M(i) := h^{-1}(c_{i-1}, c_{i+1}).$$

$P_i \equiv$ trajectories (possible broken) from M_i^- to M_i^-

P_i **compact smooth manifold with boundary**.

Let $h(\Sigma) = c_{r+1},\ h(\Sigma') = c_{r-k-1}$ and take

$\mathcal{P} := P_{r,r-k} := P_r \times P_{r-1} \times \cdots \times P_{r-k}$,

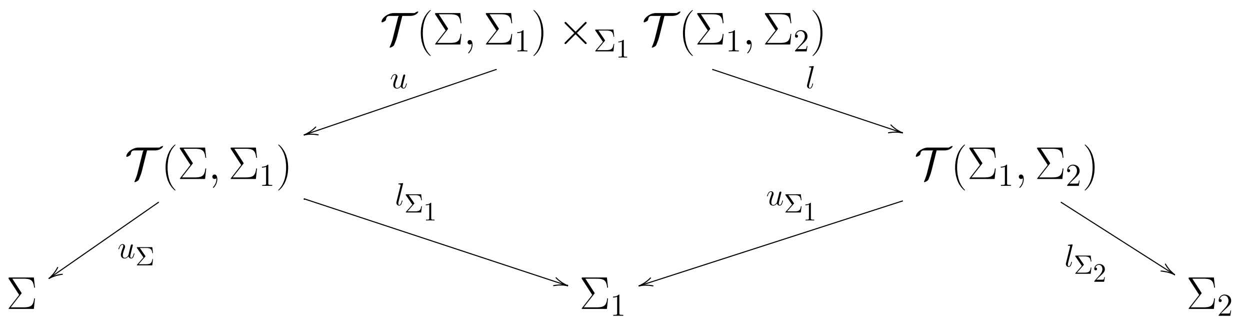
$\mathcal{O} := \prod_{i=r}^{r-k}(M_i^+ \times M_i^-)$,

$\mathcal{S} := S_\Sigma^- \times M_r^- \times \cdots \times M_{r-k+1}^- \times S_{\Sigma'}^+$.

Define $p : \mathcal{P} \to \mathcal{O}$ $s : \mathcal{S} \to \mathcal{O}$.

Show p and s are smooth and transversal.

$$\Rightarrow \boxed{\hat{T}(\Sigma, \Sigma') = p^{-1}(s(S))}$$



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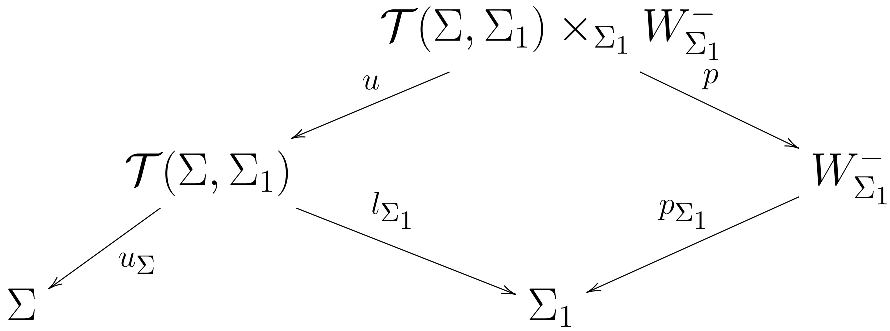

$\mathcal{T}(\Sigma,\Sigma_1)\times_{\Sigma_1}W^-_{\Sigma_1}$
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