DYNAMICS, Geometry and Topology

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Goettingen, November, 2010
Prior work:

S. Smale, D. Fried, S. P. Novikov, A. V. Pajitnov, M. Hutchins- Y. J. Lee

Influential work:

E. Witten, B. Hellfer - J. Sjöstrandt, T. Kato, M. Shubin

Work of Burghelea- Haller reported in this talk:
1. Dynamics, Laplace transform and spectral geometry, *Journal of Topology, No 1 2008*
2. Torsion, as a function on the space of representations, *in C*-algebras and Elliptic Theory II, Trends in Mathematics, 2008*
3. Papers in preparation
Dynamics = Smooth vector field $X$ on a closed manifold $M$.

Elements of Dynamics =

- Rest points
- Visible trajectories
  
  1. Instantons,
  
  2. Closed trajectories
Under nondegeneracy hypothesis = Property G

1. The set of rest points is finite
2. The set of instantons is at most countable,
3. The set of closed trajectories is at most countable.

Theorem (Kupka - Smale) The set of vector fields which satisfy G is residual in any $C^r$ – topology
PROBLEM:
- Count rest points, instantons and closed trajectories,
- Relate the results of the counting to geometry and topology,
- Provide new invariants for a vector field,
• $M$ a smooth closed manifold

• $X : M \rightarrow TM$ a smooth vector field

• $\psi_t : M \rightarrow M$ the flow of $X$ a one parameter group of diffeomorphisms.

• $\theta_m(t) := \psi_t(m)$ the trajectory with $\theta_m(0) = m$.

• A parametrized trajectory $\theta : \mathbb{R} \rightarrow M$, is characterized by

$$\frac{d\theta(t)}{dt} = X(\theta(t)).$$
\[ \mathcal{X} := \{ x \in M | X(x) = 0 \} \]

For \( x \in \mathcal{X} \) consider \( D_x(X) : T_x(M) \to T_x(M) \) the composition \( T_x M \xrightarrow{D_x X} T_x(TM) \xrightarrow{pr} T_x M \).

- \( x \in \mathcal{X} \) non degenerate = hyperbolic if
  \[ \lambda \in \text{Spec } D_x(X) \Rightarrow \Re \lambda \neq 0. \]
  - \( x \in \mathcal{X} \) non degenerate \( \Rightarrow \) has \textbf{Morse index}

\[ \text{index} = \# \{ \lambda \in \text{Spec } D_x(X) | \Re \lambda > 0 \} \]
for $x \in \mathcal{X}$

$$W^\pm_x := \{ y \in M \mid \lim_{t \to \pm \infty} \theta_x(t) = x \}$$

If $x \in \mathcal{X}_q$ then $W^\pm_x$ are images of one to one immersions

$$W^-_x = \text{im}(i^-_x : \mathbb{R}^q \to M)$$

$$W^+_x = \text{im}(i^-_x : \mathbb{R}^{n-q} \to M)$$
Red – Stable manifold
Blue – Unstable manifold
TRAJECTORIES

• parametrized trajectory: $\theta : \mathbb{R} \rightarrow M$

• equivalency of parametrized trajectories: $\theta_1 \equiv \theta_2$
  iff $\theta_1(t) = \theta_2(t + A)$ (for some $A \in \mathbb{R}$)

• nonparametrized trajectory: is an equivalency class $[\theta]$ of parametrized trajectories.

• instanton between two rest points $x$ and $y$: is an Isolated non parametrized trajectory $[\theta]$, with
  $$\lim_{t \to -\infty} \theta(t) = x \text{ and } \lim_{t \to \infty} \theta(t) = y$$

• closed trajectory: is a pair $[\hat{\theta}] = ([\theta], T)$ s.t $\theta(t + T) = \theta(t)$. 
Definition

- An instanton $[\theta]$ (from $x$ to $y$, $x$, $y$ non degenerate rest points) is non degenerate if $i_x^- \pitchfork i_y^+$ along $[\theta]$, in which case

$$\text{index } x - \text{index } y = 1.$$  

- Orientations $o_x$ and $o_y$ of the unstable manifolds $W_x^-$, and $W_y^-$ provide a sign

$$\epsilon^{o_x, o_y}([\theta]) = \pm 1$$

For a closed trajectory $[\hat{\theta}] = ([\theta], T)$ and $m \in \Gamma$, ($\Gamma = \theta(\mathbb{R})$), consider:

$$D_m(\psi_T) : T_m(M) \rightarrow T_m(M)$$

and

$$\underline{D_m(\psi_T)} : T_m(M) / T_m(\Gamma) \rightarrow T_m(M) / T_m(\Gamma)$$
\[ \Theta(s) = F(s, t - \delta(s)) \]
Definition

The closed trajectory \([\hat{\theta}] = ([\theta], T)\) is non degenerate iff

\[ (Id - D_m(\psi_T)) \]

is invertible; if so

\[ \epsilon([\hat{\theta}]) := \text{sign} \ det(Id - D\psi_T(m)) \]

\[ \rho([\hat{\theta}]) = \sup k \text{ so that } [\hat{\theta}] : S^1 \to M \text{ factors by a self map of } S^1 \text{ of degree } k. \]

Assign to\([\hat{\theta}]\) the rational number

\[ \epsilon([\hat{\theta}]) / \rho([\hat{\theta}]). \]
Property $G = H + MS + NCT$

1. **Property H**: All rest points are hyperbolic.
2. **Property MS**: For any two rest points $x, y \in X$, $i^-_x \cap i^+_y$.
3. **Property NCT**: All closed trajectories are nondegenerate.

Denote by:

- $C$ the set of closed trajectories and
- $I_{x,y}$ the set of instantons from $x$ to $y$.

For $X$ satisfying $G$ the sets $C$ and $I_{x,y}$ are countable and naturally filtered by finite sets.
Choose $\xi \in H^1(M; \mathbb{C})$ and $\omega \in \Omega^1(M)$, $d\omega = 0$ representing $\xi$. For a closed trajectory $[\hat{\theta}]$ define the functions

$$Z^\xi_{[\hat{\theta}]}(z) := \frac{e([\hat{\theta}])}{p([\hat{\theta}])} e^{z f_{[\hat{\theta}]} \xi}$$

and for an instanton from $x$ to $y$ the function

$$I^{o_x, o_y, \omega}_{[\theta]}(z) := e^{o_x, o_y([\theta])} e^{z f_{[\theta]} \omega}$$
and then the formal power series

\[
Z^\xi(z) = \sum_{[\hat{\theta}] \in C_{x,y}} Z_{[\hat{\theta}]}(z)
\]

\[
I^{\omega}_{[\theta]}(z) = \sum_{[\theta] \in \mathcal{I}_{x,y}} I^{\omega}_{[\theta]}(z)
\]

WHEN THE SUMS DEFINED ABOVE ARE ACTUALLY HOLOMORPHIC FUNCTIONS?
A Riemannian metrics on $M$ induces Riemannian metrics on $W_x^-$ if $x$ nondegenerate. Denote by $B_x(R)$ the ball of radius $R$ in $W_x^-$. 

**Definition**

$X$ satisfies \((\text{EG})\) (exponential growth property) if w.r. to some (and then with any) Riemannian metric if $\text{Vol}(B_x(R)) \leq e^{CR}$ (for some $C > 0$).
Definition

A closed 1-form $\omega \in \Omega^1(M)$ is called Lyapunov for $X$ if $\omega(X)(m) \leq 0$ and $\omega(X)(m) = 0$ only when $m \in \mathcal{X}$.

A cohomology class $[\omega]$ is Lyapunov for $X$ if it contains a closed 1-form Lyapunov for $X$.

$X$ satisfies (L) (Lyapunov property) if the set of cohomology classes Lyapunov for $X$ is nonempty.
Theorem

Suppose $X$ satisfies $\mathbf{L}$ and $\mathbf{EG}$, $\xi$ is a Lyapunov cohomology class for $X$ and $\omega \in \xi$. Choose a collection of orientations of the unstable manifolds $\mathcal{O} = \{o_x, x \in X\}$.

1. There exists $\rho \in \mathbb{R}$ so that the formal sums $Z^\xi(z)$ and $I^{o_x,o_y,\omega}_{[\theta]}(z)$ are absolutely convergent for $\Re z > \rho$ and define holomorphic functions in $\{z \in \mathbb{C}|\Re z > \rho\}$.

2. For any $x \in X_{k+1}, z \in X_{k-1}$ one has

$$\sum_{y \in X_k} I^{\mathcal{O},\omega}_{x,y} \cdot I^{\mathcal{O},\omega}_{y,z} = 0.$$
Suppose $X$ satisfies $G$, $L$ and $EG$. Define

$$C^*(M; X, \omega)(z) := (C^*(M; X), \delta^*_O, \omega(z))$$

- $C^k(M; X) := \text{Maps}(X_k, \mathbb{C})$
- $\delta^*_O, \omega(z) : C^*(M; X) \rightarrow C^{*+1}(M; X))$

\[
(\delta^*_O, \omega(z)(f))(u) := \sum_{v \in X} I_{u,v}(z)f(v)
\]

This is a holomorphic family of cochain complexes with a base.
• If $\omega_1$ and $\omega_2$ represent $\xi \in H^1(M; \mathbb{R})$, $O_1$ and $O_2$ are two sets of orientations, there exists a canonical isomorphism between $(C^*(M; X), \delta^*_{O_1, \omega_1(z)})$ and $(C^*(M; X), \delta^*_{O_2, \omega_2(z)})$.

$$\Rightarrow C^*(M; X, \xi)(z)$$

• The cohomology changes in dimension at finitely many $z$. 
$M$ closed manifold

$X$ smooth vector field satisfying $G$

$\omega$ Lyapunov closed one form for $X$

There exists $g$ Riemannian metric so that $X = -\text{grad}_g \omega$

Introduce Witten deformation $(\Omega^* M, d^\omega(z) = d + z \omega \wedge \cdots)$ equipped with the scalar products on $\Omega(M)$ defined by $g$. When $z$ is a real number It Induces Laplacians

$$\Delta^\omega_q(z) : \Omega^*(M) \rightarrow \Omega^*(M) = \Delta_q + z(L_X + L^*_X) + z^2 |\text{grad}_g \omega|^2$$
One can show that for $\Re z > \rho$

- Exactly $n_k$ eigenvalues of $\Delta^\omega_k(z)$ go to zero exponentially fast while the remaining have real part go to $\infty$ linearly fast as $\Re z \to \infty$. As a consequence

- One has a canonical orthogonal decomposition

$$(\Omega^*(M); d^\omega(z)) := (\Omega^*(M)_{sm}(M), d^\omega(z)) \oplus (\Omega^*(M)_{la}(M), d^\omega(z))$$

which diagonalizes $\Delta^\omega_*(z) = \Delta^\text{sm}_*(z) \oplus \Delta^\text{la}_*(z)$. 
Theorem

1. The above decomposition has an analytic continuation in the neighborhood of $[0, \infty)$.

If in addition $X$ satisfies $\text{EG}$ then:

2. For $\Re z > \rho$ the Integration of differential forms on unstable manifolds is well defined and provides an isomorphism from $(\Omega^*(M)_{\text{sm}}; d^\omega(z))$ to $(C^*(M; X), \delta^*_{\Omega_1,\omega_1}(z))$. The isomorphism is canonical.

3. For $\Re z > \rho$ large the $(\Omega^*(M)_{\text{la}}; d^\omega(z))$ is acyclic and

$$\log T_{la}(M, g, \omega)(z) - Z^{X,[\omega]}(z) = zR(g, X, \omega)$$

where $T_{la}(M, g, \omega)(z)$ is the complex Ray Singer torsion and $R(g, X, \omega)$ a numerical invariant defined below.
$X$ vector field which satisfies $L$ with $n_k$ rest points of index $k$ on a Riemannian manifold $(M, g)$.

Consider

$$\Delta^X_q(z) : \Omega^*(M) \to \Omega^*(M) = \Delta_q + z(L_X + L^*_X) + z^2||X||^2$$

Results of Kato and Shubin $\Rightarrow$

*For any $x \in \mathcal{X}_k$ one can associate a the function $\lambda(z) \in \text{Spec}_{k}^{\mathcal{X}}(z)$, holomorphic in a neighborhood of $[0, \infty)$ with $\lim_{\Re z \to \infty} \lambda_x(z) = 0$."

Consider $x \mapsto \lambda_x(0)$ and

$$P_k(z) = \prod_{x \in \mathcal{X}_k} (z - \lambda_x(0)).$$
Theorem

For any $k$ there exists a collection $\lambda^k_n(z)$, $n \in N$, each holomorphic function in a neighborhood of $[0, \infty)$ so that for each $t \in [0, \infty)$ the family $\{\lambda^k_r(t), r = 1, 2, \cdots\}$ represent the entire family of eigenvalues of $\Delta^X_k(t)$ (with their multiplicity).

Mild improvement of Witten methods shows that each rest point $x$ of index $k$ provides for $t$ very large an eigenvalue which go exponentially fast to zero. In fact much more is true.
Twisted cohomology

- Is a graded vector space $H^*(M; \xi)$ associated to $(M, \xi \in H^1(M; \mathbb{C})$.
  Denote $\beta_i^\xi := \dim H^i(M; \xi)$.

- For $\Re z$ large enough $\beta_i^{z\xi}$ is constant in $z$.
  Denote $\underline{\beta}_i^\xi := \lim_{\Re z \to \infty} \beta_i^{z\xi}$.
If $X$ satisfies $H$, $MS$, $L$ with $\xi$ a Lyapunov class for $X$ then:

1. For $\Re z$ large enough the cohomology of the cochain complex $\mathbb{C}^*(M; X, \xi)(z)$ and $H^*(M; z\xi)$ are naturally isomorphic.

2. One has

$$n_k \geq \beta^\xi_k$$

$$\sum_{0 \leq k \leq r} (-1)^k n_k \geq \sum_{0 \leq k \leq r} \beta^\xi_k, \ r \text{ even}$$

$$\sum_{0 \leq k \leq r} (-1)^k n_k \geq \sum_{0 \leq k \leq r} \beta^\xi_k, \ r \text{ odd.}$$
• The cochain complex $\mathbb{C}^*(M; X, \xi)(z)$ is equipped with a base so it is possible to define a torsion (in case that the $z\xi$-twisted cohomology for is trivial is a holomorphic function in $z$.)

• The cohomology class $z\xi$ can be interpreted as a rank 1-complex representation of the fundamental group and together with a so-called Euler structure there is a topological invariant the Reidemeister torsion (in case that the $z\xi$-twisted cohomology for is trivial is a holomorphic function in $z$.)

• The vector field $X$ and some minor additional data determine an Euler structure.
It is possible to show that the sum of the log of the torsion of \( \mathbb{C}^*(M; X, \xi)(z) \) and of the function \( Z^\xi(z) \) is exactly the Reidemeister torsion of \( M z^\xi \) and the Euler class defined by \( X \). This relates the functions \( I_{x,y}^O,\omega(z) \) and \( Z^\xi(z) \) and the topology
• The torsion \( \log T \in \mathbb{C} \) associated with a cochain complex \((C^*, d^*)\) with a nondegenerate bilinear form each \( C^i \), (hence with Laplacians \( \Delta_k : C^k \to C^k \)) is defined by the formula

\[
\log T = \frac{1}{2} \sum_{i \geq 0} (-1)^i i \log \det \Delta_i
\]

• \( \mathcal{R}(g, X, \omega) \in \mathbb{R} \) is an invariant associated with a Riemannian manifold \((M, g)\), a vector field \( X \) with isolated rest points and a closed one form \( \omega \) defined as follows:
For an oriented Riemannian manifold $M$, the Riemannian metric produces a $(n - 1)$ differential form (the angular momentum form)

$$\Psi(g) \in \Omega^{n-1}(TM \setminus M)$$

If the vector field $X$ has no rest points it defines the map $X : M \to T(M) \setminus M$ and

$$\mathcal{R}(g, X, \omega) := \int_M \omega \wedge X^*(\Psi(g))$$
When the vector field $X$ has singularities the integration is done over $M \setminus \mathcal{X}$ and the integral might not be convergent and requires regularization. This regularization is always possible. **Geometric regularization.**

Since the eigenvalues of the Laplacians $\Delta^i_{la}$ are infinitely many the definition of $\det \Delta^i_{la}$ also requires regularization. This regularization is again possible. **Zeta function regularization.**