DYNAMICS, Geometry and Topology

Dan Burghelea (and Stefan Haller)

Department of mathematics
Ohio State University, Columbuus, OH

Goettingen, November, 2010

Prior work:

S. Smale, D.Fried, S.P.Novikov, A.V. Pajitnov,
M. Hutchins- Y. J. Lee

Influential work :

E Witten, B.Hellfer - J.Sjöstrandt, T.Kato, M. Shubin

Work of Burghelea- Haller reported in this talk:
1. Dynamics, Laplace transform and spectral geometry, *Journal of Topology, No 1 2008*
2. Torsion, as a function on the space of representations, *in C*algebras and Elliptic Theory II, Trends in Mathematics, 2008*
3. Papers in preparation

CONTENTS

Dynamics = Smooth vector field X on a closed manifold M.

Elements of Dynamics=

- Rest points
- Visible trajectories

  1. Instantons,
  2. Closed trajectories

Under nondegeneracy hypothesis = Property G

1. The set of rest points is finite
2. The set of instantons is at most countable,
3. The set of closed trajectories is at most countable.

Theorem

(Kupka -Smale) The set of vector fields which satisfy G is residual in any C^r-topology

PROBLEM:

- Count rest points, instantons and closed trajectories,
- Relate the results of the counting to geometry and topology,
- Provide new invariants for a vector field,

Crash course in Dynamics

- M a smooth closed manifold
- $X : M \to TM$ a smooth vector field
- $\Psi_t : M \to M$ the flow of X a one parameter group of diffeomorphisms.
- $\theta_m(t) := \Psi_t(m)$ the trajectory with $\theta_m(0) = m$.
- A parametrized trajectory $\theta : \mathbb{R} \to M$, is characterized by

$$\boxed{d\theta(t)/dt = X(\theta(t))}.$$

REST POINTS:

$$\boxed{\mathcal{X} := \{x \in M | X(x) = 0\}}$$

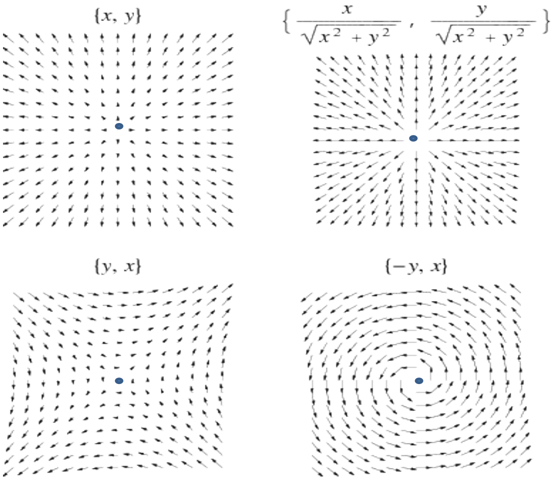
For $x \in \mathcal{X}$ consider $D_x(X) : T_x(M) \to T_x(M)$
the composition $T_xM \stackrel{D_xX}{\to} T_x(TM) \stackrel{pr}{\to} T_xM$.

- $x \in \mathcal{X}$ **non degenerate= hyperbolic** if

$$\boxed{\lambda \in \text{Spec } D_x(X) \Rightarrow \Re\lambda \neq 0.}$$

- $x \in \mathcal{X}$ non degenerate $\Rightarrow$ has **Morse index**

$$\boxed{\text{ind}x = \sharp\{\lambda \in \text{Spec } D_x(X) | \Re\lambda > 0\}.}$$


{x, y}
{x/√(x² + y²), y/√(x² + y²)}
{y, x}
{−y, x}

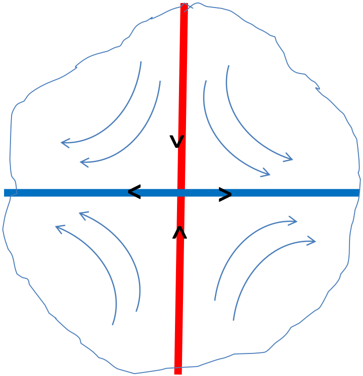
STABLE/UNSTABLE SET

for $x \in \mathcal{X}$

$$\boxed{W_x^{\pm} := \{y \in M | \lim_{t \to \pm\infty} \theta_x(t) = x\}}$$

If $x \in \mathcal{X}_q$ then $W_x^{\pm}$ are images of *one to one* immersions

$$\begin{aligned} W_x^- &= \mathrm{im}(i_x^- : \mathbb{R}^q \to M) \\ W_x^+ &= \mathrm{im}(i_x^- : \mathbb{R}^{n-q} \to M) \end{aligned}$$



Red – Stable manifold
Blue – Unstable manifold

TRAJECTORIES

• **parametrized trajectory**: $\theta : \mathbb{R} \to M$

• **equivalency of parametrized trajectories**: $\theta_1 \equiv \theta_2$ iff $\theta_1(t) = \theta_2(t + A)$ (for some $A \in \mathbb{R}$)

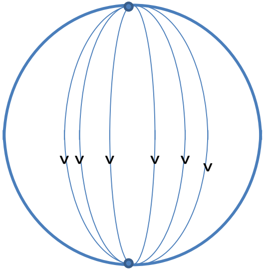
• **nonparametrized trajectory**: is an equivalency class $[\theta]$ of parametrized trajectories.

• **instanton between two rest points** x **and** y: is an Isolated non parametrized trajectory $[\theta]$, with

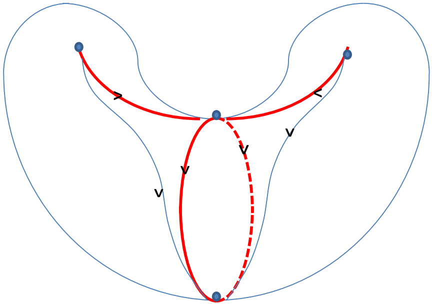
$$\lim_{t\to-\infty} \theta(t) = x \text{ and } \lim_{t\to\infty} \theta(t) = y$$

• **closed trajectory** : is a pair $[\hat{\theta}] = ([\theta], T)$ s.t $\theta(t + T) = \theta(t)$.

No Instantons



Four Instantons



Definition

• An instanton $[\theta]$ (from x to y, x, y non degenerate rest points) is non degenerate if $i_x^- \pitchfork i_y^+$ along $[\theta]$, in which case

$$\text{index } x - \text{index } y = 1.$$

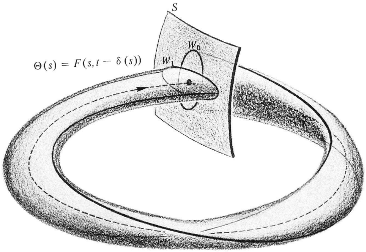
• Orientations o_x and o_y of the unstable manifolds W_x^-, and W_y^- provide a sign

$$\boxed{\epsilon^{o_x, o_y}([\theta]) = \pm 1}$$

For a closed trajectory $[\hat{\theta}] = ([\theta], T)$ and $m \in \Gamma$, $(\Gamma = \theta(\mathbb{R}),)$ consider:

$$D_m(\Psi_T) : T_m(M) \to T_m(M) \text{ and}$$

$$\underline{D_m(\Psi_T)} : T_m(M)/T_m(\Gamma) \to T_m(M)/T_m(\Gamma)$$


S
W_0
W_1
$\Theta(s) = F(s, t - \delta(s))$

Definition

The closed trajectory $[\hat{\theta}] = ([\theta], T)$ is non degenerate iff

$$(Id - \underline{D_m(\Psi_T}))$$

is invertible; if so

- $\epsilon(\underline{[[\hat{\theta}]]}) :=$ sign $det(Id - \underline{D\Psi_T}(m))$
- $p([\hat{\theta}]) = \sup k$ so that $\underline{[\hat{\theta}]} : S^1 \to M$ factors by a self map of S^1 of degree k.
- Assign to$[\hat{\theta}]$ the rational number

$$\boxed{\epsilon([\hat{\theta}])/p([\hat{\theta}])}.$$

Property G= H + MS + NCT

1. **Property H**: All rest points are hyperbolic.
2. **Property MS**: For any two rest points $x, y \in \mathcal{X}$ $i_x^- \pitchfork i_y^+$.
3. **Property NCT**: All closed trajectories are nondegenerate.

Denote by :

$\mathcal{C}$ the set of closed trajectories and
$\mathcal{I}_{x,y}$ the set of instantons from x to y.

For X satisfying **G** the sets $\mathcal{C}$ and $\mathcal{I}_{x,y}$ are countable and naturally filtered by finite sets

Choose $\xi \in H^1(M;\mathbb{C})$ and $\omega \in \Omega^1(M), d\omega = 0$ representing ξ. For a closed trajectory $[\hat{\theta}]$ define the functions

$$\boxed{Z^{\xi}_{[\hat{\theta}]}(z) := \frac{\epsilon([\hat{\theta}])}{p([\hat{\theta}])} e^{z\int_{[\hat{\theta}]}\xi}}$$

and for an instanton from x to y the function

$$\boxed{I^{o_x,o_y,\omega}_{[\theta]}(z) := \epsilon^{o_x,o_y}([\theta])e^{z\int_{[\theta]}\omega}}$$

and then the formal power series

$$\boxed{Z^{\xi}(z) = \sum_{[\hat{\theta}] \in \mathcal{C}_{x,y}} Z_{[\hat{\theta}]}(z)}$$

$$\boxed{I_{[\theta]}^{o_x, o_y, \omega}(z) = \sum_{[\theta] \in \mathcal{I}_{x,y}} I_{[\theta]}^{o_x, o_y}(z)}$$

WHEN THE SUMS DEFINED ABOVE ARE ACTUALLY HOLOMORPHIC FUNCTIONS?

A Riemannian metrics on M induces Riemannian metrics on W_x^- if x nondegenerate. Denote by $B_x(R)$ the ball of radius R in W_x^-.

Definition

X satisfies (**EG**) (exponential growth property) if w.r. to some (and then with any) Riemannian metric if $\boxed{\mathrm{Vol}(B_x(R)) \leq e^{CR}}$ (for some $C > 0$).

Definition

- A closed 1-form $\omega \in \Omega^1(M)$ is called *Lyapunov for X* if $\omega(X)(m) \leq 0$ and $\omega(X)(m) = 0$ only when $m \in \mathcal{X}$.
- A cohomology class $[\omega]$ is *Lyapunov for X* if it contains a closed 1-form Lyapunov fo X.
- X satisfies **(L)** (Lyapunov property)if the set of cohomology classes Lyapunov for X is nonempty.

Theorem

Suppose X satisfies **L** *and* **EG**, ξ *is a Lyapunov cohomology class for X and $\omega \in \xi$. Choose a collection of orientations of the unstable manifolds $\mathcal{O} = \{o_x, x \in \mathcal{X}\}$.*

1. There exists $\rho \in \mathbb{R}$ so that the formal sums $Z^{\xi}(z)$ and $I_{[\theta]}^{o_x,o_y,\omega}(z)$ are absolutely convergent for $\Re z > \rho$ and define holomorphic functions in $\{z \in \mathbb{C} | \Re z > \rho\}$.

2. For any $x \in \mathcal{X}_{k+1}, z \in \mathcal{X}_{k-1}$ one has

$$\sum_{y \in \mathcal{X}_k} I_{x,y}^{\mathcal{O},\omega} \cdot I_{y,z}^{\mathcal{O},\omega} = 0.$$

A holomorphic family of cochain complexes

Suppose X satisfies **G, L** and **EG.** Define

$$\mathbb{C}^*(M;X,\omega)(z) := (C^*(M;X), \delta^*_{\mathcal{O},\omega}(z))$$

- $C^k(M;X) := \mathrm{Maps}(\mathcal{X}_k, \mathbb{C})$
- $\delta^*_{\mathcal{O},\omega}(z) : C^*(M;X) \to C^{*+1}(M;X))$

$$\boxed{(\delta^k_{\mathcal{O},\omega}(z)(f))(u) := \sum_{v \in \mathcal{X}} I^{\mathcal{O},\omega}_{u,v}(z) f(v)}$$

This is a holomorphic family of cochain complexes with a base.

• If ω_1 and ω_2 represent $\xi \in H^1(M;\mathbb{R})$, $\mathcal{O}_1$ and $\mathcal{O}_2$ are two sets of orientations, there exists a canonical isomorphism between

$$(C^*(M;X), \delta^*_{\mathcal{O}_1,\omega_1}(z)) \text{ and } (C^*(M;X), \delta^*_{\mathcal{O}_2,\omega_2}(z)).$$

$$\Rightarrow \quad \boxed{\mathbb{C}^*(M;X,\xi)(z)}$$

• The cohomology changes in dimension at finitely many z.

Riemannian geometry

- M closed manifold
- X smooth vector field satisfying **G**
- ω Lyapunov closed one form for X

There exists g Riemannian metric so that $X = -\mathrm{grad}_g \omega$

Introduce Witten deformation $(\Omega^* M, d^\omega(z) = d + z\omega \wedge \cdots)$ equipped with the scalar products on $\Omega(M)$ defined by g. When z is a real number It Induces Laplacians

$$\boxed{\Delta^\omega_q(z) : \Omega^*(M) \to \Omega^*(M) = \Delta_q + z(L_X + L^*_X) + z^2 |\mathrm{grad}_g \omega|^2}$$

One can show that for $\Re z > \rho$

- Exactly n_k eigenvalues of $\Delta_k^\omega(z)$ go to zero exponentially fast while the remaining have real part go to ∞ linearly fast as $\Re z \to \infty$. As a consequence
- One has a canonical orthogonal decomposition

$$(\Omega^*(M); d^\omega(z)) := (\Omega^*(M)_{\text{sm}}(M), d^\omega(z)) \oplus (\Omega^*(M)_{\text{la}}(M), d^\omega(z))$$

which diagonalizes $\Delta_*^\omega(z) = \Delta_*^{\text{sm}}(z) \oplus \Delta_*^{\text{la}}(z)$.

Theorem

1. The above decomposition has an analytic continuation in the neighborhood of $[0, \infty)$.

If in addition X satisfies **EG** *then:*

2. For $\Re z > \rho$ *the Integration of differential forms on unstable manifolds is well defined and provides an isomorpfism from* $(\Omega^*(M)_{\mathrm{sm}}; d^\omega(z))$ *to* $(C^*(M; X), \delta^*_{\mathcal{O}_1, \omega_1}(z))$. *The isomorphism is canonical.*

3. For $\Re z > \rho$ *large the* $(\Omega^*(M)_{\mathrm{la}}; d^\omega(z))$ *is acyclic and*

$$\boxed{\log T_{la}(M, g, \omega)(z) - Z^{X,[\omega]}(z) = z\mathcal{R}(g, X, \omega)}$$

where $T_{la}(M, g, \omega)(z)$*is the complex Ray Singer torsion and* $\mathcal{R}(g, X, \omega)$ *a numerical invariant defined below.*

Other invariants

X vector field which satisfies **L** with n_k rest points of index k on a Riemannian manifold (M, g).

Consider

$$\Delta_q^X(z) : \Omega^*(M) \to \Omega^*(M) = \Delta_q + z(L_X + L_X^*) + z^2||X||^2$$

Results of Kato and Shubin $\Rightarrow$
For any $x \in \mathcal{X}_k$ *one can associate a the function* $\lambda(z) \in \mathrm{Spec}\,\acute{}_{\mathrm{k}}^{(}\mathrm{z})$, *holomorphic in a neighborhood of* $[0, \infty)$ *with* $\lim_{\Re z \to \infty} \lambda_x(z) = 0$.

Consider $x \rightsquigarrow \lambda_x(0)$ and

$$P_k(z) = \prod_{x \in \mathcal{X}_k} (z - \lambda_x(0)).$$

Theorem

For any k there exists a collection $\lambda_n^k(z)$, $n \in N$,*each holomorphic function in a neighborhood of* $[0, \infty)$ *so that for each* $t \in [0, \infty)$ *the family* $\{\lambda_r^k(t), r = 1, 2, \cdots$ *represent the entire family of eigenvalues of* $\Delta_k^X(t)$ *(with their multiplicity)*

Mild improvement of Witten methods shows that each rest point x of index k provides for t very large an eigenvalue which go exponentially fast to zero. In fact much more is true

Twisted cohomology

- Is a graded vector space $\boxed{H^*(M;\xi)}$ associated to $(M, \xi \in H^1(M;\mathbb{C})$.
  Denote $\beta_i^{\xi} := \dim H^i(M;\xi)$.

- For $\Re z$ large enough $\beta_i^{z\xi}$ is constant in z.
  Denote $\underline{\beta}_i^{\xi} := \lim_{\Re z \to \infty} \beta_i^{z\xi}$.

Theorem

(Novikov)
If X satisfies **H, MS, L** *with ξ a Lyapunov class for X then:*

1. For $\Re z$ large enough the cohomology of the cochain complex $\mathbb{C}^*(M; X, \xi)(z)$ and $H^*(M; z\xi)$ are naturally isomorphic.

2. One has

$$n_k \geq \underline{\beta}_k^{\xi}$$

$$\sum_{0 \leq k \leq r} (-1)^k n_k \geq \sum_{0 \leq k \leq r} \underline{\beta}_k^{\xi}, \ r \text{ even}$$

$$\sum_{0 \leq k \leq r} (-1)^k n_k \geq \sum_{0 \leq k \leq r} \underline{\beta}_k^{\xi}, \ r \text{ odd.}$$

Torsion

• The cochain complex $\mathbb{C}^*(M; X, \xi)(z)$ is equipped with a base so it is possible to define a torsion (in case that the $z\xi$- twisted cohomology for is trivial is a holomorphic function in z.)

• The cohomology class $z\xi$can be interpreted as a rank 1-complex representation of the fundamental group and together with a so called Euler structure there is a topological invariant the Reidemeister torsion (in case that the $z\xi$- twisted cohomology for is trivial is a holomorphic function in z.)

• The vector field X and some minor additional data determine an Euler structure.

It is possible to show that the sum of the log of the torsion of $\mathbb{C}^*(M; X, \xi)(z)$ and of the function $Z^\xi(z)$ is exactly the Reidemeister torsion of M $z\xi$ and the Euler class defined by X. This relates the functions $I_{x,y}^{\mathcal{O},\omega}(z)$ and $Z^\xi(z)$ and the topology

• The torsion $logT \in \mathbb{C}$ associated with a cochain complex (C^*, d^*) with a nondegerate bilinear form each C^i, (hence with Laplacians $\Delta_k : C^k \to C^k$,) is defined by the formula

$$\log T = 1/2 \sum_{i \geq 0} (-1)^i i \log \det \Delta_i$$

• $\mathcal{R}(g, X, \omega) \in \mathbb{R}$ is an invariant associated with a Riemannian manifold (M, g), a vector field X with isolated rest points and a closed one form ω defined as follows:

For an oriented Riemannian manifold M, the Riemannian metric produces a $(n-1)$ differential form (the angular momentum form)

$$\Psi(g) \in \Omega^{n-1}(TM \setminus M)$$

If the vector field X has no rest points it defines the map $X : M \to T(M) \setminus M$ and

$$\mathcal{R}(g, X, \omega) := \int_M \omega \wedge X^*(\Psi(g))$$

When the vector field X has singularities the integration is done over $M \setminus \mathcal{X}$ and the integral might not be convergent and requires regularization. This regularization is always possible. Geometric regularization .

Since the eigenvalues of the Laplacians Δ_i^{la} are infinitely many the definition of $\det \Delta_i^{la}$ also requires regularization. This regularization is again possible.
Zeta function regularization.