Traditionally topology uses finite dimensional linear algebra over $\mathbb{R}$ or $\mathbb{C}$ or its analytic version, elliptic operators, to provide numerical invariants for triangulated (compact) spaces
or for (compact) smooth manifolds.

In this lecture I will disscus the use of a sligtly more sofisticated linear algebra (linear algebra á la von Newmann associated with a finitely presented group Γ) and the corresponding elliptic theory to provide invariants for compact manifolds with fundamental group Γ. I will illustrate my discussion with an invariant of particular interest which in my view might have a fate as exciting as of the Novikov higher order signatures.

In the case that Γ is infinite, the linear algebra á la von Newmann lead to:
1) new invariants which can not be recovered from traditional ones
2)analogues of traditional invariants but with different geometric significance
3)to new mathematics and mathematical relationships between topology and other fields.

The use of linear algebra á la von Newmann in topology and geometry was pioneered by :
Atiyah-Singer, Novikov-Shubin, Cheeger Gromov.
These days there are many researchers doing exciting work. The invariant I will discuss is:

L_2-TORSION

The invariant

1: Is a scalar product in a one dimensional real vector space associated with the compact manifold M^n.

$$M \to |\det \mathbb{H}^*_{sing}(M)|$$

2: If M^{2n+1} is a closed manifold which admits a hyperbolic metric g, the above vector space is cannonically isomorphic to $\mathbb{R}$, and the invariant became a positive real number,

$$= \exp((-1)^n C(n) \cdot \mathrm{Vol}_g(M))$$

$C(n) \neq 0$ (T Schick)
$C(1) = 1/6\pi, C(2) = 62/(90\pi^2) \cdots$ (J.Lott)

3: Is defined for a large class of manifolds
(Conj.1: for all manifolds)

4:Is a HOMOTOPY INVARIANT for a large class of manifolds (Conj.2: for all manifolds)

5:Defined analytically (with a RIEMANNIAN METRIC) and combinatorially (with a TRIANGULATION).

In the first case one uses

LINEAR ALGEBRA Á LA VON NEWMANN

of $\Gamma-$Hilbert modules of finite type.

In the second case uses the theory of

PDO's IN BUNDLES OF $\Gamma-$HILBERT

MODULES OF FINITE TYPE.

CONTENTS

CLASSICAL LINEAR ALGEBRA $k = \mathbb{R}, \mathbb{C}$

OBJECTS: (f.d) Vector spaces,$V_1, V_2, \cdots$

and f.d. Hilbert spaces $(V, \langle , \rangle)$

MORPHISMS: $\alpha : V_1 \to V_2$ linear map

NUMERICAL INVARIANTS:

$\dim V \in \mathbb{Z}_+$ $\mathrm{tr}(\alpha : V \to V) \in k$,

$\log \mathrm{Vol}(\alpha : (V_1, \langle , \rangle_1) \to (V_2, \langle , \rangle_2)) \in \mathbb{R}$

SPECTRAL DENSITY FUNCTION $F(\alpha)(\lambda)$, $\lambda \in \mathbb{R}_+$
$\alpha : (V_1, \langle , \rangle_1) \to (V_2, \langle , \rangle_2)$

REMARKS:

(i)$\dim V_1 = \dim V_2 \quad \Rightarrow V_1 \cong V_2$

(ii)$\alpha : V_1 \to V_2$ injective and $\overline{\alpha(V_1)} = V_2 \Rightarrow$ α invertible.

ELEMENTARY DEFINITIONS

F. O. FREE Γ-SETS :

$$\mu : \Gamma \times X \to X$$

free action with finitely many orbits.

REGULAR TYPE REPRESENTATION:

$\tilde{\mu} : \Gamma \to U(L_2(X))$

induced by an f.o. free $\Gamma-$ setμ.

$\mu_i : \Gamma \times X_i \to X_i,\ i = 1, 2$

KERNEL:

$$\underline{\alpha} : X_1 \times X_2 \to \mathbb{C},\ \ \underline{\alpha}(g \cdot x_1, g \cdot x_2) = \underline{\alpha}(x_1, x_2)$$

$\Rightarrow \qquad \alpha : L_2(X_1) \to L_2(X_2)$

$\Gamma-$equivariant and bounded.

Any $\alpha : L_2(X_1) \to L_2(X_2)$ is induced by an $\underline{\alpha}$.

If $X_1 = X_2$, $\ \mathrm{tr}_\Gamma(\alpha) := \sum_{\hat{x} \in X/\Gamma} \alpha(i(\hat{x}), i(\hat{x}))$.

$i : X/\Gamma \to X$ section. $\pi \cdot i = id$

LINEAR ALGEBRA Á LA VON NEUMANN

(Γ countable)

OBJECTS:

$\Gamma-$ Hilbertian modules of finite type $(\mathbb{W}_1, \mathbb{W}_2 \cdots)$

$\Gamma-$ Hilbert modules of finite type$(\mathcal{W}_1, \mathcal{W}_2, \cdots)$,

$\mathcal{W} = (\mathbb{W}, \langle, \rangle)$

DEF.1.: A $\Gamma-$Hilbert module of finite type is a unitary representation isometric to a subrepresentation of a regular representation

DEF.2.: A $\Gamma-$Hilbertian module of finite type is a linear representation of Γ on a topological vector space which is isomorphic to a $\Gamma-$ Hilbert module of finite type.

MORPHISMS: $\alpha : \mathbb{W}_1 \to \mathbb{W}_2 \therefore$ continuous $\Gamma-$ equivariant

linear maps.

NUMERICAL INVARIANTS:

$\mathrm{tr}(\alpha : \mathbb{W} \to \mathbb{W}) \in \mathbb{C}$, $\dim(\mathbb{W}) := \mathrm{tr}(id_{\mathbb{W}}) \in \mathbb{R}_+$

SPECTRAL DENSITY FUNCTION for $\alpha : \mathcal{W}_1 \to \mathcal{W}_2$,

$F(\alpha)(\lambda) := \sup\{\dim \mathcal{N} / \ker \alpha | \mathcal{N} \subset \mathcal{W}_1, ||\alpha|_{\mathcal{N}}|| \leq \lambda, \lambda \in \mathbb{R}_+$

$F(\alpha)(\lambda)$ is nondecreasing + continous from the right

DETERMINANT CLASS:

DEF.3: A morphism $\alpha : \mathcal{W}_1 \to \mathcal{W}_2$, is of determinant class

iff $\int_0^1 \log \lambda dF(\alpha)(\lambda) \geq -\infty$

α of determinant class $\Rightarrow \log \mathrm{Vol}(\alpha) := \int_0^\infty \log \lambda dF(\alpha)(\lambda)$

REMARKS:

(i) $\dim \mathbb{W}_1 = \dim \mathbb{W}_2 \ \Rightarrow$ (when Center $\Gamma = 0$) $\mathbb{W}_1 \cong \mathbb{W}_2$

(ii)$\alpha : \mathcal{W}_1 \to \mathcal{W}_2$ injective with dense image is invertible

iff the germ of $F(\alpha)(\lambda) = 0$

(iii) (Farber, Lück) The category of $\Gamma-$Hilbert(ian)
modules of finite type is equivalent to the category
of finitely generated projective modules over $\mathcal{N}(\Gamma)$.

DETERMINANT LINE OF $\mathbb{W}$ (Carey, Farber, Mathai)

$|det\mathbb{W}| := \mathbb{R}\{\langle,\rangle | \text{s.t} \mathcal{W} = (\mathbb{W}, \langle,\rangle)\Gamma - \text{Hilbert module}\}// \sim$

with $\langle,\rangle_1 \sim a\langle,\rangle_2$ if there exists $\alpha : \mathcal{W} \to \mathcal{W}$, selfadjoint, positive,

injective with dense image of determinant class s.t

$\langle\alpha(x), \alpha(y)\rangle_2 = \langle x, y\rangle_1$ and $\log \mathrm{Vol}(\alpha) = -\log a$

Hilbert module structure on $\mathbb{W} \Rightarrow$

element in $|det\mathbb{W}| \Rightarrow$ a scalar product on $|det\mathbb{W}|$

PDO's IN BUNDLES OF $\Gamma-$HILBERTIAN MODULES

$\pi : E \to M, \pi :\ F \to M$ bundles of $\Gamma-$Hilbertian modules.

Pseudodifferntial operator $A : C^\infty(E) \to C^\infty(F)$

if $\cdots$

They have principal symbol $\sigma(A) \in C^\infty(\pi^* Hom_\Gamma(E, F)$

(1)Smoothing operators $\Rightarrow$ smooth kernel $\Rightarrow$ $\mathrm{tr}(A) \in \mathbb{C}$

(2)Elliptic operators $\Rightarrow$ $F(A)(\lambda)$ defined.

DEF.4 : The elliptic operator A is of determinant class
iff $\int_0^1 \log(\lambda) dF(A)(\lambda) > -\infty$.

Elliptic of determinant class $\Rightarrow$ $\log \mathrm{Vol} A \in \mathbb{R}$

defined using $\zeta-$regularization (Singer Ray).

II.

L_2- SINGULAR COHOMOLOGY

X compact ANR with $\pi_1(X) = \Gamma$

$\mathbb{H}^n_{sing}(X)$ a $\Gamma-$Hilbertian module of finite type

$|det \mathbb{H}^*_{sing}(X)| := \otimes |\det \mathbb{H}^n_{sing}(X)|^{\epsilon(n)}$,

$|\cdots|^{\epsilon(n)} = |V|$ if n even, $= |\cdots|^*$ if n odd.

L_2- COMBINATORIAL COHOMOLOGY

(X,τ) triangulation of X $\Rightarrow$ $(\tilde{X},\tilde{\tau})$ triangulation of $\tilde{X}$ $\Rightarrow$

$$\tilde{\mathcal{X}}_k \equiv \{\text{k-cells}\tilde{\tau}\} \quad \text{f.o.}\Gamma-\text{set}$$

$$I_k : \tilde{\mathcal{X}}_k \times \tilde{\mathcal{X}}_{k+1} \to \mathbb{Z}, \quad I(gx_k, gx_{k+1}) = I(x_k, x_{k+1})$$

$(\mathbb{C}^*(X,\tau), \partial_*)$ and $\partial_*^{\#}$ complex of $\Gamma-$Hilbert modules

$$\mathbb{C}^k(X,\tau) \xrightarrow{\partial_k} \mathbb{C}^{k+1}(X,\tau) \quad \mathbb{C}^k(X,\tau) \underset{\partial^{\#}_{k+1}}{\longleftarrow} \mathbb{C}^{k+1}(X,\tau)$$

$$\mathbb{C}^k(X,\tau) := L_2(\mathcal{X}_k) \quad \Gamma-\text{Hilbert module}$$

$$\Delta^k_\tau := \partial^{\#}_{k+1}\partial_k + \partial_{k-1}\partial^{\#}_k \Rightarrow$$

$\mathcal{H}^n_\tau(X) := \ker \partial_k / \overline{\Im \partial_{k-1}} \cong \ker \Delta^k_\tau$ $\quad \Gamma-$ Hilbert module

Canonical isomorphism $\theta^n : \mathcal{H}^n_\tau(X) \to \mathbb{H}^n_{sing}(X)$ $\Rightarrow$

ω_τ scalar product in $|\det \mathbb{H}^n_{sing}(X)|$

DEF.5: (X,τ) of determinant class iff Δ_τ^k's are.

(X,τ) of determinant class $\Rightarrow$

$$T_\tau(X,\tau) := \exp(1/2 \sum_k (-1)^{k+1} k \log \mathrm{Vol}\Delta_\tau^k)$$

$\mathbb{T}_\tau(X,\tau) := T_\tau(X,\tau)\omega_\tau$ $\therefore$ scalar product in $|\det \mathbb{H}^n_{sing}(X)|$.

Theorem. *If Γ satisfies Conj.2' then $\mathbb{T}_\tau(X,\tau)$ is a homotopy invariant*

All residually ameanable groups satisfy Conj.3'.

L_2-DE RHAM COHOMOLOGY

M closed manifold, $\pi_1(M) = \Gamma$, $\tilde{M} \xrightarrow{\pi} M$ universal cover$\Rightarrow$

$\mathcal{M} \to M$ FLAT BUNDLE fibers $L_2(\pi^{-1}(m) \cong L_2(\Gamma)$

$\Omega^r(M;\mathcal{M}) = C^\infty(\Lambda^r(T^*M \otimes \mathcal{M}))$

$d_r : \Omega^r(M;\mathcal{M}) \to \Omega^{r+1}(M;\mathcal{M})$

$\mathbb{H}^n_{DR}(X) := \ker d_r / \overline{d(\Omega^{r-1}(M;\mathcal{M}))}$ $\Gamma-$Hilbertian module

(M, g) Riemannian manifold $\Rightarrow$

(1)scalar product in $\Omega^r(M;\mathcal{M})$

(2)formal adjoint operators $d^\#_{r+1} : \Omega^{r+1}(M;\mathcal{M}) \to \Omega^r(M;\mathcal{M}) \Rightarrow$

$$\Delta^k_g := d^\#_{k+1} d_k + d_{k_1} d^\#_k \text{ elliptic}$$

$$\mathcal{H}^q_{DR}(M, g) := \ker \Delta^q_g \quad \Gamma-\text{Hilbert module}$$

Hodge theory and integration theory $\Rightarrow$ canonical isomorphism

$$\theta^q_g : \mathcal{H}^q_{DR}(M, g) \to \mathbb{H}^q_{DR}(M) \to \mathbb{H}^q_{sing}(M) \Rightarrow$$

ω_g scalar product in $|\det \mathbb{H}^n_{sing}(X)|$

DEF.6 : (M, g) is of determinant class if Δ_g^q's are

(M, g) of determinant class $\Rightarrow$

$$T_g(M) := \exp(1/2 \sum (-1)^{k+1} k \log \mathrm{Vol} \Delta_g^k)$$

$\mathbb{T}_g(M) = T_g(M)\omega_g$ scalar product in $|\det \mathbb{H}^*_{sing}(X)|$.

REMARK: $\mathbb{H}^*_{sing}(X) = 0, \Rightarrow |\det \mathbb{H}^*_{sing}(X)|$ cannonically isoomorphism to $\mathbb{R}$ $\rightarrow$

$\mathbb{T}(M)$ is a positive real number.

REMARKS:

(1)(M^{2n+1}, g), g hyperbolic $\Rightarrow \mathbb{H}^*(M)_{sing} = 0$

(2)(W. Lück)X compact ANR, $f : X \rightarrow X$ $\Rightarrow \mathbb{H}^*_{sing}(X_f) = 0$,
X_f the mapping torus of f.

CONJ.3(I. Singer)M^{2n+1} closed and aspherical $\Rightarrow \mathbb{H}^*_{sing}(M) = 0$

Conjecture true for ......

Theorem:. *1)* (M, τ) *of determinant class iff* (M, g) *is.*

2)Determinant class property is a homotopy invariant.

3)$\mathbb{T}_{an}(M, g) = \mathbb{T}_{comb}(M, \tau)$

1) Egorov, 2) Gromov-Shubin, 3) BFKM.

$\theta^q(g, \tau) = (\theta_g^q)^{-1} \cdot (\theta_\tau^q) : \mathcal{H}_\tau^q(M) \to \mathbb{H}_{sing}^q(M) \to \mathcal{H}_{DR}^q(M, g)$

The proof of Theorem reduces to

$\log T_{an}(M, g) = \log T_{comb}(M, \tau) + \sum (-1)^k \log \mathrm{Vol}(\theta_{g,\tau}^k)$

CONJ.2: All compact manifolds (ANR's)

are of determinant class.

CONJ.3: $\mathbb{T}(M)$ is a homotopy invariant

Both conjectures are true for Γ residually amenable.

Conj. 2, 3 true for:
(1) residually finite groups (Lück + BFKM),
(2) ameanable groups (Dodziuck -Mathai),
(3) residually ameanable groups (B.Clair).

Extension to larger class of groups T Schick.

V PROOF OF THE THEOREM (SKETCH)

$(M, g, \tau) \Rightarrow \tau$ given by (g, h) i.e.

$h : M \to \mathbb{R}$ Morse function s.t.

(1) the vector field $Y = -\mathrm{grad}_g h$ satisfies Morse Smale

(2) UNSTABLE MANIFOLDS of Y $\equiv$ OPEN SIMPLEXES of τ

$$\begin{array}{ccc}
(\Omega^*(M, \mathcal{M}), d_*) & \xrightarrow{\mathrm{Int}} & (\mathbb{C}^*(M, \tau), \partial_*) \\
\Big\uparrow e^{th} & & \Big\uparrow I+O(1/t) \\
(\Omega^*(M, \mathcal{M}), d_*(t)) & \xleftarrow[S(t)^*]{} & (\Omega^*(M, \mathcal{M}), \underline{d}_*(t))
\end{array}$$

$d_k(t) = e^{-th} \cdot d_k \cdot eth,$

$S^q(t) = (\pi/t)^{(n-2q)/4} e^{tq}$

$\Delta_g^k(t) := d^{\#}_{k+1}(t) d_k(t) + d_{k_1}(t) d^{\#}_k(t)$

STEP 1: WHS-THEORY:

Fact 1: There exists $T_0, C_1, C_2, C_3 > 0$ s.t

Spect $\Delta^q(t) \cap (C_1 e^{-tC_2}, C_3 t) = \emptyset,\ t \geq T_0, \Rightarrow$ (by elliptic theory)

$(\Omega^*(M, \mathcal{M}), d_*(t)) = (\Omega^*(M, \mathcal{M})_{sm}, d_*(t)) \oplus (\Omega^*(M, \mathcal{M})_{la}, d_*(t))$

Fact 2: $Int \cdot e^{th} \cdot S(t)|_{\Omega^*(M,\mathcal{M})_{sm}} = I + O(1/t)$

$(\Omega^*(M, \mathcal{M}), \underline{d}_*(t)) \xrightarrow{S(t)} (\Omega^*(M, \mathcal{M}), d_*(t)) \xrightarrow{\mathrm{Int}\cdot e^{th}} (\mathbb{C}^*(M, \tau), \partial_*)$

STEP 2: WHITTEN DEFORMATION:

$$\log T_g(M, g)(h, t) = 1/2 \sum (-1)^{k+1} k \log \mathrm{Vol}\Delta_g^k(t)$$

Prove that $\log T_g(M, h, t)$ is constant in t.

For $t >> 0$ $\log T_g(M, h, t) = \log T_{g,sm}(M, h, t) + \log T_{g,la}(M, h, t)$

STEP 3:

$\log T_{g,la}(M, h, t) - 1/2 \sum (-1)^{k+1} k \log \mathrm{Vol}(\Delta_g^k + 1) = O(1/t),\ t \to \infty$

STEP 4 (THE HARD PART):

$A(t) := 1/2 \sum (-1)^{k+1} k \log \mathrm{Vol}(\Delta_g^k + 1) =$

has an asymptotic expansion $t \to \infty$ $F.P.A(t) = 0$

Fact 2 in Step 1 $\Rightarrow$

$\log T_{g,sm}(M,h,t) = \log T_{comb}(M,\tau) + 1/2 \sum (-1)^k (\theta_{g,\tau}^k) + O(1/t)$

$\lim_{t\to\infty} \log T_{g,sm}(M,h,t) = \log T_{comb}(M,\tau) + 1/2 \sum (-1)^k (\theta_{g,\tau}^k)$

Step 2+ Step 3 +Step4 $\Rightarrow$

$\log T_g(M,g) = \log T_{g,la}(M,h,t=0) = \lim_{t\to\infty} \log T_{g,sm}(M,h,t)$

q.e.d

THE CONJECTURES

Let Γ be a finitely presented group.

$\mu : \Gamma \times X \to X$ a f.o.free $\Gamma-$set

$\underline{\alpha} : X \times X \to \mathbb{Z} \subset \mathbb{C}$ satisfying $\underline{\alpha}(g \cdot x_1, g \cdot x_2) = \underline{\alpha}(x_1, x_2)$

CONJ.2' : The induced morphism $\alpha : L_2(X_1) \to L_2(X_2)$

is of determinant class.

CONJ.3' :If the induced map $\tilde{\alpha} : \mathbb{Z}[X] \to \mathbb{Z}[X]$ is an

isomorphism of $\mathbb{Z}[\Gamma]$ modules then

$$\log \mathrm{Vol}(\alpha) = 0$$

Extensions and future research:

$\mathcal{M} \to M$ is the flat bundle associated to the regular

reprezentation of Γ on the Γ Hilbert module $L_2(\Gamma)$.

(a) The regular reprezentation can be replaced by any unitary representation of Γ on an $\mathcal{A}-$Hilbert module, where $\mathcal{A}$ is a finite von Neumann algebra.
The resulting invariant is only a topological invariant.

(b) Such unitary representation can be replaced by an arbitrary representation in which case one has to choose a Hermitian structure in the flat bundle associated with the representation and a Riemannian metric (or a triangulation). The result depends on these choices; this dependance can be removed at the expence of a base point in M and an Euler structure on M.
The result is a topological invariant.

(c)At least in the case $\mathbb{H}^*_{sing}(M)$ the above invariants can be improved to complex numbers whose absolute value is the L_2- torsion defined above.