REFINMENTS OF HOMOLOGY (homological spectral package).

Dan Burghelea

Department of Mathematics
Ohio State University, Columbus, OH

In analogy with linear algebra over $\mathbb{C}$ we propose

REFINEMENTS

of the simplest topological invariants

HOMOLOGY of X

NOVIKOV HOMOLOY / L_2-HOMOLOGY of $(X, \xi \in H^1(X, \mathbb{Z}))$,

associated to a continuous

REAL or ANGLE VALUED MAP

This work is :

• implicit in joint work with **S.Haller** (Vienna)

• influenced by joint work with **Tamal Dey** (OSU- Columbus) and **Du Dong** (Shanghai)

HOMOLOGY for a fixed field

Homology :

Space X :

$$\therefore \ (1.) \ \boxed{H_r(X), \quad \beta_r(X) = \dim H_r(X)}$$

Novikov homology :

Pair $(X; \xi \in H^1(X; \mathbb{Z}))$:

$$\therefore \ (2.) \ \boxed{H_r^N(X, \xi), \ \beta_r^N(X, \xi) = \text{rank } H_r^N(X, \xi)}$$

- $(X, \xi) \Rightarrow \tilde{X} \to X$ infinite cyclic cover associated to ξ.
- X compact ANR $\Rightarrow H_r(\tilde{X})$ a f.g. $\kappa[t^{-1}, t]$ module
- X compact ANR $\Rightarrow TH_r(\tilde{X})$ a $\kappa[t^{-1}, t]-$module which is a f.d. $\kappa-$vector space.
- X compact ANR $\Rightarrow \boxed{H^N(X; \xi) := H_r(\tilde{X}) / TH_r(\tilde{X})}$ a f.g. free $\kappa[t^{-1}, t]$ module

L_2-**homology**

If $\kappa = \mathbb{C}$ by "completion"

- $\mathbb{C}[t^{-1}, t] \rightsquigarrow L^\infty(\mathbb{S}^1)$ a von Neumann algebra.
- $H_r^N(X : \xi) \rightsquigarrow H_r^{L_2}(\tilde{X})$ finite type $L^\infty(\mathbb{S}^1)-$Hilbert module

and $\beta^N(X;\xi) = \dim_{vn}(H_r^{L_2}(\tilde{X}))$.

New invariants - configurations

Configurations of points in Y with multiplicities in $\mathbb{N}$

$$\mathcal{C}onf_n(Y) := \{\delta : Y \to \mathbb{N}_0 \mid \sharp(\mathrm{s}upp\ \delta) < \infty\} = Y^n/\Sigma_n$$

$$\mathcal{C}onf_n(Y) := \{\delta : Y \to \mathbb{N}_0 \mid \sum_y \delta(y) = n\} = Y^n/\Sigma_n$$

Configurations of points in Y indexing subspaces of V

$$\mathcal{C}onf_{vect}(Y) := \{\hat{\delta} : Y \to \mathcal{V}ect \mid \sharp(\mathrm{s}upp\ \hat{\delta}) < \infty\}$$

$$\mathcal{C}onf_V(Y) := \{\hat{\delta} : Y \to \mathcal{S}(V) \mid \bigoplus_y \hat{\delta}(y) = V\}$$

- $Y = \mathbb{C} \Rightarrow \mathcal{C}onf_n(Y) = \mathbb{C}^n$ and identifies with degree $n-$ monic polynomials.
- $Y = \mathbb{C} \setminus 0 \Rightarrow \mathcal{C}onf_n(Y) = \mathbb{C}^{n-1} \times (\mathbb{C} \setminus 0)$ and identifies with degree $n-$ monic polynomials with nonzero free coefficient.

SYSTHEM $(V, T : V \to V)$

$$\begin{cases} V \text{ f.d. complex vector space.} \\ T : V \to V \text{ linear map} \end{cases}$$

$$\Longrightarrow$$

SPECTRAL PACKAGE

$$\begin{cases} \dim V = n & \in \mathbb{N} \\ z_1, z_2, \cdots, z_{k-1}, z_k & \in \mathbb{C}; \text{ eigenvalues} \\ n_1, n_2, \cdots, n_{k-1}, n_k & \subseteq \mathbb{N}; \text{ multiplicities} \\ V_1, V_2, \cdots, V_{k-1}, V_k & \subseteq V; \text{ generalized eigenspaces} \end{cases}$$

PROPERTIES : $\dim V = \sum n_i, \ \dim V_i = n_i, \ V = \bigoplus V_i$

geometrization

$$\boxed{\delta^T :\equiv \begin{cases} z_1, z_2, \cdots, z_{k-1}, z_k \\ n_1, n_2, \cdots, n_{k-1}, n_k \end{cases}} \Rightarrow \begin{cases} \text{finite configurations of} \\ \text{points with multiplicities} \\ \text{s.t.} \sum_i \delta^T(z_i) = \dim V \end{cases}$$

$\delta^T \equiv P^T(z) = (z - z_1)^{n_1}(z - z_2)^{n_2} \cdots (z - z_k)^{n_k}$
the *characteristic polynomial*, $n = \dim V$.

$$\boxed{\hat{\delta}^T :\equiv \begin{cases} z_1, z_2, \cdots, z_{k-1}, z_k \\ V_1, V_2, \cdots, V_{k-1}, V_k \end{cases}} \Rightarrow \begin{cases} \text{finite configurations of} \\ \text{points indexinig "disjoint"} \\ \text{subspaces of V} \\ \text{s.t. } \oplus_i \hat{\delta}^T(z_i) = V \end{cases}$$

$$\delta^T \in \mathcal{C}onf_n(\mathbb{C}), \ \hat{\delta}^T \in \mathcal{C}onf_V(\mathbb{C})$$

basic properties

1. (Stability) $L(V,V) \ni T \rightsquigarrow \delta^T(z) = P^T(z) \in \mathbb{C}^n$ is continuous
2. (Duality) $\delta^T = \delta^{T^*}$
3. For an open and dense set of $T \in L(V,V)$, $\delta^T(z) = 0$ or 1
4. (Computability) $P^T(z)$ can be calculated with arbitrary accuracy.

One regards $\delta^T \equiv P^T(z)$ as a *refinement* of dim V,

One regards $\hat{\delta}^T$ as an *implementation* of the refinement δ^T.

TOPOLOGY ; HOMOLOGICAL SPECTRAL PACKAGE (real valued map)

SYSTHEM $(X, f : X \to \mathbb{R})$

$$\begin{cases} X \text{ a compact ANR}, \\ f \text{ continuous}, \\ \kappa \text{ a field}, \mathrm{H_r(X) := H_r(X;} \kappa), \\ r \in \mathbb{N}_0. \end{cases}$$

HOMOLOGICAL SPECTRAL PACKAGE

$$\begin{cases} \dim H_r(X) = \beta_r(X) \; ; \; \text{ Betti number} \\ z_1, z_2, \cdots, z_{k-1}, z_k \in \mathbb{C}; \text{barcodes} \\ n_1, n_2, \cdots, n_{k-1}, n_k \in \mathbb{N}; \text{multiplicities} \\ V_1, V_2, \cdots, V_{k-1}, V_k; \text{quotients of subspaces of V.} \end{cases}$$

$\beta_r = \sum_i n_i, \; \dim V_i = n_i, V \simeq \oplus V_i$

- $V_i = L_i / L'_i$ are quotients of subspaces of $H_r(X)$, $L'_i \subset L_i \subseteq H_r(X)$, *essentially disjoint*
- If $H_r(X)$ has an inner product then V_i is *canonically* realizable as a subspace of $H_r(X)$ with $V_i \perp V_j$, $i \neq j$.

geometrization

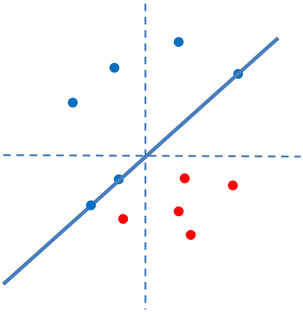
$$\boxed{\delta^f_r :\equiv \begin{cases} z_1, z_2, \cdots, & z_{k-1}, z_k \\ n_1, n_2, \cdots, & n_{k-1}, n_k \end{cases}} \Rightarrow \begin{cases} \text{finite configurations of} \\ \text{points with multiplicities} \\ \text{s.t. } \sum \delta^f_r(z_i) = \beta_r(X) \end{cases}$$

$$\delta^f_r \equiv P^f_r(z) = (z - z_1)^{n_1}(z - z_2)^{n_2} \cdots (z - z_k)^{n_k}$$

the *homological characteristic polynomial*.

$$\boxed{\hat\delta^f_r :\equiv \begin{cases} z_1, z_2, \cdots, & z_{k-1}, z_k \\ V_1, V_2, \cdots, & V_{k-1}, V_k \end{cases}} \Rightarrow \begin{cases} \text{finite configurations of points} \\ \text{indexing subspaces of } H_r(X) \\ \text{s.t. } \oplus \hat\delta^f_r(z_i) = H_r(X) \end{cases}$$

$$\delta^f_r \in Conf_{\beta_r(X)}(\mathbb{C}),\ \hat\delta^f_r \in Conf_{H_r(X)}(\mathbb{C})$$



TOPOLOGY ; HOMOLOGICAL SPECTRAL PACKAGE (angle valued map)

SYSTHEM $(X, f : X \to \mathbb{S}^1)$

$(X, f : X \to \mathbb{S}^1) \to \xi_f \in H^1(X; \mathbb{Z})$

$$\begin{cases} X \text{ a compact ANR,} \\ f \text{ continuous,} \\ \kappa \text{ a field, } H^N_r(X;\xi_f) := H_r(\tilde{X})/TH_r(\tilde{X}) \\ r \in \mathbb{N}_0. \end{cases}$$

$\tilde{f} : \tilde{X} \to \mathbb{R}$ denotes the infinite cyclic cover of f

HOMOLOGICAL SPECTRAL PACKAGE

$$\begin{cases} \mathrm{rank}\ \mathrm{H}^{\mathrm{N}}_{\mathrm{r}}(\mathrm{X};\xi_{\mathrm{f}}) = \beta^{\mathrm{N}}_{\mathrm{r}}(\mathrm{X};\xi_{\mathrm{f}})\ ;\quad \mathrm{Novikov-Betti\ number} \\ z_1, z_2, \cdots, z_{k-1}, z_k \in \mathbb{C}\setminus 0; \mathrm{exponentiated\ barcodes} \\ n_1, n_2, \cdots, n_{k-1}, n_k \in \mathbb{N}; \mathrm{multiplicities} \\ V_1, V_2, \cdots, V_{k-1}, V_k; \mathrm{free}\ \kappa[\mathrm{t}^{-1},\mathrm{t}] - \mathrm{modules.} \end{cases}$$

$V_i = L_i/L'_i$ quotients of free $\kappa[t^{-1}, t]-$submodules of $H^N_r(X;\xi_f)$, $L'_i \subset L_i \subseteq H_r(X)$, *essentially disjoint*

$\beta^N_r = \sum_i n_i,\ \mathrm{rank}\ \mathrm{V_i} = \mathrm{n_i}, \mathrm{H^N_r}(\mathrm{X};\xi_\mathrm{f}) \simeq \oplus \mathrm{V_i}$

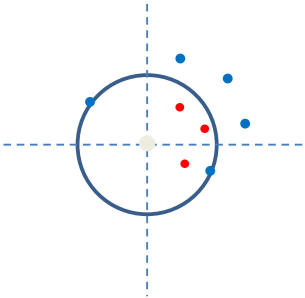
If $\kappa = \mathbb{C}$ and a $\mathbb{C}[t^{-1}, t]-$inner product by *completion* $H_r^N(X;\xi)$ is replaced by the Hilbert module $H_r^{L_2}(\tilde{X})$ and the configuration of free $\mathbb{C}[t^{-1}, t]$ by configurations of mutually orthogonal closed Hilbert $L^\infty(\mathbb{S}^1)-$submodules.

$$\boxed{\delta^f_r :\equiv \begin{cases} z_1, z_2, \cdots, & z_{k-1}, z_k \\ n_1, n_2, \cdots, & n_{k-1}, n_k \end{cases}} \Rightarrow \begin{cases} \text{finite configurations of} \\ \text{points with multiplicities in} \\ \mathbb{C}\setminus 0 \text{ s.t. } \sum \delta^f_r(z_i) = \beta^N_r(X;\xi) \end{cases}$$

$\delta^f_r \equiv P^f_r(z) = (z - z_1)^{n_1}(z - z_2)^{n_2} \cdots (z - z_k)^{n_k}$
the *homological characteristic polynomial* of degree $\beta^N_r(X;\xi)$.

$$\boxed{\hat\delta^f_r :\equiv \begin{cases} z_1, z_2, \cdots, & z_{k-1}, z_k \\ V_1, V_2, \cdots, & V_{k-1}, V_k \end{cases}} \Rightarrow \begin{cases} \text{finite configurations of points} \\ \text{indexing submodules of} \\ H^N_r(X;\xi) \text{s.t.} \\ \oplus\hat\delta^f_r(z_i) = H^N_r(X;\xi) \end{cases}$$

$$\delta^f_r \in \mathcal{C}onf_{\beta^N_r(X;\xi_f)}(\mathbb{C}\setminus 0),\ \hat\delta^f_r \in \mathcal{C}onf_{H^N_r(X;\xi_f)}(\mathbb{C}\setminus 0)$$



Definitions of δ^f_r and $\hat{\delta}^f_r$

For $f : X \to \mathbb{R}$ a proper map
$a, b \in \mathbb{R}$, κ a field

Denote:

$X_a := f^{-1}((-\infty, a])$ sub-level
$X^b := f^{-1}([b, \infty))$ over-level
Define:

- $\mathbb{I}_a(r) := \mathrm{i}mg(H_r(X_a) \to H_r(X))$
- $\mathbb{I}^b(r) := \mathrm{i}mg(H_r(X^b) \to H_r(X))$
- $\mathbb{F}_r(a, b) := \mathbb{I}_a(r) \cap \mathbb{I}^b(r)$

- Observe
  $a \leq a', b' \leq b$ imply $\mathbb{F}_r(a', b') \subseteq \mathbb{F}_r(a, b)$.
- Prove
  $\mathbb{F}_r(a, b)$ finite dimensional.
- Define $\boxed{\mathbb{F}_r(a, b, \epsilon) := \mathbb{F}_r(a, b)/\mathbb{F}_r(a-\epsilon, b) + \mathbb{F}_r(a, b+\epsilon)}$ for $\epsilon > 0$
- Observe that $\epsilon' > \epsilon''$ induces a surjective map

$$\boxed{\mathbb{F}_r(a, b; \epsilon') \to \mathbb{F}_r(a, b; \epsilon'').}$$

Definition

$$\hat{d}^f_r(a, b) := \varinjlim_{\epsilon \to 0} \mathbb{F}_r(a, b, \epsilon)$$

$$d^f_r(a, b) := \dim \hat{d}^f_r(a, b)$$

The case of $f : X \to \mathbb{R}$, X compact.

Define

$$\delta^f_r(z) = d^f_r(a, b), z = a + ib$$

$$\hat{\delta}^f_r(z) = \hat{d}^f_r(a, b), z = a + ib.$$

The case of $f : X \to \mathbb{S}^1$, X compact.

- Consider $\tilde{f} : \tilde{X} \to \mathbb{R}$ an infinite cyclic cover of f.
- Observe that
  1. $t : \hat{d}^{\tilde{f}}_r(a, b) \to \hat{d}^{\tilde{f}}_r(a + 2\pi, b + 2\pi)$ is an isomorphism and
  2. $$d^f_r(a, b) = d^{\tilde{f}}_r(a + 2\pi, b + 2\pi)$$
- Define

$$\boxed{\delta^f_r(z) = d^{\tilde{f}}_r(a, b)}, z = e^{(b-a)+ia}$$

$$\boxed{\hat{\delta}^f_r(z) = \bigoplus_k \hat{d}^{\tilde{f}}_r(a + 2\pi k, b + 2\pi k)}, z = e^{(b-a)+ia}.$$

Main results

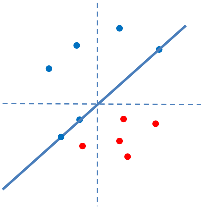
The values $t \in \mathbb{R}$ is CRITICAL if the homology of the levels of f changes at t. $Cr(f) =$ the set of critical values of f.

Theorem

1. $\sharp \mathrm{supp} \delta^f_r \leq \beta_r(X)$, $\sum_{z \in \mathrm{supp} \delta^f_r} \delta^f_r(z) = \beta^f_r$
2. *For an open sense set of maps* f, $\delta^f_r(z) = 0$ *or* $\delta^f_r(z) = 1$.

Proposition

1. *If* $z = a + ib \in \mathrm{supp}\delta^f_r$ *then both* $a, b \in Cr(f)$.
2. $z = (a + ib) \in \mathrm{supp}\delta^f_r$ *above or on diagonal implies* $[a.b]$ *is a closed* $r-$*bar code.*
3. $z = (a + ib) \in \mathrm{supp}\delta^f_r$ *below diagonal implies* (b, a) *is an open* $(r - 1)-$*bar code.*



Theorem

The assignment $C(X,\mathbb{R}) \ni f \rightsquigarrow \delta^f_r \in \mathbb{C}^{\beta_r}$ *is continuous.*

Theorem

If M^n *is a closed* $\kappa-$ *orientable topological* $n-$*dimensional manifold then*

$$\delta^f_r(z) = \delta^{-f}_{n-r}(-i\overline{z})$$

The same remains true for the configuration $\hat{\delta}^f_r$

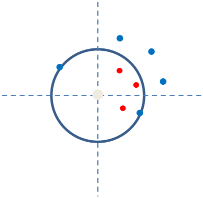
Suppose f is an angle valued map

Theorem

1. $\sharp \mathrm{supp}\delta^f_r \leq \beta^N_r(X)$, $\sum_{z\in \mathrm{supp}\delta^f_r} \delta^f_r(z) = \beta^N_r(X;\xi)$
2. *For an open sense set of maps f*, $\delta^f_r(z) = 0$ *or* $\delta^f_r(z) = 1$.

Proposition

1. *If* $z = e^{ia+(b-a)} \in \mathrm{supp}\delta^f_r$ *then both* $a, b \in Cr(\tilde{f}) = \{t \mid e^{it} \in Cr(f)\}$.
2. z *outside or on the unit circle implies* $[a, b]$ *is a closed* $r-$*bar code.*
3. z *inside the unit circle implies* (b, a) *is an open* $(r-1)-$*bar code.*



Theorem

The assignment $C(X, \mathbb{S}^1) \ni f \rightsquigarrow \delta^f_r \in \mathbb{C}^{\beta^N_r - 1} \times \mathbb{C} \setminus 0$ *is continuous*

Theorem

If M^n *is a closed* $\kappa-$ *orientable topological* $n-$*dimensional manifold then*

$$\delta^f_r(z) = \delta^{-f}_{n-r}(-i\overline{z})$$

The same remains true for the configuration $\hat{\delta}^f_r$

The proofs are done in two steps.

Step 1: Establish the results for X a finite simplicial complex and f a simplicial map

Step 2. Use results about the topology of Hilbert cube manifolds and show that the statements are true for f iff are true for $\overline{f}_Q = f \cdot p_X, p_X : X \times Q \to X$, Q the Hilbert cube I^∞.

Step 1

- Introduce the concepts:
  1. *tame map*,
  2. *homological critical value* and for a tame map,
  3. the number $\epsilon(f) = \inf |c - c'|$ c, c' different homological critical values
- Verify that for $f : X \to \mathbb{R}$ proper tame map
  1. For f proper tame map $d^f_r(a, b) \neq 0 \Rightarrow a, b$ critical values,
  2. For a box $B = (a, b] \times [c, d)$ and f proper tame map $\mathbb{F}^f_r(B) =\simeq \oplus_{(a'b') \in B \cap \mathrm{s}upp\delta} \hat{d}^f_r(a', b')$
- Let f a tame map, X compact and $\epsilon < \epsilon(f)$. Then $g : X \to \mathbb{R}$ proper map with $|g - f| < \epsilon \Rightarrow$ the support of δ^g is in an $2\epsilon-$ neighborhood of the support of δ^f and of equal cardinality as $\mathrm{s}upp\delta^f$.

Applications

In view of effective computability of the configuration δ^f_r the result can be used to :

- **Applications in topology:** Calculation of Betti numbers, Novikov Betti numbers Refinements of Morse inequalities .
- **Applications in data analysis:** Homological recognition of shapes which can be manifolds. Homological differentiations of shapes.
- **Applications in geometric analysis:** Refinement of Hodge de Rham theorem on compact Riemannian maniflods, Canonical base in the space of Harmonic forms
- **Applications in dynamics:** for dynamics of flows which admit an *action*

References

1.D. Burghelea and T. K. Dey, *Persistence for circle valued maps.* Discrete and Computational Geometry, Vol 50 2013, pp 69-98

2.Dan Burghelea, Stefan Haller, *Topology of angle valued maps, bar codes and Jordan blocks* arXiv:1303.4328

3.Dan Burghelea, *A refinement of Betti numbers in the presence of a continuous function (I),* arXiv:1501.01012

4.Dan Burghelea, *A refinement of Betti numbers in the presence of a continuous function (II),* to be posted soon

$\mathbb{P}H_r(X; f)$ (Persistent homology) are

If f real valued $\kappa-$ vector spaces
If f angle valued $\kappa[t^{-1}, t]-$ free modules / $L^\infty(\mathbb{S}^1)-$ Hilbert modules.

Refinments

$\delta_r^{P,f}$ and $\hat{\delta}_r^{P,f}$ (Configurations)

For f real valued configurations on $\mathbb{C} \setminus \Delta$
For f angle valued configurations on $\mathbb{C} \setminus \{0 \sqcup \mathbb{S}^1\}$

Based on Fredholm cross ratio of
$\alpha : A \to B, \beta : B \to C, \gamma : C \to D$ Fredholm maps.

Isomorphism classes of pairs $(T_r, V_r) \mid V_r = TH_r(\tilde{X}), T_r = t\cdot)$

Invariants $\therefore$ JORDAN CELLS
$\mathcal{J}_r(f) := \{(\lambda^r_i, n^r_i), i = 1 \cdots k_r \mid \lambda^r_i \in \overline{\kappa}, n^r_i \in \mathbb{N}\}$

Based on Linear relations.