Monodromy of a pair $(X, \xi \in H^1(X : \mathbb{Z}))$.

Dan Burghelea

Department of Mathematics
Ohio State University, Columbus, OH

This work is implicit in joint work with **S.Haller** (Vienna)

The novelty of this work is the involvement of :

- the linear algebra of *linear relations*
- the topology of *Hilbert cube manifolds*

for the benefit of computational topology.

- Consider $f : X \to \mathbb{S}^1$, $f \Rightarrow \xi_f \in H^1(X;\mathbb{Z})$

  Always:

  - X a compact ANR,
  - κ a field, $\overline{\kappa}$ its algebraic closure.

- For any $r \in \mathbb{N}_0$ one provides a collection of pairs

  $$\boxed{\mathcal{J}_r(f) := \{(\lambda_1, k_1), (\lambda_2, k_2) \cdots (\lambda_{i_r}, k_{i_r}) \mid \lambda_{\dots} \in \overline{\kappa} \setminus 0, k_{\dots} \in \mathbb{N}\}}$$

  (introduced first by Burghelea-Dey by alternative methods) called **Jordan cells** and prove:

the result

Theorem

1. f_1 *homotopic to* $f_2 \Rightarrow \mathcal{J}_r(f) = \mathcal{J}_r(f_2)$ $\therefore$
   $\mathcal{J}_r(X;\xi) := \mathcal{J}_r(f),\ \xi = \xi_f$

2. $\omega : X_1 \to X_2$ *homotopy equivalence with* $\omega^*(\xi_2) = \xi_1$,
   $\xi_i \in H^1(X;\mathbb{Z}), i = 1, 2 \Rightarrow \mathcal{J}_r(X_1, \xi_1) = \mathcal{J}_r(X_2;\xi_2)$.

3. $\prod_{1\leq l\leq i_r}(z-\lambda_l)^{k_l} = A_r(X;\xi)(r)$ *is a degree* $\sum_{1\leq l\leq i_r} k_l$ *monic polynomial with coefficients in* κ.

Homological definition of $A_r(X,\xi)$

- $(X,\xi) \Rightarrow \tilde{X} \to X$ infinite cyclic cover associated to ξ.
- X compact ANR $\Rightarrow H_r(\tilde{X})$ a f.g. $\kappa[t^{-1},t]$ module
- $V_r := TH_r(\tilde{X})$ the set of torsion elements is a f.g. $\kappa[t^{-1},t]$ module which is a f.d. $\kappa-$vector space.
  $T_r : V_r \to V_r$ is the multiplication by t.

r-Homological monodromy:

$$(V_r, T_r : V_r \to V_r) \equiv \begin{cases} V_r = TH_r(\tilde{X}) \\ T = \text{ multiplication by t} \end{cases}$$

Alexander polynomial:
The *characteristic polynomial* of T_r.

Jordan decomposition

A Jordan cell:

$$T(\lambda, k) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{pmatrix}. \tag{1}$$

JORDAN DECOMPOSITION . With respect to some base in V any $T : V \to V$ is written as

$$\begin{pmatrix} T(\lambda_1, k_1) & 0 & 0 & 0 \\ 0 & T(\lambda_2, k_2) & 0 & \vdots \\ 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & T(\lambda_r, k_r) \end{pmatrix}. \tag{2}$$

Crash course on linear relations

- A linear relation $R : V_1 \rightsquigarrow V_2$ is $R \subseteq V_1 \times V_2$.
  write $v_1 R v_2$ iff $(v_1, v_2) \in R$, $v_i \in V_i$.
- Two liner relations $R_1 : V_1 \rightsquigarrow V_2$ and $R_2 : V_2 \rightsquigarrow V_2$ can be composed
  $(v_1 (R_2 \cdot R_1) v_3$ iff $\exists v_2$ with $v_1 R_1 v_2$ and $v_2 R_2 v_3$.
  The diagonal $\Delta \subset V \times V$ is the identity.
- The reverse of $R : V_1 \rightsquigarrow V_2$ is $R^\dagger : V_2 \rightsquigarrow V_1$
  $v_2 R^\dagger v_1$ iff $v_1 R v_2$.
  One has $(R_1 \cdot R_2)^\dagger = R_2^\dagger \cdot R_1^\dagger$ and $R^{\dagger\dagger} = R$.
- Direct sum $R' \oplus R'' : V_1' \oplus V_1'' \rightsquigarrow V_2' \oplus V_2''$ of two relations $R' : V_1' \rightsquigarrow V_2'$ and $R'' : V_1'' \rightsquigarrow V_2''$ is defined by:
  $(v_1', v_1'')(R' \oplus R'')(v_2', v_2'')$ iff $(v_1' R' v_2')$ and $(v_1'' R'' v_2'')$.

Example :

1. $f : V_1 \to V_2 \Rightarrow R(f) = \mathrm{graph\ f} := \{(\mathrm{v}_1, \mathrm{f}(\mathrm{v}_1)) \in \mathrm{V}_1 \times \mathrm{V}_2\}$
2. $\alpha : V_1 \to W, \beta : V_2 \to W$
   $\Rightarrow R(\alpha, \beta) : V_1 \rightsquigarrow V_2 = \{(v_1, v_2 \mid \alpha(v_1) = \beta(v_2)\}$

For a linear relation R: $V \rightsquigarrow W$ define:

$$\mathrm{dom}\,(R) := \{v \in V \mid \exists w \in W : vRw\} = pr_V(R)$$
$$\mathrm{img}\,(R) := \{w \in W \mid \exists v \in V : vRw\} = pr_W(R)$$
$$\ker(R) := \{v \in V \mid vR0\} \cong (V \times 0) \cap R$$
$$\mathrm{mul}\,(R) := \{w \in W \mid 0Rw\} \cong (0 \times W) \cap R$$

Observation

1. $R\colon V \rightsquigarrow W = R(f)$ *iff*
$\mathrm{dom} R = V$ *and* $\mathrm{mul} R = 0$.

2. $R\colon V \rightsquigarrow V = R(T)$ *for* $T : V \to V$ *a linear isomorphism iff*
$\mathrm{dom} R = V$ *and* $\ker R = 0$.

For $R\colon V \rightsquigarrow V$ define

1. $D : \{v \in V \mid \exists v_i \in V, i \in \mathbb{Z}, v_i R v_{i+1}, v_0 = v\}$. The relation R restricts to a relation $R_D : D \rightsquigarrow D$
2. $K_+ := \{v \in V \mid \exists v_i, 0 \leq i \leq N, v_i R v_{i+1}, v_0 = v, v_N = 0\}$
3. $K_- := \{v \in V \mid \exists v_i, -M \leq i \leq 0, v_i R v_{i+1}, v_0 = v, v_{-M} = 0\}$
4. $V_{reg} := \frac{D}{D \cap (K_+ + K_-)}, \pi : D \to \frac{D}{D \cap (K_+ + K_-)}$

and consider the composition of relations

$$R_{reg} := R(\pi) \cdot R_D \cdot R(\pi)^\dagger : V_{reg} \rightsquigarrow V_{reg}.$$

Proposition

1. *There exists a linear isomorphism* $T^R : V_{reg} \to V_{reg}$, *s.t.* $R_{reg} = R(T^R)$.
2. *If* $(R\colon V \rightsquigarrow V) \sim (R'\colon V' \rightsquigarrow V')$ *are similar relations*[a] *then* $T^R \sim T^{R'}$ *are similar linear isomorphisms*
3. $(T^R)^{-1} = T^{R^\dagger}$
4. $(R' \oplus R'')_{reg} = R'_{reg} \oplus R''_{reg}$.
5. *Suppose* $R_1 : V_1 \rightsquigarrow V_2, R_2 : V_2 \rightsquigarrow V_1$, *then* $(R_2 \cdot R_1)_{reg} \sim (R_1 \cdot R_2)_{reg}$.

[a] there exists an isomorphism of vector spaces $\omega : V \to V'$ such that $R' = R(\omega) \cdot R \cdot R(\omega^{-1})$

In fact given $R\colon V \rightsquigarrow V$ one has
$R = R_{\text{reg}} \oplus R_{\text{sing}}$
$R_{\text{sing}}\colon V_{\text{sing}} \rightsquigarrow V_{\text{sing}}$

The components are unique up to isomorphism but not the decomposition.

The cut at θ (with respect to the map $f : X \to \mathbb{S}^1$)

For θ a weakly regular value (i.e. $f^{-1}(\theta)$ is an ANR.)

- $\overline{X}^f_\theta$, the two sided compactification of $X \setminus f^{-1}(\theta)$ with sides $f^{-1}(\theta)$.
- One has $f^{-1}(\theta) \xrightarrow{i_1} \overline{X}_\theta \xleftarrow{i_2} f^{-1}(\theta)$.
- It induces in $r-$ homology

$$H_r(f^{-1}(\theta)) \xrightarrow{i_1(r)} \overline{H}_r(X_\theta) \xleftarrow{i_2(r)} H_r(f^{-1}\theta)) \ .$$

- Consider the linear relation $R(i_1(r), i_2(r)) := R^f_\theta(r)$ and then the relation $(R^f_\theta(r))_{\mathrm{reg}}$.

Definition

The $r-$ *topological monodromy* of $f : X \to \mathbb{S}^1$ at $\theta \in \mathbb{S}^1$, (θ a weakly regular value), is the similarity class $[(R^f_\theta(r))_{\rm reg}]$, equivalently of the **similarity class of the linear isomorphism** denoted by $\boxed{[T^f_\theta(r)]}$. The Jordan cells $\mathcal{J}^f_\theta(r)$ are the Jordan cells of the linear isomorphism $T^f_\theta(r)$

Denote by $\overline{f}_K : X \times K \to \mathbb{S}^1$ the composition of f with the projecttion $X \times K \to X$.

Proposition

1. $[T^f_{\theta_1}(r)] = [T^f_{\theta_2}(r)]$ *for* θ_1, θ_2 *two weakly regular angles* $\therefore [T^f(r)] := [T^f_\theta(r)]$.
2. $f, g : X \to \mathbb{S}^1$ *homotopic maps imply* $[T^f(r)] = [T^g(r)]$.
3. $[T^{\bar{f}_{S^1}}(r)] = [T^f(r) \oplus T^f(r-1)]$ *for* $r > 0$ *and* $[T^{\bar{f}_{S^1}}](0) = [T^f(0)]$.
4. K *acyclic compact ANR implies* $[T^{\bar{f}_K}(r)] = [T^f(r)]$
5. *If* $f_i : X_i \to \mathbb{S}^1$ $i = 1, 2$, $\omega : X_1 \to X_2$ *a homeomorphism and* $f_2 \cdot \omega$ *is homotopic to* f_1 *then* $[T^{f_1}(r)] = [T^{f_2}(r)]$.

Hilbert cube manifolds

$Q = I^\infty$ the product of infinitely many copies of I.

Theorem

1. *(R Edwards) X is a compact ANR iff $X \times Q$ is a Hilbert cube manifold.*
2. *(T Chapman) If $\omega : X \to Y$ is a homotopy of equivalence between two finite simplicial complexes with Whitehead torsion $\tau(\omega) = 0$ then there exists a homeomorphism $\omega' : X \times Q \to Y \times Q$ such that ω' and $\omega \times id_Q$ are homotopic.*
3. *(folklore) If ω is a homotopy equivalence between two finite dimensional complexes then $\omega \times id_{\mathbb{S}^1}$ has the Whitehead torsion $\tau(\omega \times id_{\mathbb{S}^1}) = 0$.*

Theorem

1. f_1 *homotopic to* $f_2 \Rightarrow \mathcal{J}_r(f) = \mathcal{J}_r(f_2) \therefore \mathcal{J}_r(X;\xi_f)$
2. $\omega : X_1 \to X_2$ *with* $\omega^*(\xi_2) = \xi_1, \xi_i \in H^1(X;\mathbb{Z}), i = 1, 2$
   $\Rightarrow \mathcal{J}_r(X_1, \xi_1)) = \mathcal{J}_r(X_2;\xi_2)$.
3. $\prod_{1\leq l\leq i_r}(z - \lambda_l^r)^{k_l} = A_r(X;\xi)(r)$ *is a degree* $\sum_{1\leq l\leq i_r} k_l^r$ *monic polynomial with coefficients in* κ.

About proof :

- The proposition leads to verification of item 1 based on *topological disjunction tricks*.
- The proposition and the theorem (on Hilbert cute manifolds) lead to item 2.
- Elementary algebraic topology (Mayer-Vietoris properties + direct limit arguments + $\cdots$) lead to item 3.

- **Applications:** $\cdots$ (Novikov Betti numbers, new algorithm for Alexander polynomial, conclusions on closed trajectories of some vector fields.)
- **Computability:** (see references 2 and 3 for algorithms)

References

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3.D. Burghelea and T. K. Dey, *Persistence for circle valued maps.* Discrete and Computational Geometry, Vol 50 2013, pp 69-98