A computer friendly alternative to Morse-Novikov theory

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Classical MN-theory

Input:

- M^n smooth manifold,
- (X, ω)
  - X smooth vector field which is Morse Smale
  - ω closed differential degree one form Lyapunov for X
- κ a field.

For a vector field X the dynamics consists of

- **rest points** $R(X) := \{x \in M \mid X(x) = 0\}$
- **instantons** $I(x, y)$ visible (isolated) trajectories from x to y, $x, y \in R(X)$
- **visible closed trajectories**

• $\omega \Rightarrow \xi_\omega \in H^1(M;\mathbb{R}) = hom(H_1(X;\mathbb{Z}) \to \mathbb{R})$

$H^1(X:\mathbb{R}) \ni \xi \Rightarrow img(\xi) = \boxed{\Gamma_\xi \simeq Z^{n(\xi)} \subset \mathbb{R}}$
$n(\xi)$ the degree of irrationality

• **Novikov field** $\kappa(\xi)$ the completion of $\kappa[\mathbb{Z}^n(\xi)]$ w.r. to the embeding $\mathbb{Z}^{n(\xi)} \subset \mathbb{R}$.

For a generic set of vector fields one has

1. The set $R(X)$ is finite and each $x \in R(X)$ has a Morse index$i(x)$,
2. Instantons exists only from x to y with $i(x) - i(y) = 1$
3. The Poincaré return map of any closed trajectories has no eigenvalue one
4. in each homotopy class of paths from x to y, $x, y \in R(X)$ or of closed curves there are only finitely many instantons or closed trajectories

and the Morse-Novikov theory outputs

- A field $\kappa(\xi_\omega)$, the **Novikov field** and for any (M, X, ω) as above with X generic
- The **chain complex** $(C_*(X, \partial_*(X, \omega)$ of $\kappa(\xi_\omega)$ vector spaces the **Novikov complex** generated by the rest points and the visible (=isolated) trajectories of X

The main results of MN-theory state that

• - The homology $H_r(C_*, \partial_*)$ calculates the $r-$th Novikov-Betti number,

• - The complex depends only on ω and is determined up to a non canonical isomorphism by two of the three collections of numbers:

$\beta_r^N := \dim H_r(C_*, \partial_r), \quad \rho_r^N = rank(\partial_r), \quad c_r^N := \dim C_r.$

Relevance

The chain complex (C_*, ∂_*)

1. *provides the exact number of rest points of Morse index r*
2. *give a good information about the instantons between rest points,*
3. *establishes the existence of closed trajectories.* (when combined with standard Betti numbers)

Computer friendly means computable by **computer implementable algorithms**; $\kappa(\xi_\omega)$ and ∂_r are not computer friendly.

In view of potential interest outside mathematics we like:

1. to derive the collection of numbers $\beta_r,\ \rho_r,\ c_r$ as **computer friendly invariants** and without any reference to the field $\kappa(\xi_\omega)$,
2. to extend MN-theory from

closed smooth manifolds		compact ANR's
vector fields	to	flows on compact ANR's
differential closed one form		TC1-form
Lyapunov for the vector field		Lyapunov for the flow

in particular get rid of genericity hypothesis. Done in [1] and [3] in case the TC1-form ω is exact and in case ξ_ω is an integral cohomology class.

("TC1-form" abbreviation for "topological closed one form")

TC1-form,

- Let $\xi \in H^1(X;\mathbb{R}) = Hom(H_1(X;\mathbb{Z}) \to \mathbb{R})$ and $\Gamma = \Gamma_\xi := (img\xi) \subset \mathbb{R}$.
- The homomorphism $\xi : H_1(X;\mathbb{Z}) \to \mathbb{R}$ defines the associated $\Gamma-$principal cover, $\pi : \tilde{X} \to X$ i.e. the free action $\mu : \Gamma \times \tilde{X} \to \tilde{X}$ with π the quotient map $\tilde{X} \to \tilde{X}/\Gamma = X$.
- Consider $f : \tilde{X} \to \mathbb{R}$, $\Gamma-$equivariant maps, i.e. $f(\mu(g,x)) = f(x) + g$.

Definition

A TC1- form of cohomology class ξ is an equivalence class of $\Gamma-$equivariant maps with $f \sim g$ iff $f - g$ is locally constant.

One denotes:

$\mathcal{Z}^1(X;\xi)$ the set of TC-1 forms in the class ξ

$\mathcal{Z}^1(X) = \bigcup_{\xi \in H^1(X;\mathbb{R})} \mathcal{Z}(X;\xi)$

Examples

1. A smooth closed one form on a manifold
2. A simplicial one co-cycle on a simplicial complex

Let $f : \tilde{X} \to \mathbb{R}$ and $t \in \mathbb{R}$. Denote by :

$$\tilde{X}^f_t = f^{-1}((-\infty, t]), \quad \tilde{X}^f_{<t} = f^{-1}((-\infty, t)) = \cup_{t'<t}\tilde{X}_{t'},$$

$$\tilde{X}^t_f = f^{-1}([t, \infty)), \quad \tilde{X}^{>t}_f = f^{-1}((t, \infty)) = \cup_{t'>t}\tilde{X}^{t'}).$$

$$\boxed{R^f(t) := \dim H_*(\tilde{X}^f_t, \tilde{X}^f_{<t}) + \dim H_*(\tilde{X}^t_f, \tilde{X}^{>t}_f).}$$

Definition

1. t regular value if $R^f(t) = 0$
2. t critical value if $R^f(t) \neq 0$
3. $CR(f) := \{t \in \mathbb{R} \mid R^f(t) \neq 0\} \quad \mathcal{O}(f) := CR(f)/\Gamma$

Tameness

Definition

Supposed X compact, $\xi \in H^1(X;\mathbb{Z})$

- The $\Gamma -$ equivariant map $f : \tilde{X} \to \mathbb{R}$ is tame if for any $t \in \mathbb{R}$ $f^{-1}(t)$ is an ANR.
- The TC1-form $\omega \in \mathcal{Z}^1(X)$ is tame if one (and then any) $f \in \omega$ is tame.

Proposition

1. *X is a compact ANR.*
2. *$R^f(t) < \infty$ for any $t \in \mathbb{R}$,*
3. *the set $\mathcal{O}(f)$ is a finite set.*

Denote by $\mathcal{Z}^1_{tame}(\cdots) := \{\omega \in \mathcal{Z}^1(\cdots) \mid \omega \text{ tame}\}$

Closed one form which are locally polynomials on a closed manifold and simplicial one cocycle on a finite simplicial complex are tame TC1-forms

For any tame TC-1 form $\omega \in \mathcal{Z}(X;\xi)$, field κ and nonnegative integer r we define two configurations

$$\boxed{\underline{\delta}^{\omega}_{r} : \mathbb{R} \to \mathbb{Z}_{\geq 0}} \text{ and } \boxed{\underline{\gamma}^{\omega}_{r} : \mathbb{R}_{+} \to \mathbb{Z}_{\geq 0}}$$

and the numbers $\beta^{\omega}_{r}, \rho^{\omega}_{r}$ and c^{ω}_{r},

$$\boxed{\beta^{\omega}_{r} := \sum_{t\in\mathbb{R}} \underline{\delta}^{\omega}_{r}(t)}, \quad \boxed{\rho^{\omega}_{r} := \sum_{t\in\mathbb{R}_{+}} \underline{\gamma}^{\omega}_{r}(t)}$$

and

$$\boxed{c^{\omega}_{r} := \beta^{\omega}_{r} + \rho^{\omega}_{r} + \rho^{\omega}_{r-1}}.$$

Topological properties

Theorem

Let $\omega \in \mathcal{Z}^1_{tame}(X)$. One has

1. $\underline{\delta}^\omega_r(t) \neq 0$ *or* $\underline{\gamma}^\omega_r(t) \neq 0$ *implies that for any* $f \in \omega$, $t = c' - c''$ *with* $c', c'' \in CR(f)$,
2. $$\beta^\omega_r = \boxed{\sum \underline{\delta}^\omega_r(t) = \beta^N_r(X;\xi)}$$
3. *for any* $f \in \omega$ *and* $a_o \in o \in \mathcal{O}(f)$,

$$c^\omega_r = \boxed{\begin{array}{l} \sum_{t\in\mathbb{R}} \underline{\delta}^\omega_r(t) + \sum_{t\in\mathbb{R}_+} \underline{\gamma}^\omega_r(t) + \sum_{t\in\mathbb{R}_+} \underline{\gamma}^\omega_{r-1}(t) = \\ = \sum_{o\in\mathcal{O}(f)} \dim H_r(\tilde{X}^f_{a_o}, \tilde{X}^f_{<a_o}) \end{array}}$$

Recall $\tilde{X}^f_{a_o} = f^{-1}((-\infty, a_0]), \tilde{X}^f_{<a_o} = f^{-1}((-\infty, a_0))$.

Corollary

The Novikov complex of the Morse form ω in Hodge form is given by $C_r = C_r^- \oplus C_r^+ \oplus \mathcal{H}_r$ with

$$\mathcal{H}_r = \kappa^{\beta_r^\omega},$$

$$C_r^- = C_{r-1}^+ = \kappa^{\rho_r^\omega}$$

$$\partial_r = \begin{pmatrix} 0 & 0 & 0 \\ Id & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Poincaré Duality property

Theorem

Suppose X is a closed topological $n-$dimensional manifold and $\omega \in \mathcal{Z}^1_{tame}(M)$.

1. $\underline{\delta}_r^\omega(t) = \underline{\delta}_{n-r}^\omega(-t)$,
2. $\underline{\gamma}_r^\omega(t) = \underline{\gamma}_{n-r-1}^{-\omega}(t)$.

• The set of TC-1 forms $\mathcal{Z}^1(X;\xi)$ is an $\mathbb{R}-$vector space equipped with an obviously defined distance

$$||\omega - \omega'|| := \inf_{\begin{cases} f \in \omega \\ g \in \omega' \end{cases}} \sup_{x \in \tilde{X}} |f(x) - g(x)|.$$

•The space of configurations $Conf_N(Y) := \{\delta : Y \to \mathbb{Z}_{\geq 0} \mid \sum \delta(y) = N\}$ is equipped with the obvious *collision topology*.

• For K closed subset of Y the space of configurations $Conf(Y \setminus K)$ is equipped with the *bottleneck topology* provided by the pair $K \subset Y$.

Stability property

Theorem

1. *the assignment* $\mathcal{Z}^1_{tame}(X;\xi) \ni \omega \rightsquigarrow \underline{\delta}^\omega_r \in Conf_{\beta^N_r(X;\xi)}(\mathbb{R})$ *is continuous and extend continuously to* $\mathcal{Z}(X;\xi)$.
2. *the assignment* $\mathcal{Z}^1_{tame}(X;\xi) \ni \omega \rightsquigarrow \underline{\gamma}^\omega_r \in Conf([0,\infty) \setminus 0)$ *is continuous.*

For $f \in \omega$, $f : \tilde{X} \to \mathbb{R}$ with $f(\mu(g, x)) = f(x) + g$ one defines the maps

$$\delta_r^f : \mathbb{R}^2 \to \mathbb{Z}_{\geq 0} \text{ and } \gamma_r^f : \mathbb{R}_+^2 \to \mathbb{Z}_{\geq 0},$$

$\mathbb{R}_+^2 := \{(x, y) \mid x < y\}$ as follows.

Denote :

1. $\boxed{\mathbb{I}_a^f(r) := img(H_r(\tilde{X}_a) \to H_r(\tilde{X}))}$, $\quad \mathbb{I}_a^f(r) \subset H_r(\tilde{X})$

   $$\mathbb{I}_{<a}^f(r) := \cup_{a'<a} \mathbb{I}_{a'}^f(r), \quad \mathbb{I}_{-\infty}^f(r) = \cap_a \mathbb{I}_a^f(r),$$

2. $\boxed{\mathbb{I}_f^a(r) := img(H_r(\tilde{X}^a) \to H_r(\tilde{X}))}$, $\quad \mathbb{I}_f^a(r) \subset H_r(\tilde{X})$

   $$\mathbb{I}_f^{>a}(r) := \cup_{a'>a} \mathbb{I}_f^{a'}(r), \quad \mathbb{I}_f^{+\infty}(r) = \cap_a \mathbb{I}_f^a(r),$$

3. For $a < b$

   $$\boxed{\mathbb{T}_r^f(a,b) := \ker(H_r(\tilde{X}_{\leq a}) \to H_r(X_{\leq b}))},$$

   and for $a' < a$ the inclusion induced map

   $$\boxed{i : \mathbb{T}_r(a', b) \to \mathbb{T}_r(a, b)}\ .$$

4. $$\boxed{\mathbb{T}_r(a, < b) := \cup_{a<b'<b} \mathbb{T}_r(a, b') \subseteq \mathbb{T}_r(a, b)},$$

   $$\boxed{\mathbb{T}_r(< a, b) := \cup_{a'<a} i(\mathbb{T}_r(a', b)) \subseteq \mathbb{T}_r(a, b)}.$$

- Define

for $a, b \in \mathbb{R}$

$$\boxed{\hat{\delta}^f_r(a,b) := \frac{\mathbb{I}_a(r) \cap \mathbb{I}^b(r)}{\mathbb{I}_{<a}(r) \cap \mathbb{I}^b(r) + \mathbb{I}_a(r) \cap \mathbb{I}^{>b}(r)}, \ \delta^f_r(a,b) := \dim \hat{\delta}^f_r(a,b)}$$

for $a, b \in \mathbb{R}$, $a < b$

$$\boxed{\hat{\gamma}^f_r(a,b) := \frac{\mathbb{T}_r(a,b)}{\mathbb{T}_r(<a,b) + T_r(a,<b)}, \quad \gamma^f_r(a,b) := \dim \hat{\gamma}^f_r(a,b).}$$

and for $t \in \mathbb{R}$

$$\boxed{\mathbb{F}^f_r(t) := \sum_{\begin{cases} a-b \leq t, \\ a,b \in CR(f) \end{cases}} \mathbb{I}_a(r) \cap \mathbb{I}^b(r)}$$

One can show that:

Proposition

1. $\delta^f_r(a,b) = \delta^f_r(a+g, b+g)$ *and* $\gamma^f_r(a,b) = \gamma^f_r(a+g, b+g)$ *for any* $g \in \Gamma$,
2. *for a regular value* $\mathrm{supp}\delta^f_r \cap (a \times \mathbb{R})$ *and* $\mathrm{supp}\gamma^f_r \cap (a \times (a, \infty))$ *is empty,*
3. *for a critical value* $\mathrm{supp}\delta^f_r \cap (a \times \mathbb{R})$ *and* $\mathrm{supp}\gamma^f_r \cap (a \times (a, \infty))$ *is finite,*
4. *for b regular value* $\mathrm{supp}\delta^f_r \cap (\mathbb{R} \times b)$ *and* $\mathrm{supp}\gamma^f_r \cap ((-\infty, b) \times b)$ *is empty,*
5. *for b critical value* $\mathrm{supp}\delta^f_r \cap (\mathbb{R} \times b)$ *and* $\mathrm{supp}\gamma^f_r \cap ((-\infty, b) \times b)$ *is finite.*
6. $\mathbb{F}^f_r(t)$ *is a f.g.* $\kappa(\Gamma)-$ *module.*

For $\omega \in \mathcal{Z}^1_{tame}(X)$ define

$$\boxed{\mathcal{O}(\omega) := \sqcup_{f\in\omega}\mathcal{O}(f)/\sim,}$$

$\mathcal{O}(f_1) \ni o_1 \sim o_2 \in \mathcal{O}(f_2)$, $f_2 = f_1 + s$, iff $\theta(s) : \mathcal{O}(f_1) \to \mathcal{O}(f_2)$, the bijective correspondence induced the translation $\theta(s)(t) = t + s$ satisfies $\theta(s)(o_1) = o_2$.

Observation

For $a_o \in o \in \mathcal{O}(f)$, $f \in \omega$, $\delta^f_r(a_o, a_o + t)$ and $\gamma^f_r(a_o, a_o + t)$ are independent on a_o and f.

Define $\boxed{\underline{\delta}^{\omega}_{o,r}(t) := \delta^{f}_{r}(a_o, a_o + t)}$ and then

$$\boxed{\underline{\delta}^{\omega}_{r}(t) = \sum_{o \in \mathcal{O}(\omega)} \underline{\delta}^{\omega}_{o,r}(t)}.$$

Define $\boxed{\underline{\gamma}^{\omega}_{o,r}(t) := \gamma^{f}_{r}(a_o, a_o + t)}$ and then

$$\boxed{\underline{\gamma}^{f}_{r}(t) = \sum_{o \in \mathcal{O}(\omega)} \underline{\gamma}^{f}_{o,r}(t)}.$$

If ω has degree of rationality ≤ 1 (i.e. $\Gamma \subset \mathbb{R}$ is a discrete subset) then the calculations of $\mathcal{B}_r(f)$ proceed as in[2] or [1]

One hopes that the barcodes of tame TC1-forms of degree of rationality one approximate arbitrary close the bar codes of any tame TC1-form ω. This turns out to be true in the smooth case when the space is a compact smooth manifold and ω is a Morse closed one form. THIS IS WORK IN PROGRESS.

Applications

1. **MN theory on** M^{S^1}
2. **Data analysis** Point cloud data with a skew symmetric two point map.
3. **Dynamics** on flows on compact spaces which are not manifolds.

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