The Virtually small spectral package of a closed Riemannian manifold with respect to a Morse function

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The spectral Package of a Riemannian manifold

For a closed Riemannian manifold (M, g) consider

- $\Delta_q : \Omega^q(M) \to \Omega^q(M)$, the Laplace Beltrami operators, $0 \leq q \leq \dim M$.
- $\mathbb{S}_q(M, g) := \{\lambda^q_\alpha \in \mathbb{R}_{\leq 0}, \omega^q_\alpha \in \Omega^q(M)\}, \;\; \Delta_q(\omega^q_\alpha) = \lambda^q_\alpha \omega^q_\alpha$ the "spectral package"
- *Spec*$\Delta_q := \{\lambda^q_\alpha \in \mathbb{R}_{\leq 0}, \;\; q = 0, \cdots n\}$ *the spectrum of (M,g) = "music" of (M,g)*.

Geometry/Topology -some basic invariants

For a closed Riemanian manifold (M, g) consider

- the Betti numbers $\beta_q = \dim(H^r(M;\mathbb{C}))$
  $\therefore$ the E-P characteristic $\chi(M) := \sum(-1)^q\beta_q$,
- the torsion numbers $T_i(M) :=$
  $\sharp\{\text{ finite order elements of } \mathrm{H^i(M;\mathbb{Z})}\}$ $\therefore T(M) = \prod T_i^{(-1)^i}$,
- the volume numbers: $V_i := \mathrm{Vol}(\mathcal{H}^i(M, g)/\mathrm{img}(H^i(M;\mathbb{Z}))$ $\therefore$
  $\mathcal{V}(M, g) := \prod V_i^{(-1)^i}$, $0 \leq i \leq \dim M = n$ $\mathcal{H}^i = \ker \Delta_i$,
  $V_n = Vol(M)$, $V_0 = 1$.
- the Reidemeister torsion $T_{Re}(M, g) := T(M) \cdot \mathcal{V}(M, g)$

Spectral geometry -Two basic results

The basic functions derived from the spectrum:

- Heat trace $h_i(t) := \sum e^{-t\lambda_i^q}$ and then $h(t) = (-1)^i h_i(t)$,
- Zeta function $\zeta_i(s) := \sum (\lambda_i^q)^{(-s)}, \therefore \; \zeta(s) = \sum (-1)^i i \zeta_i(s)$,
- $\zeta_i(s)$ is the Mellin transform of $h_i(t)$.

Results of Herman Weyl, ... Ray-Singer.....

Theorem

1. $\lim_{t\to 0} h_0(t) \cdot t^{n/2} = V_n(M)/4\pi$, $n = \dim M$,
   $\lim_{t\to 0}(h_0(t) - Vol(M^2)/(4\pi t) = (1/6)\chi(M^2)$
2. $\boxed{-\zeta'(0) = \log T_{re}(M, g)}$.

All these results involve the knowledge of spectrum as an infinite set of numbers and can not be used directly because of that.

This work intends to get such invariants computable from a finite part of the spectral package + some additional data (for example a Morse function)

I will discuss some
NEW MATHEMATICS INJECTED IN THE FIELD OF SPECTRAL GEOMETRY

1. Witten deformation

$f : M \to R$ smooth,

Witten deformation of the de Rham complex

$$\boxed{(\Omega^*(M), d^*(t)),\ t \in \mathbb{R}, \mathbb{C}}$$

$$d(t)(\cdots) = e^{-tf} d(e^{tf} \cdots) = (d + tdf \wedge)(\cdots)$$
$$\Delta_q^f(t) = d(t) \cdot d(t)^\sharp + d(t)^\sharp \cdot d(t),$$

$$\boxed{\Delta_q^f(t) = \Delta_q + t(L_X + L_X^\star) + t^2 ||X||^2}$$

L_X the Lie derivative w. r. to the vector field $X = -\mathrm{grad_g f}$.

2. Reilich-Kato perturbation theory

Theorem

Let $A(z)$ **be a holomorphic "self adjoint" family** *(i.e. self adjoint for z real) of unbounded invertible operators with compact resolvent in the complex Hilbert space H, (z in a neighborhood of* $\mathbb{R}$ *in* $\mathbb{C}$*).*

There exists the families of **real valued analytic functions** $\lambda_i(t)$ **and** $H-$**valued analytic functions** $\omega_i(t)$, *each having a holomorphic extension* $(\lambda_i(z), \omega_i(z))$ *to an open neighborhood of* $\mathbb{R}$ *inside* $\mathbb{C}$ *so that*

1. *each* $\lambda_i(z)$ *is an eigenvalue of* $A(z)$ *and* $\lambda_i(z)$ **exaust all eigenvalues**
2. $\omega_i(t)$ **are orthonormal eigenvectors** *corresponding to the eigenvalues* $\lambda_i(z)$.

Apply to $A(z) = \Delta_q^f(z) + P(z)$, $P(z)$ the projection on $\ker \Delta_q^f(z)$.

3. Morse theory

Let (M, g) be closed Riemannian manifold, $f : M \to \mathbb{R}$ Morse-Smale.

$Cr_q(f) = \{x \in M \mid \mathrm{grad_g f} = 0, \mathrm{index\ x} = \mathrm{q}\}$, $N_q := \sharp Cr_q(f)$.
$X := -\mathrm{grad_g f}$

For $x \in Cr_q(f)$ let $W_x^- \subset M$ be the unstable manifold of X at the rest point x, $W_x \approx \mathbb{R}^q$.

Theorem

For any $\omega \in \Omega^q(M)$ and any $x \in Cr_q(f)$ **the integral $\int_{W_x^-} \omega$ is convergent** *and defines a continuous linear map*

$$Int_x : \Omega^q(M) \to \mathbb{C}.$$

for the Frechet topology on $\Omega^q(M)$.

Morse-Smale complex and integration

Let $(C^*(M,X),\delta^*_X)$ be the the Morse-Smale complex,
$C^q := \mathrm{Maps(Cr_q(f);\mathbb{C}),\ \delta^q_X = \cdots}$

For any $z \in \mathbb{C}$ one has a morphism of cochain complexes

$$\boxed{Int^*(z) : (\Omega^*(M), d^*(t)) \to (C^*, \delta^*)}$$

$$(Int(z)(\omega))(x) := \int_{W_x^-} e^{zh}\omega.$$

This induces an isomorphism in cohomology denoted by

$$\boxed{Hint^q(t) : H^q(\Omega^*(M), d^*(t)) \to H^q(C^*, \delta^*).}$$

Combine Morse Theory and Witten deformation

Witten theorem.

Theorem

Let (M^n, g) be a closed Riemannian manifold and $f : M \to \mathbb{R}$ a Morse function.

There exists the constants $C_1, C_2, C_3, T_0 > 0$ depending on g and f so that for $t \geq T_0$

(1) Spect$(\Delta_q^f(t) \cap (C_1 e^{-C_2 t}, C_3 t) = \emptyset$, and

(2) the number of eigenvalues of $\Delta_q^f(t)$ (counted with multiplicity) in the interval $[0, C_1 e^{-C_2 t}]$ is equal to N_q.

$\therefore for T \geq T_0$

$$\boxed{(\Omega^*(M), d^*(t)) = (\Omega^*_{sm}(t), d^*(t)) \oplus (\Omega^*_{la}(t), d^*(t))}.$$

The Virtually small spectral package w.r. to a Morse function f.

- Reilich- Kato theory put eigenvalues of $\Delta^f_q(t)$ in families $\lambda^q_\alpha(t)$, real analytic in t for $-\infty < t < \infty$
- Witten Theorem selects exactly N_q of them are convergent to 0 when $t \to \infty$ (all others are go to ∞). Call them the"small families"

$$\boxed{\lambda^q_i(t) \quad i \leq 1 \leq N_q}.$$

- The corresponding set of eigenforms define the finite dimensional subcomplex $(\Omega^*_{sm}(t), d(t)) \subset (\Omega^*, d(t))$ depending analytically in t which coincides with the one specified before for $t > T_0$.

- The integration $Int^*(t) : (\Omega^*(M), d^*(t)) \to (C^*(M,X), \delta^*_X)$ restricts to

$$Int^*(t) : (\Omega^*_{sm}(M)(t), d^*(t)) \to (C^*, \delta^*)$$

a real analytic family of quasi-isomorphisms of f.d. cochain complexes .

Definition

The collection

$$\boxed{\mathbb{S}_{q,vsm}(M,g,f) := \{\lambda_i^q(0), \omega_i^q(0) \in \Omega^q(M)\}}$$

for $1 \leq i \leq N_q, 0 \leq q \leq \dim M$ defines the **virtually small spectral package** of the Riemannian manifold (M, g) w.r. to the Morse function f.

TO WHAT EXTENT $\mathbb{S}_{q,vsm}(M, g, f)$ CARRY RELEVANT GEOMETRIC AND TOPOLOGICAL INFORMATION

Suppose (M, g) is a closed Riemannian manifold and f a Morse-Smale function.

One says f **is good** if every C^2- close enough Morse-Smale function g has the same virtually small spectral package.

A few results

Theorem

The meromorphic function $a(t)$ has an holomorphic extension to a neighborhood of the real line no poles or zeros.

Theorem

1. *β_q of the N_q functions $\lambda_i^q(t)$ are identically zero.*

2. $T_{Re} = \prod_{q=0}^{\dim M}(\prod_{i=1}^{N_q} \lambda_i^q)^{i(-1)^i} \cdot a(0)$

3.The set of good Morse-Smale functions is open and dense in the C^r topology for $r \geq 2$.

CONJECTURES:

A. All Morse-Smale functions are good.

B. The virtually eigenfunctions of Max Warjewski are exactly the virtually small eigenfunctions of some Morse function.

C. There exist estimates of type Cheeger-Busser for the virtually small eigenfunctions.