A COMPUTATIONAL ALTERNATIVE TO MORSE-NOVIKOV THEORY

Dan Burghelea

Department of Mathematics
Ohio State University, Columbus, OH

Shanghai, July 2017

summary

- A brief summary of what AMN-theory does
- The AMN theory invariants
- From AMN invariants to the algebraic topology
- Complements of complex hyper surfaces via AMN theory
- Computational results
- Description of AMN invariants

The AMN-theory considers:

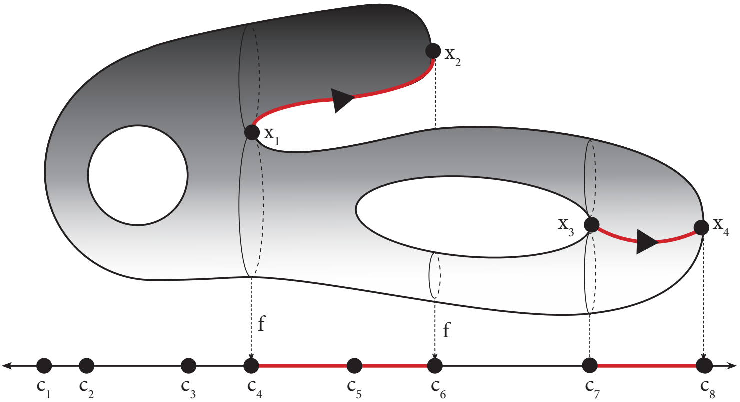
$f : X \to \mathbb{R}$ and $f : X \to \mathbb{S}^1$

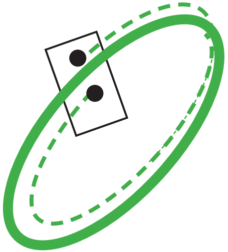
X a compact ANR, f a tame map

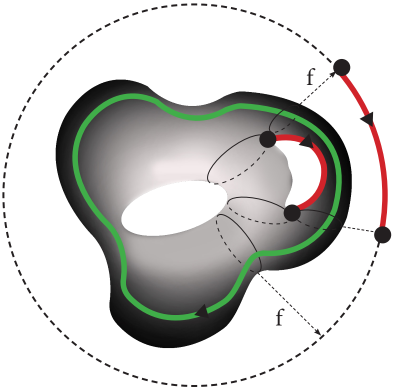
Examples:

finite simplicial complex , simplicial map

compact topological manifold, local polynomial maps


x_1
x_2
x_3
x_4
f
f
c_1
c_2
c_3
c_4
c_5
c_6
c_7
c_8




f
f

AMN invariants for real-valued maps $f : X \to \mathbb{R}$

Fix a field κ

<u>For $f : X \to \mathbb{R}$</u>

a) **critical values** $Cr(f) := \cdots c_i < c_{i+1} < c_{i+2} \cdots$

b) **barcodes**

1. **closed barcodes** $\mathcal{B}^c_r(f)$, $[a, b] \subset \mathbb{R}$,
   $a, b \in Cr(f)$ regarded as a complex number $z = a + ib \in \mathbb{C}$
2. **open barcodes** $\mathcal{B}^o_r(f)$, $(a, b) \subset \mathbb{R}$,
   $a, b \in Cr(f)$ regarded as a complex number $z = b + ia \in \mathbb{C}$
3. **closed-open barcodes** $\mathcal{B}^{c,o}_r(f)$, $[a, b) \subset \mathbb{R}$,
   $a, b \in Cr(f)$ regarded as a complex number $z = a + ib \in \mathbb{C}$
4. **open-closed barcodes** $\mathcal{B}^{o,c}_r(f)$, $(a, b] \subset \mathbb{R}$,
   $a, b \in Cr(f)$ regarded as a complex number $z = b + ia \in \mathbb{C}$

For $f : X \to \mathbb{S}^1$

a) **critical values** $Cr(f) := 0 \leq \theta_1 < \theta_2, \cdots, < \theta_N < 2\pi$

b) **barcodes**

1. **closed barcodes** $\mathcal{B}^c_r(f)$, $[a, b] \subset \mathbb{R}_{\geq 0}$
   $a, b \in Cr(f)$ mod 2π regarded as a complex number $z = e^{i(a+b)+(b-a)} \in \mathbb{C}$
2. **open barcodes** $\mathcal{B}^o_r(f)$, $(a, b), \subset \mathbb{R}_{\geq 0}$
   $a, b \in Cr(f)$ regarded as a complex number $z = e^{i(a+b)+(a-b)} \in \mathbb{C}$
3. **closed-open barcodes** $\mathcal{B}^{c,o}_r(f)$, $[a, b) \subset \mathbb{R}_{\geq 0}$
   $a, b \in Cr(f)$ regarded as a complex number $z = e^{i(a+b)+(b-a)} \in \mathbb{C}$
4. **open-closed barcodes** $\mathcal{B}^{o,c}_r(f)$, $(a, b] \subset \mathbb{R}_{\geq 0}$
   $a, b \in Cr(f)$ regarded as a complex number $z = e^{i(a+b)+(a-b)} \in \mathbb{C}$

c) **Jordan cells** $\mathcal{J}_r(f) := \{(\lambda, k) \mid \lambda \in \overline{\kappa} \setminus 0, k \in \mathbb{Z}_{\geq 1}\}$

(λ, k) corresponds to a Jordan matrix

$$T(\lambda, k) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{pmatrix} \qquad (1)$$

if $k \geq 2$, and $T(\lambda, 1) = (\lambda)$ if $k = 1$.

$$f : X \to \mathbb{S}^1 \Rightarrow \xi_f \in H^1(X;\mathbb{Z})$$

- **Betti numbers** $\beta_r(X) = \dim H_r(X)$
- **Twisted Betti numbers** $\beta_r(X : (\xi_f, u))$ for $u \in \kappa \setminus 0$
- **Novikov-Betti numbers** $\beta_r^N(X, \xi) = rank\ FH_r(\tilde{X})$
- **Monodromy** $[T_r] := (V_r, T_r) = Tor\ H_r(\tilde{X})$

• $\xi \Rightarrow \tilde{X} \to X \Rightarrow H_r(\tilde{X})$ a f.g. $\kappa[t^{-1}, t]$-module, $H_r(\tilde{X}) = FH_r(\tilde{X}) \oplus Tor\ H_r(\tilde{X})$

• $u \in \kappa \setminus 0, \xi \in H^1(X;\mathbb{Z}) \Rightarrow \ (\xi, u) : H_1(X;\mathbb{Z}) \xrightarrow{\xi} \mathbb{Z} \xrightarrow{\hat{u}} \kappa \setminus 0$ with $\hat{u}(n) = u^n$

• $Tor\ H_r(\tilde{X}) = (V(r), T(r))$ f.g. torsion module hence, $V(r)$ a f.d κ-vector space, $T(r) : V(r) \to V(r)$ linear isomprphism.

AMN-invariants of $f : X \to \mathbb{R}/\mathbb{S}^1$

$\swarrow \qquad \searrow$

Algebraic topology of X
or(X, ξ_f)

Dynamics of a flow on X
Ψ with f Lyapunov

Dynamics= rest points, instantons, closed trajectories

Results AMN-invariants $\Rightarrow$ Algebraic topology

Theorem

For $f : X \to \mathbb{S}^1$ the following holds:

1. $\beta^N_r(X, \xi_f) = \sharp\mathcal{B}^c_r(f) + \sharp\mathcal{B}^o_{r-1}(f)$
2. $\beta_r(X, (\xi_f, u)) = \beta^N_r(X, \xi_f) + \sharp\mathcal{J}_{r,u}(f) + \sharp\mathcal{J}_{(r-1),1/u}(f)$
3. $\beta_r(X) = \beta^N_r(X, \xi_f) + \sharp\mathcal{J}_{r,u}(f) + \sharp\mathcal{J}_{(r-1),1/u}(f)$
4. $[T_r(\xi_f)] = [\oplus_{(\lambda,k)\in\mathcal{J}_r(f)} T(\lambda, k)]$
5. *For any $\theta \in \mathbb{S}^1$ we have $\beta_r(f^{-1}(\theta)) \leq \sum_{(\lambda,n)\in\mathcal{J}_r(f)} n$*

The Morse or the Novikov complex associated with f (up to isomorphism) w.r.t. to a field is completely determined by the closed, open and closed-open bar codes.

Complement of hyper surfaces

$P(z_1, z_2, \cdots z_n)$ polynomial, $z = (z_1, z_2, \cdots z_n) \in \mathbb{C}^n$.

- $V = P^{-1}(0) \subset \mathbb{C}^n$,
- $B_P := \{z^1, z^2, \cdots z^N\}$, $z^i \in \mathbb{C}^n$ the bifurcation set
- $\boxed{Y := \mathbb{C}^n \setminus V,\ \ P : Y \to \mathbb{C} \setminus 0}$,

  $\boxed{f_1 : Y \to \mathbb{R}_{>0},\ f_1(z) = |P(z)|}$

  $\boxed{f_2 : Y \to \mathbb{S}^1,\ f_2(z) := P(z)/|P(z)|}$
- $\boxed{X_R := \{z \in \mathbb{C}^n \mid |z| \leq R\} \setminus \{z \mid |P(z)| < 1/R\}}$,

$P : C^n \setminus P^{-1}(B) \to C \setminus B_P$ is a smooth bundle.

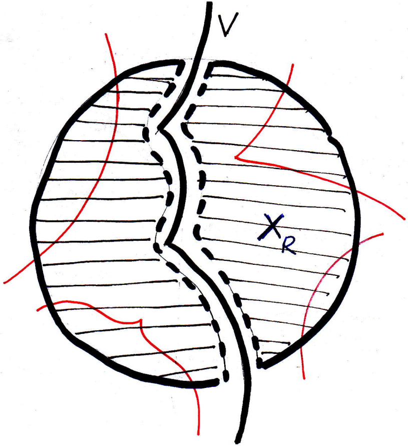


Figure : X_R

Proposition

Suppose $R > \sup_{i=1,2,\cdots N}\{|z_i|\}$. *Then*

1. X_R *is a compact manifold with boundary, with* $Y \setminus IntX_R$ *is a collar of the boundary* $\partial(X_R)$, *i.e.* $Y \sim X_R \cup \partial X_R \times [0,\epsilon)$
2. *the critical values of* f_1 *are* $\{|z_i|, z_i \in B_P \setminus 0\}$
3. *the critical angles go* f_2 *are* $\{argz_i.z_i \in B_P \setminus 0\}$
4. *If* P *is good at* ∞ *then* $f_2 : \partial X_R \to \mathbb{S}^1$ *is a bundle.*

Good at ∞ means V is in general position at ∞ cf. L.Maxim (i.e., its projective completion of V intersects the hyperplane at infinity transversally) or is tame at ∞ cf. Nemethi-Zaharia.

Theorem

Let P a polynomial as above and let $\xi = \xi_{f_2} \in H^1(Y;\mathbb{Z})$ *the induced cohomology class. Suppose that P is good at* ∞

1. $$\beta^N_r(Y;\xi) = \begin{cases} 0 \text{ if r} \neq \text{n} \\ \chi(Y) \text{ if r} = \text{n} \end{cases} \quad (2)$$
2. $\mathcal{J}_r(f) = \emptyset$ *for* $r \geq n$
3. *For any* $u \in \kappa \setminus 0$
$$\beta_r(X : (\xi, u)) = \begin{cases} 0, r > n \\ \sharp\mathcal{J}_{r,u}(f) + \sharp\mathcal{J}_{(r-1),1/u}(f),\ r < n \\ \chi(Y) + \sharp\mathcal{J}_{(n-1),1/u}(f),\ r = n \end{cases}$$
4. *For* $\theta \in \mathbb{S}^1$, $\beta_r(f_2^{-1}(\theta)) \geq \sum_{(\lambda,k)\in\mathcal{J}_r(f)} k$

An oriented graph has:

- Vertices : $\{x_\alpha,\ \alpha \in \mathcal{A}\}$
- Oriented edges; $\{\ x_\alpha \xrightarrow{a_{\alpha,\beta}} x_\beta\ \}$,

A graph representations ρ assigns:

$x_\alpha \rightsquigarrow V_\alpha$ finite dimensional κ-vector space

$a_{\alpha,\beta} \rightsquigarrow V_\alpha \to V_\beta$, κ-linear map

The graph $\mathcal{Z}$

$$\cdots \xleftarrow{b_{i-1}} x_{2i-1} \xrightarrow{a_i} x_{2i} \xleftarrow{b_i} x_{2i+1} \xrightarrow{a_{i+1}} x_{2i+2} \xleftarrow{b_{i+1}} \cdots$$

Figure : The graph $\mathcal{Z}$.

Compactly supported indecomposable representations are

Barcodes:
indexed by four type of Intervals in $\mathbb{R}$

$$\begin{cases} [a, b], (a, b), \\ [a, b), (b, a] \end{cases}$$

with ends , $a \leq b, \;\; a, b \in \mathbb{Z}$

the graph G_{2m}

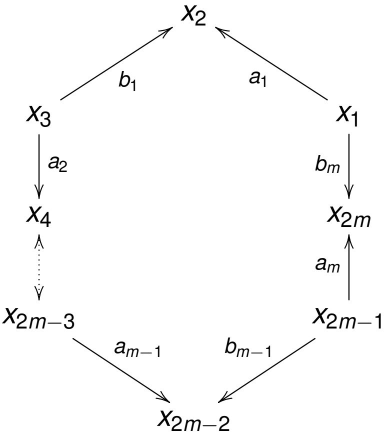
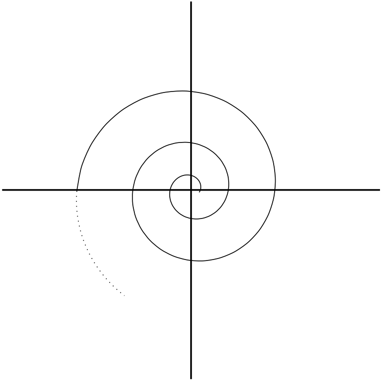


Figure : The graph G_{2m}.

The spiral

$$z(t) = \begin{cases} x(t) = (t+1)\cos t \\ y(t) = (t+1)\sin(t) \end{cases}, t \in [0, \infty)$$



Indecomposable representations are

- **Barcodes**

four type of Intervals $\subset [0, \infty)$

$$\begin{cases} [a, b], (a, b), \\ [a, b), (b, a] \end{cases}$$

with
$a \in \{2\pi k/m \mid 1 \leq k \leq m\}$,
$b \in \{2\pi k/m + 2\pi r \mid 1 \leq k \leq m, r \in \mathbb{Z}_{\geq 0}\}$

- **Jordan cells**

pairs (λ, k), $\lambda \in \overline{\kappa} \setminus 0, k \in \mathbb{Z}_{\geq 1}$

Construct the representation $\rho_r(f), r \geq 0$,

1. Find $\cdots \leq c_i < c_{i+1} <, \cdots, < c_{i+m} < \cdots$ critical values
2. Choose $\cdots, < t_i < t_{i+1}, \cdots, t_{i+m}, c_i < t_i < c_{i+1}, \cdots$
3. Denote $X_{2i-1} = f^{-1}(t_{i-1}), X_{2i} = f^{-1}([t_{i-1}, t_i])$
4. Define $\rho_r(f)$:

$$V_{2i-1} = H_r(X_{2i-1}) \xrightarrow{a_i(r)} H_r(X_{2i}) = V_{2i} \xleftarrow{b_i(r)} H_r(X_{2i+1}) = V_{2i+1}$$

$a_i(r), b_i(r)$ induced by the inclusions $X_{2i-1} \xrightarrow{\subset} X_{2i} \xleftarrow{\supset} X_{2i+1}$

- The r-barcodes of f are the barcodes of $\rho_r(f)$ with the ends i and j replaced by c_i and c_j

The AMN invariants for $f : X \to \mathbb{S}^1$

Construct the representation $\rho_r(f), r \geq 0$,

1. Find $0 \leq \theta_1 < \theta_2 <, \cdots, < \theta_m < 2\pi$ critical angles
2. Choose $t_1 < t_2, \cdots, t_m, \theta_i < t_i < \theta_{i+1}, \cdots \theta_m < t_m < 2\pi$
3. Denote by $X_{2i-1} = f^{-1}(t_{i-1}), X_{2i} = f^{-1}([t_{i-1}, t_i])$
4. Define $\rho_r(f)$:

$$V_{2i-1} = H_r(X_{2i-1}) \xrightarrow{a_i(r)} H_r(X_{2i}) = V_{2i} \xleftarrow{b_i(r)} H_r(X_{2i+1}) = V_{2i+1}$$

$a_i(r), b_i(r)$ induced by the inclusions $X_{2i-1} \xrightarrow{\subset} X_{2i} \xleftarrow{\supset} X_{2i+1}$

- The r-barcodes of f are the barcodes of $\rho_r(f)$ with the ends i and $j + mk$ replaced by θ_i and $\theta_j + 2\pi k$
- The Jordan cells $\mathcal{J}_r(f)$ are the Jordan cells of $\rho_r(f)$.

A better way to calculate $J_r(f)$.

Linear relation - Regular part

$$\begin{array}{ccccc} V & \xrightarrow{\alpha} & W & \xleftarrow{\beta} & V \\ \uparrow \subseteq & & \uparrow \subseteq & & \uparrow \subseteq \\ V_{reg} & \xrightarrow{\alpha_{reg} \simeq} & W_{reg} & \xleftarrow{\beta_{reg} \simeq} & V_{reg} \end{array}$$

$$\Rightarrow (V_{reg}, T_{reg}), \;\; T_{reg} := \beta_{reg}^{-1} \cdot \alpha_{reg} .$$

Cut at θ

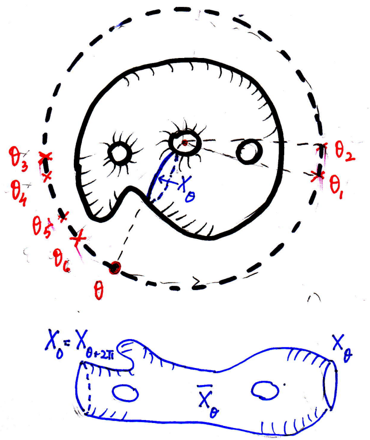


Figure : cut at θ

Jordan cells (of f)

Use any $\theta \in \mathbb{S}^1$

- Consider

$$\begin{array}{ccccc} V_r = H_r(X_{\theta+2\pi}) & \xrightarrow{i_-(r)} & W_r = H_r(\overline{X}_\theta) & \xleftarrow{i_+(r)} & V_r = H_r(X_\theta) \\ \uparrow & & \uparrow & & \uparrow \\ (V_r)_{reg} & \xrightarrow{i_-(r)_{reg}} & (W_r)_{reg} & \xleftarrow{i_+(r)_{reg}} & (V_r)_{reg} \end{array}$$

- Define $(V(r),\ T(r)) := (V(r)_{reg},\ i_+(r)^{-1}_{reg} \cdot i_-(r)_{reg})$.

$\mathcal{J}_r(f)$ are the Jordan cells of $T(r)$

References

1.Dan Burghelea, Stefan Haller, *Topology of angle valued maps, bar codes and Jordan blocks* arXiv:1303.4328 to appear 2017

2. Dan Burghelea, *New topological invariants for real- and angle-valued maps. An alternative to Morse-Novikov theory* chapter 8 (book) World Scientific 2017 see (https://people.math.osu.edu/burghelea.1/)