A STRANGE GEOMETRIC INVARIANT WITH APPLICATIONS TO TOPOLOGY AND SPECTRAL GEOMETRY

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   $H^1(M, \mathbb{R}) \times E(M, x) \to \mathcal{L}$
MATHAI- QUILLEN FORM

\( \pi : E \to M \) an oriented rank \( k \) vector bundle,

\( \tilde{\nabla} := (\nabla, \mu) \), \( \nabla \) connection, \( \mu \) parallel Hermitian structure

The form

\[
\Psi(\tilde{\nabla}) \in \Omega^{k-1}(E \setminus 0_M)
\]
defined by:

1. \( (i_b)^*(\Psi(\tilde{\nabla})) := \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \frac{1}{|x|^n} \iota_X Vol \)

where \( X \) is the Euler vector field on \( E_b \) and \( i_b : E_b \setminus 0 \to E \setminus 0_M \) the inclusion.

2. \( \Psi(\tilde{\nabla}) \) compatible with \( \nabla \).

(EULER VECTOR FIELD ON \( \mathbb{R}^n \))
PROPERTIES of $\Psi(\tilde{\nabla})$:

1. $\Psi(\tilde{\nabla})$ is the pullback of a form on $(E \setminus 0_M)/\mathbb{R}_+$

2. $d(\Psi(\tilde{\nabla})) = \pi^*(E(\tilde{\nabla}))$ where $E(\tilde{\nabla}) \in \Omega^k(M)$ is the Euler form of $\tilde{\nabla}$.

3. $\Psi(\tilde{\nabla}_1)) - \Psi(\tilde{\nabla}_2)) = \pi^*(c_{CS}(\tilde{\nabla}_1, \tilde{\nabla}_2))$ where $c_{CS}(\tilde{\nabla}_1, \tilde{\nabla}_2) \in \Omega^{n-1}/d(\Omega^{n-2}(M))$

4. For $M = \mathbb{R}^n$ equipped with $g_{i,j} = \delta_{i,j}$, $E = T(M)$, $\tilde{\nabla}$ the Levi-Civita pair and in the coordinates $x_1 \cdots x_n, \xi_1, \cdots \xi_n$

$$\Psi = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \sum (-1)^i \frac{\xi_i}{(\sum \xi_i^2)^{n/2}} d\xi_1 \cdots d\xi_i \cdots d\xi_n.$$ 

$M$ closed manifold, $g$ Riemannian metric, 
$X : M \to TM \setminus 0_M$ vector field no rest points. 
$\omega \in \Omega^1(M)$ closed form. Define:
\[ \mathcal{R}(X, \omega, g) := (-1)^{n+1} \int_M \omega \wedge X^*(\Psi(\tilde{\nabla}_g)) . \]

**PROPERTIES** of \( \mathcal{R}(X, \omega, g) \).

1: \( \mathcal{R}(X, \omega_1, g) - \mathcal{R}(X, \omega_2, g) = \int_M hE(\tilde{\nabla}_g) \)

if \( \omega_1 - \omega_2 = dh \).

2: \( \mathcal{R}(X, \omega, g_1) - \mathcal{R}(X, \omega, g_2) = \int \omega \wedge c_{CS}(g_1, g_2) \)

3: \( \mathcal{R}(X_1, \omega, g) - \mathcal{R}(X_2, \omega, g) = \int_{\Sigma(X)} p_2^* \omega \)

\( X^1, X^2 \) vector fields with no rest points,

\( \mathbb{X} := \{X_t\}, -1 \leq t \leq 1 \) homotopy,

\( p_2 : M \times [-1, 1] \rightarrow [-1, 1] \Sigma(\mathbb{X}) \) the zero set of \( \mathbb{X} \)

Application to ”mapping torus”.

\( \varphi : V \rightarrow V, \Rightarrow \varphi^i : H^i(V; \mathbb{C}) \rightarrow H^i(V; \mathbb{C}) \)

\( P_i(z) := \text{Characteristic polynomial of } \varphi^i \)

\( \zeta(z) = \frac{\prod_{i=\text{even}} P_i(z)}{\prod_{i=\text{odd}} P_i(z)} \)
\( M = V_\varphi \) the mapping torus,
g Riemannian metric on \( M \),
\( \pi : M \to S^1 \) and \( \omega := \pi^*(dt) \),
\( (\ast) \),
\[
(\Omega^*(M), d_\omega(t))
\]
with \( d_\omega(t) = d + t\omega \wedge \cdot \),
\[
\log T_{an}(M, \omega, g)(t) \text{ the Ray Singer torsion of the complex } (\ast) \text{ equipped with the scalar product induced from } g.
\]

\( X \) any vector field with \( X(\omega) < 0 \)

**Theorem.** For \( t > \sup\{||\lambda|| \lambda \text{ eigenvalue of } \varphi^i\} \)

\[
\log T_{an}(M, \omega, g)(t) = \log \zeta(e^t) + t\mathcal{R}(X, \omega, g).
\]
GEOMETRIC REGULARIZATION.

$X$ vector field with hyperbolic zeros.

$\mathcal{X}$ the set of zeros of $X$

$\omega = \omega' + dh, \quad \omega' = 0$ in the neighborhood of $\mathcal{X}$

$$(-1)^{n+1} R(X, \omega, g; \omega') := \int_{M \setminus \mathcal{X}} \omega' \wedge X^*(\Psi(g))$$

$$- \int h E(\nabla) + \sum_{x \in \mathcal{X}} (-1)^{\text{Ind}(x)} h(x)$$

PROPERTIES of $R(X, \omega, g)$.

1: $R(X, \omega_1, g) - R(X, \omega_2, g) = \int_M h E(\nabla_g) + \sum_{x \in \mathcal{X}} (-1)^{\text{Ind}(x)} h(x)$

2: $R(X, \omega, g_1) - R(X, \omega, g_2) = \int \omega \wedge c_{GS}(g_1, g_2)$

3: $R(X_1, \omega, g) - R(X_2, \omega, g) = \int_{\Sigma(X)} p_2^* \omega$

$X^1, X^2$ vector fields, $\Sigma = \{X_t\}, -1 \leq t \leq 1$

homotopy, $\Sigma(X)$ the zero set of $X$. 
: Application 2:

\( \tau \) smooth triangulation

\( X \) Euler vector field for \( \tau \), i.e.

1) \( \mathcal{X} = \) the set of baricenters
2) \( \text{Ind}(x) = \dim(\sigma_x) \)
3) unstable manifolds = open simplexes.

\( \rho \) representation of \( \pi_1(M) \), \( (E(\rho) \to M, \nabla_\rho) \)
the associated flat bundle,

\( \mu \) a hermitian structure with
\( \omega(\mu) \) the Kamber-Thondeur closed one form

Consider:

\[ \log \text{Tan}(M, \rho, g, \mu), \log \text{TRe}(M, \rho, \tau, \mu), \]

\( X \) Euler vector field.

**Theorem.** *(Bismut Zhang)*

\[ \log \text{Tan}(M, \rho, g, \mu) = \log \text{TRe}(M, \rho, \tau, \mu) + \mathcal{R}(X, \omega(\mu), g) \]
EULER STRUCTURES

\[ E(M, x) = \text{connected components of vector fields } X \text{ with } X = \{x\} \]

\[ H_1(M : \mathbb{Z}) \text{ acts (freely and transitively) on } E(M, x) \]
\[ g \in H_1(M : \mathbb{Z}), e \in E(M, x) \leadsto g \cdot e \in E(M, x) \]

\( \mathcal{L} \) the class of ”all” one dimensional Euclidean space \((L, t)\).

\( \mathbb{R} \) acts freely and transitively on \( \mathcal{L} \)
\[ \alpha \in \mathbb{R}, (L, t) \in \mathcal{L} \leadsto (L, e^{\alpha t}) \]

Theorem.

*There exists a natural pairing*

\[ T : Hq(M, \mathbb{R}) \times E(M, x) \to \mathcal{L} \]

so that \( T(\xi, g \cdot e) = e^{\xi(g)}T(\xi, e) \)