A STRANGE GEOMETRIC INVARIANT WITH APPLICATIONS TO TOPOLOGY AND SPECTRAL GEOMETRY

CONTENTS

MATHAI- QUILLEN FORM

$\pi : E \to M$ an oriented rank k vector bundle,

$\tilde{\nabla} := (\nabla, \mu)$, ∇ connection, μ parallel Hermitian structure

The form

$$\boxed{\Psi(\tilde{\nabla}) \in \Omega^{k-1}(E \setminus 0_M)}$$

defined by:

1. $$\boxed{(i_b)^*(\Psi(\tilde{\nabla})) := \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \frac{1}{|x|^n} \iota_X Vol}$$

where X is the Euler vector field on E_b and

$i_b : E_b \setminus 0 \to E \setminus 0_M$ the inclusion.

2. $\Psi(\tilde{\nabla})$ compatible with ∇.

(**EULER VECTOR FIELD ON** $\mathbb{R}^n$)

PROPERTIES of $\Psi(\tilde{\nabla})$:

1. $\Psi(\tilde{\nabla})$ is the pullback of a form on $(E \setminus 0_M)/\mathbb{R}_+$

2. $d(\Psi(\tilde{\nabla})) = \pi^*(E(\tilde{\nabla})$ where $E(\tilde{\nabla}) \in \Omega^k(M)$ is the Euler form of $\tilde{\nabla}$.

3. $\Psi(\tilde{\nabla}_1)) - \Psi(\tilde{\nabla}_2)) = \pi^*(c_{CS}(\tilde{\nabla}_1, \tilde{\nabla}_2)$ where $c_{CS}(\tilde{\nabla}_1, \tilde{\nabla}_2) \in \Omega^{n-1}/d(\Omega^{n-2}(M))$

4. For $M = \mathbb{R}^n$ equipped with $g_{i,j} = \delta_{i,j}$,

$E = T(M)$, $\tilde{\nabla}$ the Levi-Civita pair and in the coordinates $x_1 \cdots x_n, \xi_1, \cdots \xi_n$

$$\boxed{\Psi = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \sum (-1)^i \frac{\xi_i}{(\sum \xi_i^2)^{n/2}} d\xi_1 \cdots \hat{d\xi_i} \cdots d\xi_n}.$$

M closed manifold, g Riemannian metric,

$X : M \to TM \setminus 0_M$ vector field no rest points.

$\omega \in \Omega^1(M)$ closed form. Define:

$$\boxed{\mathcal{R}(X,\omega,g) := (-1)^{n+1}\int_M \omega\wedge X^*(\Psi(\tilde{\nabla}_g))}.$$

PROPERTIES of $\mathcal{R}(X,\omega,g)$.

$1:\ \mathcal{R}(X,\omega_1,g)-\mathcal{R}(X,\omega_2,g)=\int_M hE(\tilde{\nabla}_g)$
if $\omega_1-\omega_2=dh$.

$2:\ \mathcal{R}(X,\omega,g_1)-\mathcal{R}(X,\omega,g_2)=\int\omega\wedge c_{CS}(g_1,g_2)$

$3:\ \mathcal{R}(X_1,\omega,g)-\mathcal{R}(X_2,\omega,g)=\int_{\Sigma(\mathbb{X})} p_2^*\omega$

X^1, X^2 vector fields with no rest points,

$\mathbb{X} := \{X_t\}, -1\le t\le 1$ homotopy,

$p_2 : M\times[-1,1]\to[-1,1]$ $\Sigma(\mathbb{X})$ the zero set of $\mathbb{X}$

Application to "mapping torus".

$\varphi : V\to V,\ \Rightarrow \varphi^i : H^i(V;\mathbb{C})\to H^i(V;\mathbb{C})$

$P_i(z) :=$ Characteristic polynomial of φ^i

$\zeta(z)=\frac{\prod_{i=even}P_i(z)}{\prod_{i=odd}P_i(z)}$

$M = V_\varphi$ the mapping torus,

g Riemannian metric on M,

$\pi : M \to S^1$ and $\omega := \pi^*(dt)$,

$(*)$, $\boxed{(\Omega^*(M), d_\omega(t))}$

with $d_\omega(t) = d + t\omega \wedge \cdot$,

$\log T_{an}(M, \omega, g)(t)$ the Ray Singer torsion of the complex $(*)$ equipped with the scalar product induced from g.

X any vector field with $X(\omega) < 0$

Theorem. *For* $t > \sup\{|\lambda| | \lambda$ *eigenvalue of* $\varphi^i\}$

$$\boxed{\log T_{an}(M, \omega, g)(t) = \log \zeta(e^t) + t\mathcal{R}(X, \omega, g)}.$$

GEOMETRIC REGULARIZATION.

X vector field with hyperbolic zeros.

$\mathcal{X}$ the set of zeros of X

$\omega = \omega' + dh, \;\; \omega' = 0$ in the neighborhood of $\mathcal{X}$

$(-1)^{n+1}\mathcal{R}(X, \omega, g; \omega') := \int_{M \setminus \mathcal{X}} \omega' \wedge X^*(\Psi(g))$
$- \int hE(\tilde{\nabla}) + \sum_{x \in \mathcal{X}} (-1)^{\mathrm{Ind}(x)} h(x)$

PROPERTIES of $\mathcal{R}(X, \omega, g)$.

$1: \;\; \mathcal{R}(X, \omega_1, g) - \mathcal{R}(X, \omega_2, g) = \int_M hE(\tilde{\nabla}_g)$
$+ \sum_{x \in \mathcal{X}} (-1)^{\mathrm{Ind}(x)} h(x)$

$2: \;\; \mathcal{R}(X, \omega, g_1) - \mathcal{R}(X, \omega, g_2) = \int \omega \wedge c_{CS}(g_1, g_2)$

$3: \;\; \mathcal{R}(X_1, \omega, g) - \mathcal{R}(X_2, \omega, g) = \int_{\Sigma(\mathbb{X})} p_2^* \omega$

X^1, X^2 vector fields, $\mathbb{X} = \{X_t\}, -1 \leq t \leq 1$

homotopy, $\Sigma(\mathbb{X})$ the zero set of $\mathbb{X}$.

: Application 2:

τ smooth triangulation

X Euler vector field for τ, i.e.

1)$\mathcal{X} =$ the set of baricenters
2)$\mathrm{Ind}(x) = \dim(\sigma_x)$
3) unstable manifolds= open simplexes.

ρ representation of $\pi_1(M)$, $(E(\rho) \to M, \nabla_\rho)$

the associated flat bundle,

μ a hermitian structure with
$\omega(\mu)$ the Kamber-Thondeur closed one form

Consider:

$\log T_{an}(M, \rho, g, \mu)$, $\log T_{Re}(M, \rho, \tau, \mu)$,

X Euler vector field.

Theorem. *(Bismut Zhang)*

$$\log T_{an}(M, \rho, g, \mu) = \log_{Re}(M, \rho, \tau, \mu) + \\ + \mathcal{R}(X, \omega(\mu), g)$$

EULER STRUCTURES

$E(M, x)$ = connected components of vector fields X with $\mathcal{X} = \{x\}$

$H_1(M : \mathbb{Z})$ acts (freely and transitively) on $E(M, x)$

$g \in H_1(M : \mathbb{Z}), e \in E(M, x) \;\rightsquigarrow g \cdot e \in E(M, x)$

$\mathcal{L}$ the class of "all" one dimensional Euclidean space (L, t).

$\mathbb{R}$ acts freely and transitively on $\mathcal{L}$

$\alpha \in \mathbb{R}, (L, t) \in \mathcal{L} \;\rightsquigarrow (L, e^{\alpha} t)$

Theorem.

There exists a natural pairing

$T : Hq(M, \mathbb{R}) \times E(M, x) \to \mathcal{L}$

so that $T(\xi, g \cdot e) = e^{\xi(g)} T(\xi, e)$