Graphs Representations and the Topology of Real and Angle valued maps (an alternative approach to Morse-Novikov theory)

Dan Burghelea

Department of mathematics
Ohio State University, Columbus, OH

Tianjin, May 8

Based on joint work with

TAMAL K. DEY
Department of Computer Sciences
Ohio State University, Columbus, OH

STEFAN HALLER
Univ. of Vienna, Austria

Preceeded by work on Persistent Homology of H.Edelsbruner, D.Letscher, A. Zamorodian, G.Carlsson, V. de Silva

Consider TAME maps $f : X \to \mathbb{R}$ and $f : X \to \mathbb{S}^1$

Our theory recovers the topology of the UNDERLYING SPACE X from computable invariants associated with a TAME map f

In our theory the key elements are:
- **critical values, bar codes** between critical values
- **Jordan cells and the canonical long exact sequence .**

Morse theory recovers the topology of the UNDERLYING manifold M from elemets associated with a Morse map $f : M \to \mathbb{R}, f : M \to \mathbb{S}^1$

In Morse theory the key elements are:
critical points, instantons between critical points,
closed trajectories and the Morse complex).

CONTENTS

Topology

- κ a field, $\overline{\kappa}$ its algebraic closure.
- $\kappa[t, t^{-1}]$ resp. $\kappa[[t, t^{-1}]$ the ring resp. field of Laurent polynomials resp. formal power series. $\kappa[t, t^{-1}] \subset \kappa[[t, t^{-1}]$.
- X be a compact ANR.
- $H_r(X)$ the singular homology with coefficients in κ *Betti numbers*

$$\boxed{\beta_r(X) = \dim H_r(X).}$$

- $(X, \xi), \xi \in H^1(X, \mathbb{Z})$.

  $\tilde{X} \to X$ the infinite cyclic cover associated with ξ.
  $T : \tilde{X} \to \tilde{X}$ the deck transformation.

- $H_r(\tilde{X})$ is a $\kappa[t, t^{-1}]$ module with the multiplication by t induced by T.

- $NH_r(X, \xi) = H_r(\tilde{X}) \otimes_{\kappa[t,t^{-1}]} \kappa[[t, t^{-1}]$ the Novikov homology is a vector space over the field $\kappa[[t, t^{-1}]$.
  Novikov- Betti numbers

$$\boxed{\beta N_r(X; \xi) = \dim_{\kappa[[t,t^{-1}]} NH_r(X; \xi)}.$$

- Consider $V(\xi) := \ker\{H_r(\tilde{X}) \to NH_r(X;\xi)\}$ induced by tensoring $H_r(\tilde{X})$ by $\kappa[t,t^{-1}] \to \kappa[[t,t^{-1}]$.
- $V(\xi)$ is a finite dimensional vector space over κ
  $T(\xi) : V(\xi) \to V(\xi)$ is induced by the multiplication by t, $\kappa-$linear isomorphism. The *monodromy* associated with ξ is the pair

$$\boxed{(V(\xi), T(\xi))}.$$

Tame maps

Definition

A continuous map $f\colon X \to \mathbb{R}$ resp. $f\colon X \to S^1$, X a compact ANR, is *tame* if:

1. Any fiber $X_\theta = f^{-1}(\theta)$ is the deformation retract of an open neighborhood.
2. Away from a finite set of numbers/angles $\Sigma = \{\theta_1, \cdots, \theta_m\} \subset \mathbb{R}$, resp. S^1 the restriction of f to $X \setminus f^{-1}(\Sigma)$ is a fibration.

To $f : X \to \mathbb{R}$ resp. $f : X \to \mathbb{S}^1$ a tame map one associates via *graph representations* **bar codes** resp. **bar codes** and **Jordan cells** .

Bar codes and Jordan cells

Bar codes are finite intervals I of real numbers of four types:

1. Type 1, closed , $[a, b]$ with $a \leq b$,
2. Type 2, open, (a, b) with $a < b$,
3. Type 3, left open right closed$(a, b]$ with $a < b$,
4. Type 4, left closed right open $[a, b)$ with $a < b$.

Jordan cells are pairs $J = (\lambda \in \overline{\kappa}, n \in \mathbb{Z}_{\geq 0}) = (\overline{\kappa}^n, T(\lambda, n)$ It should be interpreted as a a matrix

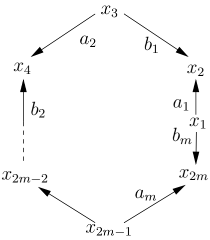
$$T(\lambda, n) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{pmatrix}$$

Graphs and graph representations

The oriented graph $\mathcal{Z}$:

$$\cdots \xleftarrow{b_{i-1}} x_{2i-1} \xrightarrow{a_i} x_{2i} \xleftarrow{b_i} x_{2i+1} \xrightarrow{a_{i+1}} x_{2i+2} \xleftarrow{b_{i+1}} \cdots$$

The oriented graph G_{2m} :



A $\mathcal{Z}$ or a G_{2m} representation ρ is given by

$$\rho = \begin{cases} x_i \rightsquigarrow V_i \\ a_i \rightsquigarrow \alpha_i : V_{2i-1} \rightarrow V_{2i} \\ b_i \rightsquigarrow \beta_i : V_{2i+1} \rightarrow V_{2i} \end{cases}$$

A finitely supported $\mathcal{Z}$-representation[1], or a G_{2m}-representation ρ can be uniquely decomposed as a sum of **indecomposable representations.**

For the graph $\mathcal{Z}$ the indecomposable representations with finite support are indexed by the four types of *bar codes* intervals with ends $i, j \in \mathbb{Z}$: **closed, open,** **left-closed right-open** and **left-open right-closed.**

[1] i.e. all but finitely many vector spaces V_x have dimension zero

The indecomposable $\mathcal{Z}-$representations with finite support :

1. $\rho([i,j]), i \leq j$ has $V_r = \kappa$ for $r = \{2i, 2i+1, \cdots 2j\}$ and $V_r = 0$ if $r \neq [2i, 2j]$,
2. $\rho((i,j)), i < j$ has $V_r = \kappa$ for $r = \{2i+1, 2i+2, \cdots 2j-1\}$ and $V_r = 0$ if $r \neq [2i+1, 2j-1]$,
3. $\rho((i,j]), i < j$ has $V_r = \kappa$ for $r = \{2i+1, 2i+2, \cdots 2j\}$ and $V_r = 0$ if $r \neq [2i+1, 2j]$,
4. $\rho([i,j)), i < j$ has $V_r = \kappa$ for $r = \{2i, 2i+1, \cdots 2j-1\}$ and $V_r = 0$ if $r \neq [2i, 2j-1]$,

with all α_i and β_i the identity provided that the source and the target are both non zero.

For G_{2m} the indecomposable representations are indexed by similar intervals (bar codes) with ends $i, j + mk$, $1 \le i, j \le m$, $k \in \mathbb{Z}_{\geq 0}$, $i \le j$ with $1 \le i \le m$ and by Jordan cells.

bar codes: For any triple of integers $\{i, j, k\}$, $1 \le i, j \le m$, $k \ge 0$, we have the representations denoted by

1. $\rho^I([i, j]; k) \equiv \rho^I([i, j + mk])$,
2. $\rho^I((i, j]; k) \equiv \rho^I((i, j + mk])$,
3. $\rho^I([i, j); k) \equiv \rho^I([i, j + mk))$,
4. $\rho^I((i, j); k) \equiv \rho^I((i, j + mk))$,

Jordan cell:

$$\rho^{II}(\lambda, k) = \{V'_r = \lambda^k, \alpha'_1 = T(\lambda, k), \alpha'_i = Id \ i \neq 1, \ \beta'_i = Id\}.$$

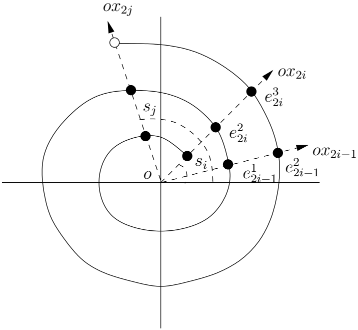


Figure: The spiral for $[i, j + 2m)$.

Notations: For a $\mathcal{Z}$ or $G_{2m}-$ representation ρ denote by

- $\mathcal{B}^c(\rho)$ closed bar codes
- $\mathcal{B}^o(\rho)$ open bar codes
- $\mathcal{B}^{co}(\rho)$ left closed right open bar codes
- $\mathcal{B}^{oc}(\rho)$ left open right closed bar codes and
- $\mathcal{B}(\rho) = \mathcal{B}^c \sqcup \mathcal{B}^o \sqcup \mathcal{B}^{co} \sqcup \mathcal{B}^{oc}$ all bar codes

For a G_{2m} representation ρ denote by

- $\mathcal{J}(\rho)$ the collection of all Jordan cells.

Invariants for a tame map

Let f be a tame map.

• Consider
(real valued) the critical values $-\infty < \theta_1 < \theta_2 < \cdots < \theta_m < \infty$

(angle valued) the critical angles $0 < \theta_1 < \cdots < \theta_m \leq 2\pi$.

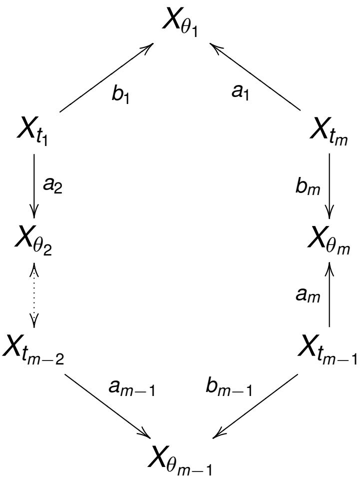
• Choose
t_i, $i = 1, 2, \ldots, m$, with $\theta_1 < t_1 < \theta_2 < \cdots < t_{m-1} < \theta_m < t_m$, and for f angle valued $2\pi < t_m < \theta_1 + 2\pi$.

•The tameness of f

- when real valued induces the diagram :

$$\cdots \xleftarrow{b_{i-1}} X_{t_{i-1}} \xrightarrow{a_i} X_{\theta_i} \xleftarrow{b_i} X_{t_i} \xrightarrow{a_{i+1}} X_{\theta_{i+1}} \xleftarrow{b_{i+1}} \cdots$$

- when angle valued induces the diagram:



• For $r \leq \dim X$ let $\rho_r(f)$ be the $\mathcal{Z}$- resp. G_{2m}-representation defined by:

$$\boxed{V_{2i} = H_r(X_{\theta_i}), V_{2i+1} = H_r(X_{t_i})}$$

$$\boxed{\alpha_i : V_{2i-1} \to V_{2i}, \quad \beta_i : V_{2i+1} \to V_{2i}}$$

induced by the continuous maps a_i and b_i.

• Consider the decomposition of ρ_r in indecomposable components.

• Convert the intervals $\{i, j\}$ into $\{\theta_i, \theta_j\}$ (for f real valued) resp. the intervals $\{i, j + km\}$, $1 \leq i, j \leq m$, into $\{\theta_i, \theta_j + 2\pi k\}$ (for f angle valued) .

• Denote the set of these intervals whose ends are critical values/angles by $\mathcal{B}_r(f)$ and the sets of Jordan cells by $\mathcal{J}_r(f)$.

The $r-$invariants.

Definition

1. For $f : X \to \mathbb{R}$ the sets $\mathcal{B}_r(f)$, $r = 1, 2, \cdots \dim X$ are the r-invariants of the map f.

2. For $f : X \to \mathbb{S}^1$ the sets $\mathcal{B}_r(f)$, and $\mathcal{J}_r(f)$, $r = 1, 2, \cdots \dim X$ are the r-invariants of the map f.

Call the pair $(V_r(f), T_r(f)) = \bigoplus_{(\lambda,k)\in \mathcal{J}_r(f)} (\overline{\kappa}^k, T(\lambda, k))$ the *r-monodromy* of the angle valued f.

Note One has two types of monodromy :

associated to an angle valued map derived from Jordan cells,

associated to $\xi \in H^1(X; \mathbb{Z})$ via algebraic topology.

The main results

Theorem

If $f\colon X \to \mathbb{R}$ is a tame map and $X_t = f^{-1}(t)$ then

1. $\beta_r(X_t) = \sharp\{I \in \mathcal{B}_r(f) \mid I \ni t\}$
2. $\dim \mathrm{im}\big(\mathrm{H_r(X_t)} \to \mathrm{H_r(X)}\big) = \sharp\{I \in \mathcal{B}^c_r(f) \mid I \ni t\}$
3. $\beta_r(X) = \sharp\mathcal{B}^c_r(f) + \sharp\mathcal{B}^o_{r-1}(f)$

Definition

For an interval $I \subset \mathbb{R}$ and an angle $\theta \in (0, 2\pi]$ denote by $n_\theta(I) = \sharp\{k \in \mathbb{Z} \mid \theta + 2\pi k \in I\}$. For $J \in \mathcal{J}_r$ write $J = (\lambda(J), k(J)$

Theorem

If $f\colon X \to S^1$ *is a tame map,* $\xi_f \in H^1(X;\mathbb{Z})$ *represents* f *and* $X_\theta = f^{-1}(\theta)$ *then:*

1. $\beta_r(X_\theta) = \sum_{I \in \mathcal{B}_r(f)} n_\theta(I) + \sum_{J \in \mathcal{J}_r(f)} k(J)$
2. $\dim im\big(H_r(X_\theta) \to H_r(X)\big) = \sharp\big\{I \in \mathcal{B}^c_r(f) \;\big|\; \theta \in I\big\} + \cdots$
3. $\beta_r(X) = \begin{cases} \sharp\mathcal{B}^c_r(f) + \sharp\mathcal{B}^o_{r-1}(f) + \\ \sharp\big\{(\lambda,k) \in \mathcal{J}_r(f) \;\big|\; \lambda = 1\big\} + \\ \sharp\big\{(\lambda,k) \in \mathcal{J}_{r-1}(f) \;\big|\; \lambda = 1\big\} \end{cases}$
4. $\beta N_r(X;\xi_f) = \sharp\mathcal{B}^c_r(f) + \sharp\mathcal{B}^o_{r-1}(f).$

Theorem

1. $V_r(\xi_f) := \ker(H_r(\tilde{X}) \to H_r^N(X; \xi_f))$ *is a finite dimensional* κ*-vector space and* $(V_r(\xi_f) \otimes \overline{\kappa}, T_r(\xi_f)) \otimes \overline{\kappa}) = (V_r(f), T_r(f))$
2. $H_r(\tilde{X}) = \kappa[t^{-1}, t]^N \oplus V_r(\xi_f)$ *as* $\kappa[t^{-1}, t]$*-modules with* $N = \beta N_r(f) = \sharp\mathcal{B}_r^c(f) + \sharp\mathcal{B}_{r-1}^o(f)$.

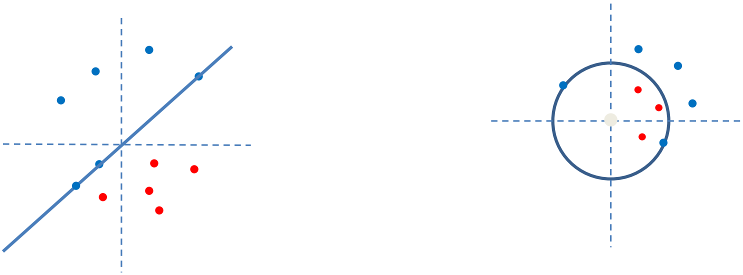
Remark:

- For f real valued the numbers $\boxed{\sharp\mathcal{B}^c_r + \sharp\mathcal{B}^o_{r-1}}$ $(= \beta_r(X))$ are homotopy invariants.
- For f circle valued te numbers $\boxed{\sharp\mathcal{B}^c_r + \sharp\mathcal{B}^o_{r-1}}$ $(= \beta N_r(X)$ and the collections $\boxed{\mathcal{J}_r(f)}$ are homotopy invariants.

Define $Cr_r(f)$ as a configuration of points in:

• $\mathbb{C}$ (for $f : X \to \mathbb{R}$) consisting of
$r-$ closed bar code $[s, s'] \rightsquigarrow z = s + s'i \in \mathbb{C}$
$(r-1)-$ open bar code $(s, s') \rightsquigarrow z = s' + is \in \mathbb{C}$

• $\mathbb{C} \setminus 0$ (for $f : X \to \mathbb{S}^1$) consisting of
$r-$ closed bar code $[s, s'] \rightsquigarrow z = e^{(s'-s)+2\pi si} \in \mathbb{C} \setminus 0$
$(r-1)-$ open bar code $(s, s') \rightsquigarrow z = e^{(s-s')+2\pi s'i} \in \mathbb{C} \setminus 0$



blue $r-$closed barcodes, red $(r-1)-$open bar codes

$Cr(f)$ can be regarded as points in the symmetric product $S^{\beta_r(X)}(R^2)$ resp. $S^{\beta N_r(X;\xi_f)}(\mathbb{C} \setminus 0)$.

$S^N(M) = (\overbrace{M \times M \times \cdots M}^{N})/\Sigma_N$, Σ_N - the $N-$symmetric group.

$C^0_{\rm tame}(M;\mathbb{R})$ resp.$C^0_{\rm tame}(M;S^1)$ the space of tame maps dense in the space $C^0(M;\mathbb{R})$ resp.$C^0(M;S^1)$ of all continuous maps.

Theorem

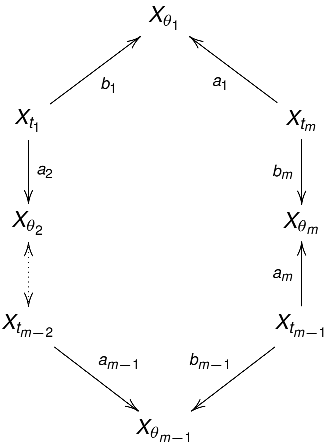
The assignments $f \rightsquigarrow Cr_r(f)$ is a continuous map on $C^0_{\rm tame}(M;\mathbb{R})$ resp. $C^0_{\rm tame}(M;\mathbb{S}^1)$ hence has a continuous extension to the entire $C^0(M;\mathbb{R})$ resp. $C^0_{\rm tame}(M;\mathbb{S}^1)$.

As a consequence

- The closed and open bar codes as well as the Jordan cells can be defined for any continuous maps.
- The monic polynomials $P_r(f)(z)$ whose roots are the points of $Cr(f)$are well defined for any continuous map and the assignment $f \rightsquigarrow P_r(f)(z)$ is continuous.
- The collection $\mathcal{J}(f)_r$ remains constant on a connected component of $C^0(M; \mathbb{S}^1)$.

About proof

- Start with $f : X \to S^1 \Rightarrow$



- Consider

$\mathcal{R} = \sqcup_{1 \le i \le m} X_{t_i}, \quad \mathcal{X} = \sqcup_{1 \le i \le m} X_{s_i}.$

• Derive
the long exact sequence (**the canonical sequence**)

$$\cdots \to H_r(\mathcal{R}) \xrightarrow{M(\rho_r)} H_r(\mathcal{X}) \to H_r(X) \to H_{r-1}(\mathcal{R}) \xrightarrow{M(\rho_{r-1})} H_{r-1}(\mathcal{X}) \to \cdot$$

with

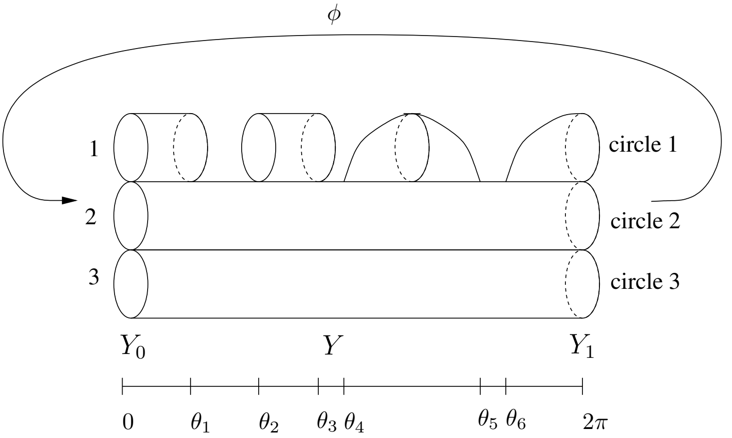
$$M(\rho_r) = \begin{pmatrix} \alpha_1^r & -\beta_1^r & 0 & \cdots & 0 \\ 0 & \alpha_2^r & -\beta_2^r & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_{m-1}^r & -\beta_{m-1}^r \\ -\beta_m^r & 0 & \cdots & 0 & \alpha_m^r \end{pmatrix}.$$

$\alpha_i^r : H_r(R_i) \to H_r(X_i)$ and $\beta_i^r : H_r(R_{i+1}) \to H_r(X_i)$ induced by the maps a_i and b_i.

Examples

Spaces : Y map $\varphi : Y_0 \to Y_1 \Rightarrow X := Y_\varphi$.

Maps: $p : Y \to [0, 2\pi] \subseteq \mathbb{R}, \quad f : X \to S^1, \quad \varphi = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$.



r-invariants for p	
dimension	bar codes
0	$[0, 2\pi]$
1	$[0, 2\pi]$ $[0, 2\pi]$ $[0, \theta_1]$ $[\theta_2, \theta_3]$ (θ_3, θ_5) $(\theta_6, 2\pi]$

Table 1:

r-invariants of f		
dimension	bar codes	Jordan cells
0		$(1, 1)$
1	$(\theta_6, \theta_1 + 2\pi]$ $[\theta_2, \theta_3]$ (θ_4, θ_5)	$(2, 2)$

Table 2.

Calculations:

Input

• Record X as a square $N \times N$ matrix, N the total number of simplicee of X with entries the numbers $I(\tau, \sigma)$, and record the values of f on vertices as an additional row.

$$\Downarrow$$

Output 1:

• $\rho_r = \{\alpha_r, \beta_r\}$

$$\Downarrow$$

Output 2

• A table of boxes with two columns (bar codes, Jordan cells) and dim X rows : $r = 1, 2, \cdots \dim X$.

The meaning of bar codes

$0 \neq x \in H_r(X_t)$:

- dead (right) at $t' > t$ if image in $H_r(X_{[t,t']})$ vanishes,
- dead (left) at $t'' < t$ if image in $H_r(X_{[t'',t]})$ vanishes,
- observable at t' if image in $H_r(X_{[t,t']})$ is not trivial and lies in the image of $H_r(X'_t)$.

$N\{s_i, s_j\} = \sharp\{$ maximal number of linearly independent elements in $H_r(X_t)$ which are dead/ observable at s_i/s_j but not before/after $s_i/s_j.\}$

REFERENCES

1. D. Burghelea and T. K. Dey, *Persistence for circle valued maps.* (arXiv:1104.5646), 2011.

2. D. Burghelea and S. Haller, *Graph representations and the topology of real and angle valued maps,* (arXiv:1204.5646), 2012.

3. D. Burghelea, *On the bar codes of continuous real and angle valued maps.* (in preparation)

4. G. Carlsson, V. de Silva and D. Morozov, *Zigzag persistent homology and real-valued functions,* Proc. of the 25th Annual Symposium on Computational Geometry 2009, 247–256.