Dynamics and Morse-Novikov theory

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Dynamics on smooth manifolds

M^n (closed) smooth manifold
X a smooth vector field on M
X generates a flow $\mu^X : \mathbb{R} \times M \to M$ ∴

$$\mu(t, \mu(s, x)) = \mu(t + s, x)$$

Elements of dynamics:

1. **Rest points** $R(X) := \{x \in M \mid X(x) = 0\}$
2. **Instantons** from a rest point x to rest point , $I(x, y)$.
3. **Closed trajectory**

- **Rest - points** For $x \in R(X)$one has
  - W_x^- unstable set
  - W_x^+ stable set

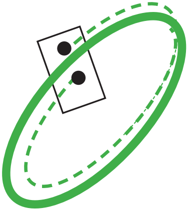
  For generic vector field and $x \in R(X)$ one has:
  $W_x^- \sim \mathbb{R}^{i(x)}$
  $W_x^+ \sim \mathbb{R}^{n-i(x)}$
  $i(x)$ the **Morse index** of x.

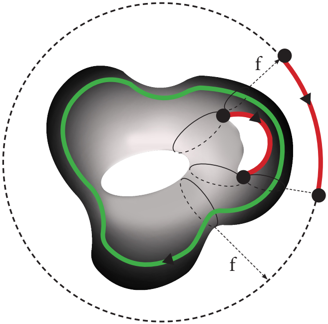
- **Instantons**= (visible trajectories)
  For generic vector field if instantons from x to y exist then $i(x) - i(y) = 1$ and

$$I(x, y) = W_x^- \cap W_y^+ / \mathbb{R}.$$

  If an orientation on each W_x^- is provided then each instanton $\gamma \in I(x, y)$ has a sign, $\boxed{\epsilon(\gamma) = \pm 1.}$

- **Closed trajectories** Each closed trajectory γ has a Poincaré return map, a linear isomorphism $T_\gamma; V_\gamma \to V_\gamma$ unique up to conjugation. The closed trajectory γ is *isolated* and therefore visible iff T has no eigenvalue 1.





The main challenge:
Count the set of rest points, the set of instantons, the set of visible closed trajectories.

Morse-Novikov theory considers the problem under an additional Hypothesis:

Lyapunov condition: There exists a differentiable closed one form $\omega \in \mathcal{Z}^1(M)$ s.t. $\omega(X)(x) < 0$ for $x \in M \setminus R(X)$ and $\omega(x) = 0$ iff $x \in R(X)$.

Notaions

Note:

• $\omega \Rightarrow \xi_\omega \in H^1(M;\mathbb{R})$
$\xi \in H^1(M;\mathbb{R}) \Rightarrow \ \Gamma_\xi := img(H_1(M;\mathbb{Z}) \to \mathbb{R}) \simeq \mathbb{Z}^{n(\xi)}$,
$n(\xi)$ **degree of irrationality.**

• $\xi \in H^1(M;\mathbb{Q}) \Rightarrow n(\xi) \leq 1 \Rightarrow \Gamma_\xi$ **discrete subset** of $\mathbb{R}$.

• $\xi \in H^1(M;\mathbb{R}) \setminus H^1(M;\mathbb{Q}) \Rightarrow n(\xi) > 1 \Rightarrow \Gamma_\xi$ **dense subset** of $\mathbb{R}$.

For a generic set of pairs (X, ω) as above one has:

1. The set $R(X)$ is finite,
2. Any $x \in R(X)$ has a Morse index $i(x)$
3. The set of instantons $I(x, y)$ if nonempty implies $i(x) - i(y) = 1$,
4. Each homotopy class $u \in \prod I(x, y)$ of paths from x to y, $x, y \in R(X)$, $\prod(x, y) =$ contains a finite number (possibly zero) of elements in $I(x, y)$ (i.e. for $i(x) - i(y) = 1$
5. Each homotopy class of closed curves contains a finite number (possibly zero) of closed trajectories.

Note: For $\omega \in \xi \in H^1(M; \mathbb{R})$ Novikov has defined

1. a field $\kappa(\xi)$, a completion of the algebra $\kappa[\Gamma_\xi]$, based on the embedding $\Gamma_\xi \subset \mathbb{R}$.
2. for a generic pair (X, ω) a chain complex of finite dimensional $\kappa(\xi_\omega))-$vector spaces

$$C(X, \omega) := \{C_r(X), \partial_r(X, \omega) : C_r(X) \to C_{r-1}\}$$

- $C_r(X) := \kappa(\xi_\omega)[\mathcal{X}_r]$ $\mathcal{X}_r$, the set of rest points of X= critical points of ω of Morse index r
- ∂_r : the matrix with entries $\partial_r(x, y)$, obtained by interpreting the formal sums $\sum_{u \in \prod(x,y)} \boxed{(\sum_{\gamma \in u} \epsilon(\gamma))}$[1] as elements of $\kappa(\xi_\omega)$.

[1] $\sum_{\gamma \in u} \epsilon(\gamma) \in \mathbb{Z}$

The main results of MN- theory state:

- $H_r(C(X,\omega)) \simeq H_r^N(X,\xi_\omega;\kappa)$,
- Up to a non canonical isomorphism the chain complex $C(x,\omega)$ depends only on ω,
- Up to a non canonical isomorphism the chain complex depends on two of the following systems of numbers: $c_r = \dim C_r$, $\beta_r = \dim H_r(C(X,\omega))$ and $d_r = rank\partial_r$ related by $c_r = \beta_r + d_r + d_{r-1}$.

The numbers c_r, β_r, d_r do not involve directly the field $\kappa(\xi_\omega)$ nor $H_r^N(M;\xi)$.

- It will be useful if all these three numbers can be calculated from a triangulation of M and the 1-simplicial cocycle induced from ω on the triangulation, data computer implementable inputs.
- It will be useful to extend MN theory from

$$\begin{cases} manifolds \\ vector\ field \\ Lyapunov\ closed\ one\ form \end{cases} \quad \text{to}$$

$$\begin{cases} nice\ space\ (\ compact\ ANR\ ) \\ flows\ on\ thespace \\ TC1-\ form = one\ cocycle\ of\ the\ space. \end{cases}$$

TC1-forms

- **multivalued map on X** $\{U_\alpha, f_\alpha : U_\alpha \to \mathbb{R}\}$ s.t. $\cup_\alpha U_\alpha = X$, $f_\alpha - f_\beta|_{U-\alpha \cap U_\beta} = constant$
- **equivalent multivalued maps** $\{U_\alpha, f_\alpha\} \sim \{V_\beta, g_\beta\}$ iff $\{U_\alpha, V_\beta, f_\alpha, g_\beta\}$ is a multivalued map
- A **TC1- form** ω is an equivalence class of multivalued maps.
- The set of all TC1-forms $\mathcal{Z}^1(X)$ is a $\mathbb{R}-$ vector space equipped with the uniform convergence topology

- The TC1-form ω defines:

1. The homomorphism $\underline{\omega} : H_1(X; \mathbb{Z}) \to \mathbb{R}$, equivalently $\xi_\omega \in H^1(X : \mathbb{R})$,
2. The group $\Gamma = img(\underline{\omega}) \subset \mathbb{R}$, $\Gamma \simeq \mathbb{Z}^n$, $n = n(\xi_\omega)$ = the degree of irrationality of ξ_ω,
3. The $\Gamma-$principal covering $\pi : \tilde{X} \to X$,
4. The collections of real-valued $\Gamma-$equivariant maps $f : \tilde{X} \to \mathbb{R}$ which differ by a constant. Write $f \in \omega$.

- The TC1-form ω is tame if $f^{-1}(t)$ is an ANR for $f \in \omega$.

- Denote by $\mathcal{Z}^1(X; \xi) := \{\omega \in \mathcal{Z}(X) \mid \xi_\omega = \xi\}$

Configurations

- Y topological space

$\mathcal{C}onf(Y) := \{\delta : Y \to \mathbb{Z}_{\geq 0} \mid \sharp(\text{ support } \delta) < \infty\}$

- $K \subset Y$, K closed

$\mathcal{C}onf(Y \setminus K)$ is equipped with the **bottleneck topology** depending on the embedding $K \subset Y$.

- When $K = \emptyset$ this topology on $\mathcal{C}onf(Y)$ is referred to as the **collision topology**

For any $\omega \in \mathcal{Z}^1(X)$ one constructs the configurations

$$\boxed{\underline{\delta}_r^\omega : \mathbb{R} \to \mathbb{Z}_{\geq 0}} \textit{ and } \boxed{\underline{\gamma}_r^\omega : \mathbb{R}_{>0} \to \mathbb{Z}_{\geq 0}}$$

which satisfy:

Theorem

1. $\sum_{t \in \mathbb{R}} \underline{\delta}_r^\omega(t) = \beta_r$
2. $\sum_{t \in \mathbb{R}_{>0}} \underline{\gamma}_r^\omega(t) = d_r$
3. $c_r = \beta_r + d_r + d_{r-1}$

Theorem

1. *The assignment* $\mathcal{Z}^1(M;\xi) \rightsquigarrow \underline{\delta}_r^\omega \in \mathcal{C}onf(R)$ *is continuous.*
2. *The assignment* $\mathcal{Z}^1(M:\xi) \rightsquigarrow \underline{\delta}_r^\omega \in \mathcal{C}onf(R)$ *is continuous.*

Theorem

Suppose that M^n *is a closed topological manifold. Then*

1. $\underline{\delta}_r^\omega(t) = \underline{\delta}_{n-r}^\omega(-t)$
2. $\underline{\gamma}_r^\omega(t) = \underline{\gamma}_{n-r-1}^{-\omega}(t)$

Let $\omega \in \mathcal{Z}^1(X)$ tame and choose $f \in \omega$.
Denote

1. $$\mathbb{F}_r(a,b) := \begin{cases} img(H_r(f^{-1}((-\infty,a]);\kappa) \to H_r(\tilde{X};\kappa)) \\ \cap \\ img(H_r(f^{-1}([b,\infty));\kappa) \to H_r(\tilde{X};\kappa)) \end{cases}$$
2. $\mathbb{F}_r((a-\epsilon]\times[b,b+\epsilon)) := \mathbb{F}_r(a,b)/(\mathbb{F}_r(a-\epsilon,b)+\mathbb{F}_r(a,b+\epsilon))$
3. $\hat{\delta}_r(a,b) := \varinjlim_{\epsilon\to 0} \mathbb{F}_r((a-\epsilon]\times[b,b+\epsilon))$

Define

- $\delta_r(a,b) := \dim \hat{\delta}_r(a,b)$
- $$\boxed{\underline{\delta}_r^\omega(t) := \sum_{b-a=t} \delta_r(a,b)}$$

Denote

1. $\mathbb{T}_r(a,b); = \ker(H_r(f^{-1}((-\infty,a]);\kappa) \to H_r(f^{-1}((-\infty,b];\kappa))$ for $a < b$
2. $\mathbb{T}_r((a-\epsilon]\times[b,b-\epsilon)) := \mathbb{T}_r(a,b)/(\mathbb{T}_r(a-\epsilon,b)+\mathbb{T}_r(a,b+\epsilon))$ for $a < b-\epsilon < b$
3. $\hat{\gamma}_r(a,b) := \varinjlim_{\epsilon\to 0} \mathbb{T}_r((a-\epsilon]\times(b-\epsilon,b])$

Define

- $\gamma_r(a,b) := \dim \hat{\gamma}_r(a,b)$
- $\boxed{\underline{\gamma}_r^{\omega}(t) := \sum_{b-a=t} \gamma_r(a,b)}$

Note that:

- $\underline{\delta}(a, b) = \underline{\delta}(a + g, b + g)$ and
  $\underline{\gamma}(a, b) = \underline{\gamma}(a + g, b + g)$ for $g \in \Gamma$
- When $n(\xi) \leq 1$ f is a proper tame map hence
  the vector spaces items (1)and (2) are f.d. , δ_r and γ_r have discrete support and $\underline{\delta}_r$ and $\underline{\gamma}_r$ are configurations. (not hard to established)
- When $n(\xi) > 1$ none of the above hold but
  the vector space item (3) is f.d., the supports of δ_r and γ_r intersected with any line $\{(x, y) \mid y - x = t\}$ or with any horizontal line in the plane is finite
  which makes $\underline{\delta}_r$ and $\underline{\gamma}_r$ well defined and configurations. (difficult)

Suppose X a finite simplicial complex an $f : \tilde{X} \to \mathbb{R}$ (representing ω) a simplicial map.

• When $n(\xi_\omega) \leq 1$ the calculation of δ_r and γ_r is described in [1] and [2].

• When $n(\xi_\omega) > 1$, by Theorem 2, $\underline{\delta}_r^\omega$ and $\underline{\gamma}_r^\omega$ are arbitrary closed to $\underline{\delta}_r^{\omega'}$ and $\underline{\gamma}_r^{\omega'}$ with $n(\xi_{\omega'}) = 1$.

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