Witten Deformation and analytic continuation

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Motivation

Witten deformation permits to select a FINITE SEGMENT of the spectrum of the Laplace-Beltrami operators (eigenvalues, eigenforms) of a closed Riemannian manifold (M, g) when additional data (Morse function/closed one form) are given. It is hoped that this segment is TOPOLOGICALLY RELEVANT and COMPUTABLE.

Notation

• When one writes $A(t)$ one understands a real, or complex or vector valued analytic function in $t \in \mathbb{R}$.

• When one writes $A(z)$ one understands a complex or vector valued holomorphic=analytic function in an open neighborhood of R in $\mathbb{C}$.

Witten deformation

The Witten Deformation is a one parameter family ($t \in \mathbb{R}$ or $\mathbb{C}$) of de Rham type complexes $\boxed{(\Omega^*(M), d^*(t))}$ where

$$\begin{gathered} d^*(t) = e^{-tf} d^*(e^{tf} \cdots) = d^* + tdf \wedge \cdots \\ f : M \to \mathbb{R} \end{gathered} \tag{1}$$

or more general

$$\begin{gathered} d^*(t) = d^* + t\omega \wedge \cdots \\ \omega \in \Omega^1(M),\ d\omega = 0 \end{gathered} \tag{2}$$

The Witten Laplacians

In the presence of a Riemannian metric g, the family $(\Omega^*(M), d^*(t))$ produces the family $\Delta_q(z)$ of elliptic operators on $\Omega^*(M)$ defined by

$$\boxed{\Delta_q(z) = \Delta_q + z(L_X + L_X^\star) + z^2||X||^2.}$$

Essential features:

- $\Delta_q(z)$ is a polynomial with coefficients self-adjoint operators
- For $t \in \mathbb{R}$, $\Delta_q(t) = (d^{q+1}(t)^* \cdot d^q(t) + d^{q-1}(t) \cdot (d^q(t))^*$ is zero order perturbation of the standard laplacian Δ_q.

Here

- L_X is the Lie derivative w. r. to the vector field $X = -\mathrm{grad_g f}$,
- $L_X^\star$ its formal adjoint .
- $L_X + L_X^*$ is of order zero.

$\Delta_q(t)$ is zero order perturbation of Δ_q

Theorem

(Relich, Kato)
There exist the real valued analytic functions $\lambda_i^q(t)$ *and of vector valued analytic functions* $\omega_i(t) \in \Omega^q(M)$ $i = 1, 2, \cdots$, $t \in \mathbb{R}$, *each with a holomorphic extension* $\lambda_i^q(z), \omega_i^q(z)$ *to an open neighborhood of* $\mathbb{R}$ *inside* $\mathbb{C}$, *so that:*

1. *each* $\lambda_i(z)$ *is an eigenvalue of* $\Delta_q(z)$ *and* $\lambda_i(z)$ *exhaust all eigenvalues*
2. $\omega_i(z)$ *are eigenforms corresponding to the eigenvalue* $\lambda_i(z)$, *of norm* 1 *for* $z = t$ *real number*

$\lambda_i^q(z)$ and $\omega_i^q(z)$ are referred to as *branches* (of eigenvalues and eigenforms).

The case of Morse functions/forms

Suppose that f or ω has all critical points non degenerate.

Theorem

(Witten) If (M^n, g) is a closed Riemannian manifold then there exist the constants $C_1, C_2, C_3, T_0 > 0$ so that for $t \geq T_0$

(1) $spec\Delta_q(t) \cap \boxed{(C_1 e^{-C_2 t}, C_3 t)} = \emptyset$, and

(2) the number of eigenvalues of $\Delta_q(t)$ counted with multiplicity in the interval $[0, C_1 e^{-C_2 t}]$ is equal to N_q, the number of critical points of index q.

Important consequences:

1. Exactely N_q branches $\lambda_i^q(t),\ i = 1, 2, \cdots N_q$ go exponentially fast to 0 = the **virtually small branches**, while all others go at least linearly fast to ∞ = the **large branches** (but no more than quadratically fast).
   Of them exactly β_q (β_q^N) are identically zero.
2. They define a canonical orthogonal decomposition

$$(\Omega^*(M), d^*(t)) = (\Omega^*(M)_{sm}(t), d^*(t)) \oplus (\Omega^*(M)_{la}(t), d^*(t)),$$

real analytic in t.

Ultimately

• Ω_{sm} extends to $\Omega^*_{sm}(z), d^*(z)$ a holomorphic family of finite dimensional complexes quasi-isomorphic to $\Omega^*(M), d^*(z)$.
$(\Omega(M)_{sm}, d(z))$, is the span of the eigen-forms corresponding to the virtually small branches

• $(\Omega_{la}(t), d^*(t))$ is a real analytic family (for $t \in \mathbb{R}$) of acyclic complexes.
$(\Omega(M)_{la}, d(t))$ is the span of the eigen-forms corresponding to large branches.

Theorem

For a residual set of smooth functions/ closed one form (in $C^r, r \geq 2$)topology

1. *each not identically zero virtually small branch of eigenvalues of is simple,*
2. *exactly β_q resp. β_q^N (Novikov Betti numbers) are identically zero and*
3. *there is a canonical (up to multiplication by ± 1) choices of virtually small branches of eigenforms, which are orthonormal, hence a canonical base (up to multiplication by ± 1) of $\Omega^*(M)_{sm}(z), d^*(z)$.*

If one restrict to virtually small branches "residual set" can be replaced by "open and dense set".

Integration theory

• Let $f : M \to \mathbb{R}$ be a Morse function s.t. (f, g) satisfies Morse Smale condition.

• For $x \in Cr_q(f)$ let $W_x^- \subset M$ be the unstable manifold of $X = \text{grad } f$ at the rest point x.

Proposition

For any $\omega \in \Omega^q(M)$ and any $x \in Cr_q(f)$ the integral $\int_{W_x^-} \omega$ is convergent and defines a continuous linear map

$$Int_x : \Omega^q(M) \to \mathbb{C}.$$

for the Frechet topology on $\Omega^q(M)$.

• Let $(C^*(M,X),\delta^*_X)$ be the Morse-Thom complex defined by the partition of M in cells (the unstable manifolds of X.)
•The integration provides the quasi isomorphism

$$Int^* : (\Omega^*(M), d^*) \to (C^*(M,X), \delta_X)$$

•Let

$$\boxed{Int^*(z) := \ (\Omega^*(M), d^*(t)) \xrightarrow{e^{-zf}} (\Omega^*(M), d^*) \xrightarrow{Int^*} (C^*(M,X), \delta_X)}$$

• $Int^*(z)$ is holomorphic in z and quasi-isomorphism for any z.

Theorem

1. $Int^*(t)$ *restricted to* $\Omega^*(M)_{sm}(t)$, *w.r. to the canonical bases satisfies* $\mathrm{Int}^q_{sm}(t) = (\mathrm{Id} + O(e^{-Ct}))D^q(t)$ *where* $D^q(t) = (t/\pi)^{1/4-q/2} Id$.

2.The maps $a_q(z) = \det Int^q_{sm}(z)$ *is holomorphic in an open neighborhood* U *of* $\mathbb{R}$ *in* $\mathbb{C}$ *and the a priory rational function*

$$a(z) = \prod_q (a_q(z))^{(-1)^q}$$

has no zeros and no poles on $\mathbb{R}$.

• **The Virtually small spectral package w.r.t. to a Morse function** f.

The restrictions of the branches convergent to 0 define the virtually small spectral package of (M, g) provided by (M, g, f).

$$VS_q(M, g, f)) = \{\lambda_1^q, \lambda_2^q, \cdots \lambda_{N_q}^q; \ \ \omega_1^q, \cdots \omega_{N_q}^q; \ \ a_q, \ a\}.$$

$a_q(z)$ is derived from integration of $\omega_i^q(z)$ on the unstable sets.

Let $T_q = \sharp(TorH_q(M;\mathbb{Z}))$. There are exactly $N_q - \beta_q$ positive real eigenvalues (counted with their multiplicity) in $VS_q(M, g, f)$.

Theorem

$$\prod T_q^{(-1)^q} = \prod_q \Big(\prod_{\lambda_i^q \in VS_q, \lambda_i^q \neq 0} \lambda_i^q \Big)^{(-1)^q} \cdot a$$

Additional application

For a generic vector field X which admits a Lyapunov cohomology class using analytic continuation and results of BH one can **regularize** the number of closed trajectories (which are actually countably many) and express this number in terms of the topology of the underlying manifold. This number a priory integer is actually real.

Conjectures.

1. $a_q(t) \neq 0$ for any $t \in \mathbb{R}$.
2. If $f_s, 0 \leq s \leq 1$ is a smooth family of Morse functios with (f_s, g) Morse Smale for any s them $VS(M, g, f_s)$ is constant in s.

Item (2) of the above conjecture became recently a theorem. This implies that the virtually small package $VS(M, g, f)$ can be calculated with arbitrary accuracy.