NEW TOPOLOGICAL INVARIANTS INSPIRED BY DATA ANALYSIS AND DYNAMICS

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This material is entirely contained in my book :

Dan Burghelea, *New topological invariants for real- and angle-valued maps; an alternative to Morse-Novikov theory*

Word scientific Publishing

LECTURE 1

The invariants

are defined for

$$(X, f : X \to \mathbb{R})$$

$$(X, f : X \to \mathbb{S}^1)$$

X a compact ANR, f a **tame** map.
and are motivated by Data analysis and Dynamics.

- are numerical
- use homology $H_r(\cdots ; \kappa)$
- are computer friendly
- related to Morse-Novikov theory

Real-valued map $f : X \to \mathbb{R}$

X is a locally compact ANR, $f : X \to \mathbb{R}$ a **tame** proper map

Critical values $CR(f) = \{c \in \mathbb{R}, H_*(f^{-1}(t);\kappa) \text{ changes}\}$

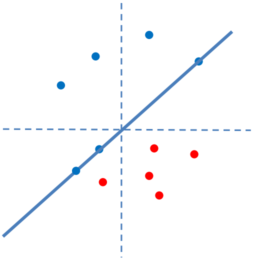
Barcodes = four multi-sets

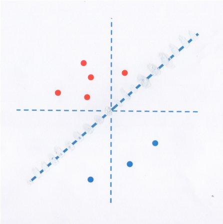
$$\begin{cases} \text{closed barcodes} - \mathcal{B}^c_r(f), \quad [a,b], a,b \in CR(f) \Rightarrow z = a + ib (a \leq b) \\ \text{open barcodes} - \mathcal{B}^c_r(f), \quad (a,b), a,b \in CR(f) \Rightarrow z = b + ia (a < b) \\ \text{closed} - \text{open barcodes} - \mathcal{B}^{c,o}_r(f), \quad [a,b), a,b \in CR(f) \Rightarrow z = a + ib (a \leq b) \\ \text{open} - \text{closed barcodes} - \mathcal{B}^{o,c}_r(f), \quad (a,b], a,b \in CR(f) \Rightarrow z = b + ia (a < b) \end{cases}$$

$$\mathcal{B}^c_r(f) \sqcup \mathcal{B}^o_{r-1}(f) \quad \Rightarrow \boxed{\delta^f_r : \mathbb{R}^2 = \mathbb{C} \to \mathbb{Z}_{\geq 0}}$$

$$\mathcal{B}^{c,o}_r(f) \sqcup \mathcal{B}^{o,c}_{r-1}(f) \Rightarrow \boxed{\gamma^f_r : \mathbb{R}^2 \setminus \Delta = \mathbb{C} \setminus \Delta_C \to \mathbb{Z}_{\geq 0}}$$

If X compact then δ^f_r and γ^f_r have finite support.



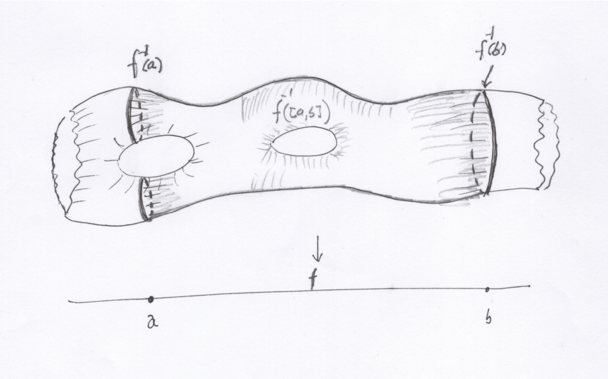


Configurations δ_r^f and γ_r^f

"death" and "observability".

For $a \leq b$ consider

$$u \in H_r(f^{-1}(a);\kappa) \xrightarrow{I_{a\,r}^{a,b}} H_r(f^{-1}([a,b]);\kappa) \xleftarrow{I_{b\,r}^{a,b}} H_r(f^{-1}(b);\kappa) \ni v\}$$



One says that :

- $u \in H_r(f^{-1}(a);\kappa)$ is
  dead at b if $i_{a\ r}^{a,b}(u) = 0$
  observable at b if $i_{a\ r}^{a,b}(u) \neq 0$ and $i_{a\ r}^{a,b}(u) \in img(i_{b\ r}^{a,b})$
- $v \in H_r(f^{-1}(b);\kappa)$ is
  dead at a if $i\,_r b^{a,b}(v) = 0$
  observable at a if $i_{b\ r}^{a,b}(v) \neq 0$ and $i_{b\ r}^{a,b}(v) \subset img(i\,_r a^{a,b})$.

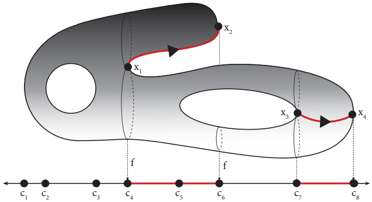
Intuitive definition

1. The interval $\begin{cases} [a,b], \\ (a,b), \\ [a,b), \\ (a,b], \end{cases}$ is $\begin{cases} \textit{closed barcode} \\ \textit{open barcode} \\ \textit{closed} - \textit{open barcode} \\ \textit{open} - \textit{closed barcode} \end{cases}$ if for any $t \in (a,b)$ there exists u in $H_r(f^{-1}(t);\kappa)$ observable at any $t' \in \begin{cases} [a,b] \\ (a,b) \\ [a,b) \\ (a,b] \end{cases}$ and

$$\begin{cases} \textit{not observable at } t' < a \textit{ and } t' > b, \\ \textit{dead at } t' \leq a \textit{ and } t' \geq b, \\ \textit{not observable at } t' < a \textit{ and dead at any } t' \geq b, \\ \textit{dead at } t' \leq a \textit{ and not observable at any } t' > b. \end{cases}.$$

2. A barcode I with the ends a, b has multiplicity m iff for any $t \in (a,b)$ there exists exactly m linearly independent homology classes $u_1, u_2, \cdots u_m \in H_r(f^{-1}(t);\kappa)$ which are observable as linearly independent classes for any t' in (a,b) and all satisfy the conditions above .

EXAMPLE



$$\begin{cases} \mathcal{B}_0^c(f) = \{[c_1, c_8]\} \\ \mathcal{B}_0^0(f) = \{(c_2, c_3)(c_5, c_7)\} \\ \mathcal{B}_0^{c,o}(f) = \emptyset \\ \mathcal{B}_0^{o,c}(f) = \{(c_4, c_6]\} \end{cases}$$

$$\begin{cases} \mathcal{B}_1^c(f) = \{[c_2, c_3], [c_5, c_7]\} \\ \mathcal{B}_1^0(f) = \{(c_1, c_8)\} \\ \mathcal{B}_1^{o,c}(f) = \emptyset \\ \mathcal{B}_1^{c,o}(f) = \{[c_4, c_6)\}. \end{cases}$$

Relevance

1. $\beta_r(X;\kappa) := \dim H_r(X;\kappa) = \sharp(\mathcal{B}^c_r \sqcup \mathcal{B}^o_{r-1}),$
2. For any $t \in \mathbb{R}$ $\dim H_r(f^{-1}(t);\kappa) = \sharp\{I \in \mathcal{B}_r(f)), t \in I\},$
3. When X a smooth compact manifold and $f : M \to \mathbb{R}$ Morse function the Morse complex $(C_r(f), \partial_r(f) : C_r(f) \to C_{r-1}(f))$ has

$$\dim C_r(f) = \sharp Crit_r(f) = \sharp\mathcal{B}^c_r(f) + \sharp\mathcal{B}^o_{r-1}(f) + \sharp\mathcal{B}^{c,o}_r(f) + \sharp\mathcal{B}^{c,o}_{r-1}(f)$$

$$rank(\partial_r) = \sharp\mathcal{B}^{c,o}_{r-1}(f).$$

$Crit_r(f)$ denotes the set of critical points of index r.

Poincaré duality property:

If M^n is a closed topological manifold which is $\kappa-$orientable then

1. $\delta_r^f(a,b) = \delta_{n-r}^f(b,a)$, *equivalently* $\delta_r^f(z) = \delta_{n-r}^f(-i\overline{z})$
2. $\gamma_r^f(a,b) = \gamma_{n-r-1}^f(b,a)$, *equivalently* $\gamma_r^f(z) = \gamma_{n-r-1}^f(-i\overline{z})$.

Notations

U topological space, $K \subset U$ closed subset, X compact topological space

1. $\mathcal{C}onf_N(U)$, configurations of N points of U with **collision topology**
2. $\mathcal{C}onf(U \setminus K)$, configurations of points on $U \setminus K$ with **bottleneck topology**
3. $C(X;\mathbb{R})$, the space of continuous real-valued tame maps with the compact-open topology
4. $C_t(X;\mathbb{R})$, the subspace of continuous real-valued tame maps with the induced (compact-open) topology.

Properties

Stability property:

1. *The assignment*
$$\mathcal{C}_t(X;\mathbb{R}) \ni f \rightsquigarrow \delta_r^f \in \mathcal{C}onf_{\beta_r(X:\kappa)}(\mathbb{R}^2 = \mathbb{C})$$
is continuous. It extends to a continuous map on $\mathcal{C}(X;\mathbb{R})$.
2. *The assignment*
$$\mathcal{C}_t(X;\mathbb{R}) \ni f \rightsquigarrow \gamma_r^f \in \mathcal{C}onf(\mathbb{R}^2 = \mathbb{C} \setminus \Delta_C)$$
is continuous.

Angle valued maps $f : X \to \mathbb{S}^1$

X a compact ANR, $f : X \to \mathbb{S}^1$ tame,
$\tilde{X} \xrightarrow{\pi} X$ the **infinite cyclic cover** of X.
$\tilde{f} : \tilde{X} \to \mathbb{R}$ the **infinite cyclic cover** of f
$\mu : \mathbb{Z} \times \tilde{X} \to \tilde{X}$ the induced action,

$$\therefore$$

$$X = \tilde{X}/\mathbb{Z}, \quad \tilde{f}(\mu(n, x)) = \tilde{f}(x) + 2\pi n, \quad \tilde{f}^{-1}(t) = f^{-1}(\theta = e^{it})$$

- $c \in CR(\tilde{f}) \Rightarrow (c + 2\pi) \in CR(\tilde{f})$,
- $CR(f) = CR(\tilde{f})/2\pi\mathbb{Z}$,
- $\{a, b\} \in \mathcal{B}_r(\tilde{f}) \Rightarrow \{a + 2\pi, b + 2\pi\} \in \mathcal{B}_r(\tilde{f})$.
- Possibly infinite barcodes $(-\infty, \infty)$

The invariants

Barcodes $\begin{cases} closed & := \mathcal{B}^c_r(f) := \mathcal{B}^c_r(\tilde{f})/2\pi\mathbb{Z}, \\ open & := \mathcal{B}^o_r(f) := \mathcal{B}^o_r(\tilde{f})/2\pi\mathbb{Z}, \\ closed - open & := \mathcal{B}^{c,o}_r(f) := \mathcal{B}^{c,o}_r(\tilde{f})/2\pi\mathbb{Z}, \\ open - closed \ : & = \mathcal{B}^{o,c}_r(f) := \mathcal{B}^{o,c}_r(\tilde{f})/2\pi\mathbb{Z}, \end{cases}$

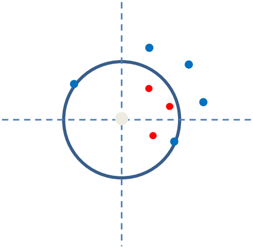
$$\mathcal{B}^c_r(f) \sqcup \mathcal{B}^o_{r-1}(f) \Rightarrow \boxed{\delta^f_r : \mathbb{R}^2/2\pi\mathbb{Z} = \mathbb{C} \setminus 0 \to \mathbb{Z}_{\geq 0}}$$

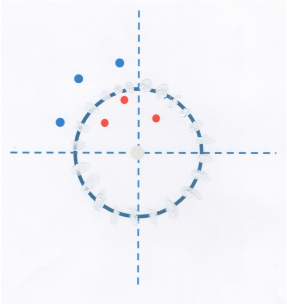
$$\mathcal{B}^{c,o}_r(f) \sqcup \mathcal{B}^{o,c}_r(f) \Rightarrow \boxed{\gamma^f_r : (\mathbb{R}^2 \setminus \Delta)/2\pi\mathbb{Z} = (\mathbb{C} \setminus 0) \setminus \mathbb{S}^1 \to \mathbb{Z}_{\geq 0}}$$

Jordan blocs = $\mathcal{J}_r(f)$ a multi-set set of conjugacy classes of indecomposable invertible matrices (described below).

When κ is algebraically closed an indecomposable matrix is conjugated with the Jordan matrix $T(\lambda, k)$,

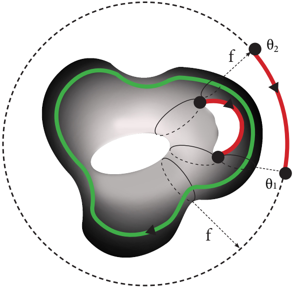
$\lambda \in \kappa \setminus 0, k \in \mathbb{Z}_{>0}$.





Configurations δ_r^f and γ_r^f

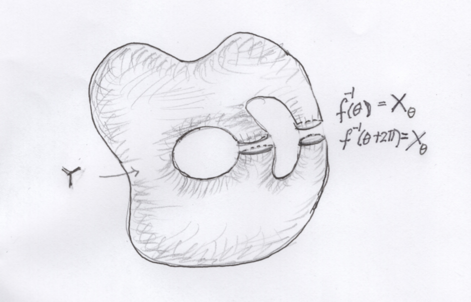
Example of angle valued map



$$\begin{cases} \mathcal{B}_0^c(f) = \emptyset, \\ \mathcal{B}_0^o(f) = \{(\theta_1, \theta_2)\}, \\ \mathcal{B}_0^{c,o}(f) = \emptyset, \\ \mathcal{B}_0^{o,c}(f) = \emptyset \end{cases} \quad \begin{cases} \mathcal{B}_1^c(f) = \emptyset, \\ \mathcal{B}_1^o(f) = \{(\theta_1, \theta_2)\}, \\ \mathcal{B}_1^{c,o}(f) = \emptyset, \\ \mathcal{B}_1^{o,c}(f) = \emptyset \end{cases} \quad \begin{cases} \mathcal{J}_0(f)\{1.1)\}, \\ \mathcal{J}_1(f)\{1.1)\} \end{cases}$$

The $\theta-$**cut.**

For $\theta = e^{it}$ consider
$X_\theta = f^{-1}(\theta) = \tilde{f}^{-1}(t) = \tilde{f}^{-1}(t+2\pi)$, $Y = \tilde{f}^{-1}([t, t+2\pi])$
and the inclusions
$f^{-1}(\theta) = \tilde{f}^{-1}(t) \subset \tilde{f}^{-1}([t, t+2\pi]) \supset \tilde{f}^{-1}(t+2\pi) = f^{-1}(\theta)$



Passing to homology one obtains

$$V_r := H_r(X_\theta) \xrightarrow{i_l} W_r := H_r(Y) \xleftarrow{i_r} V_r := H_r(X_\theta)\ .$$

a *linear relation*.

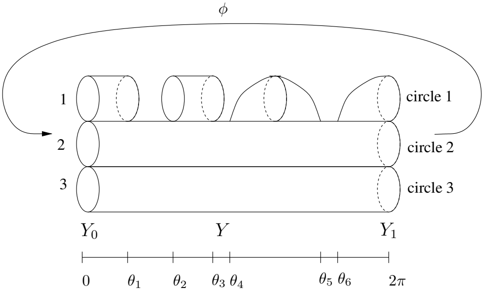
A linear relation is *invertible* if α and β are isomorphisms. Any linear relation $V \xrightarrow{\alpha} W \xleftarrow{\beta} V$ contains invertible sub relations partially ordered by inclusion. All maximal invertible sub relations are isomorphic and the composition $T := \beta^{-1} \cdot \alpha$ for a maximal invertible sub relation is unique up to a conjugation

Up to composition T decomposes as $T \sim \oplus T_J$; the conjugacy class of T_J defines the Jordan block $J \in \mathcal{J}_r(f)$.

One more example

The space X is obtained from Y described in the picture below by identifying the right end Y_1 (a union of three circles) to the left end Y_0 (a union of three circles) following the map $\phi\colon\ Y_1 \to Y_0$ given by the matrix

$$\begin{pmatrix} 3 & 3 & 0 \\ 2 & 3 & -1 \\ 1 & 2 & 3 \end{pmatrix}.$$



In this example the critical angles are $\{\theta_0 = 0 = 2\pi, \theta_1, \cdots, \theta_6\}$ $\mathcal{B}_0(f) = \mathcal{B}_2(f) = \emptyset$

$$\begin{cases} \mathcal{B}_1^c = \{[\theta_2, \theta_3]\} \\ \mathcal{B}_1^o = \{(\theta_4, \theta_5)\} \\ \mathcal{B}_1^{o,c} = \{(\theta_6, \theta_1 + 2\pi]\} \\ \mathcal{B}_1^{\bar{o},c} = \emptyset \end{cases} \qquad \begin{cases} \mathcal{J}_0(f) = \{(1,1)\} \\ \mathcal{J}_1(f) = \{(\lambda = 2, k = 2)\}. \end{cases}$$

LECTURE 2

Recall from Lecture 1

X **compact** ANR,

$f : X \to \mathbb{S}^1$ continuous **tame** map

$\Rightarrow$

- $\mathcal{B}^c_r(f), \mathcal{B}^o_r(f), \mathcal{B}^{c,o}_r(f), \mathcal{B}^{o,c}_r(f)$ barcodes,

equivalently:
δ^f_r configuration on $\mathbb{R}^2/2\pi\mathbb{Z} = \mathbb{C} \setminus 0$,
γ^f_r configuration on $(\mathbb{R}^2 \setminus \Delta)/2\pi Z = (\mathbb{C} \setminus 0) \setminus (\mathbb{S}^1)$

- $\mathcal{J}_r(f)$ Jordan blocks.

$$f : X \to \mathbb{S}^1 \Rightarrow \xi_f \in H^1(X; \mathbb{Z}),$$

$$f : X \to \mathbb{S}^1 \Rightarrow, \mu : \mathbb{Z} \times \tilde{X} \to \tilde{X}.$$

$\mu \Rightarrow H_r(\tilde{X}; \kappa)$ is a **f.g.** $\kappa[t^{-1}, t]-$**module** with $\kappa[t^{-1}, t]$ a PID.

- **Novikov-Betti numbers:** $\beta^N_r(X, \xi_f; \kappa) := rank\ H_r(\tilde{X}; \kappa)$
- **Monodromy:** $T_r(f) : V_r(f) \to V_r(f)$
    1. $V_r(f) := Tor\ H_r(\tilde{X}; \kappa)$ f.g submodule and f.d $\kappa-$vector space.
    2. $T_r(f)$ the multiplication by t
- **Novikov complex** $(C_r(f), \partial_r(f) : C_r(f) \to C_{r-1}(f))$ of $\kappa[t^{-1}, t]]-$vector spaces, for $X = M$ a smooth compact manifold, $f : M \to \mathbb{S}^1$ a **Morse map**

$\kappa[t^{-1}, t]]$ the field of Laurent power series.

Relevance

1. $\beta^N_r(X,\xi_f;\kappa) = \sharp\mathcal{B}^c_r(f) + \sharp\mathcal{B}^o_{r-1}(f).$
2. $T_r(f) \sim \oplus_{J\in\mathcal{J}_r(f)} T_J,\ \ for\ \ T_J \in J$
3. $\beta^f_r(X;\kappa) = \sharp\mathcal{B}^c_r(f) + \sharp\mathcal{B}^o_{r-1}(f) + \sharp\mathcal{J}_{r,1}(f) + \sharp\mathcal{J}_{r-1,1}(f)$

   $\mathcal{J}_{r,1}$ the collection of Jordan cells $J = (\lambda_J, n_J)$ with $\lambda_J = 1$.
4. M smooth compact manifold, $f : M \to \mathbb{S}^1$ Morse map the Novikov complex $(C_r(f), \partial_r(f) : C_r(f) \to C_{r-1}(f))$ has:

$$\dim C_r(f) = \sharp Crit_r(f) = \sharp\mathcal{B}^c_r(f) + \sharp\mathcal{B}^o_{r-1}(f) + \sharp\mathcal{B}^{c,o}_r(f) + \sharp\mathcal{B}^{c,o}_{r-1}(f)$$
$$rank(\partial_r(f)) = \sharp\mathcal{B}^{c,o}_r(f)$$

with $Crit_r(f)$ the set of critical points of Morse index r.

Properties

Poincaré duality property
If M^n is a closed κ-orientable topological manifold and $f: M \to \mathbb{S}^1$ a tame continuous map, then:

1. $\delta^f_r(z) = \delta^f_{n-r}(\tau(z))$ *where* $\tau(z) = 1/\overline{z}$, *is the inversion across the unit circle,*
2. $\gamma^f_r(z) = \gamma^f_{n-1-r}(\tau(z))$.

Stability Property
Suppose X is a compact ANR.

1. *The assignment* $C_{\xi,t}(X,\mathbb{S}^1) \ni f \rightsquigarrow \delta^f_r \in \mathcal{C}onf_{\beta^N_r(X,\xi_f;\kappa)}(\mathcal{C} \setminus 0)$ *is continuous and extends continuously to* $C_\xi(X,\mathbb{S}^1)$.
2. *The assignment* $f \rightsquigarrow \gamma^f_r$ *from* $C_{\xi,t}(X,\mathbb{S}^1)$ *to the space of configurations in* $\mathcal{C}onf((\mathbb{C} \setminus 0) \setminus \mathbb{S}^1)$ *is continuous.*

Measure theoretic approach; barcodes $\mathcal{B}^c_r(f)$, $\mathcal{B}^o_{r-1}(f)$

- For a tame continuous proper map $f : X \to \mathbb{R}$ denote
  $\mathbb{I}^f_a(r) := img(H_r(f^{-1}((-\infty, a]) \to H_r(X))$,
  $\mathbb{I}^a_f(r) := img(H_r(f^{-1}([a, \infty)) \to H_r(X))$.
- For $(a, b) \in \mathbb{R}^2$ denote

$$F^f_r(a, b) := \dim(\mathbb{I}^f_a(r) \cap \mathbb{I}^b_f(r)) < \infty$$

- Consider sets = **boxes**

$$B = (a', a] \times [b, b') \subset \mathbb{R}^2,\ a' < a,\ b' > b$$

Define

$$F^f_r(B) := F^f_r(a, b) + F^f_r(a', b') - F^f_r(a, b') - F^f_r(a', b) \geq 0$$

- For B, B_1, B_2 boxes with $B = B_1 \sqcup B_2$ one has

$$\boxed{F^f_r(B) = F^f_r(B_1) + F^f_r(B_2)} \quad (1)$$

- Denote $B(a,b;\epsilon) := (a-\epsilon, a] \times [b, b+\epsilon)$, $\epsilon > 0$ and note $\epsilon' > \epsilon'' \Rightarrow F^f_r(B(a,b;\epsilon')) \geq F^f_r(B(a,b;\epsilon''))$.
- Define

$$\boxed{\delta^f_r(a,b) := \lim_{\epsilon\to 0} F^f_r(B(a,b;\epsilon)).}$$

Since f is tame $\delta^f_r(a,b) \neq 0 \Rightarrow a, b \in CR(f)$.

• $B \rightsquigarrow F_r(B)$ defines a $\mathbb{Z}-$valued measure on the sigma-algebra generated by boxes, with density δ^f_r.

• When X is compact δ^f_r is a configuration of points in $\mathbb{C}$.

Define

- $\mathcal{B}_r^c(f) = \{[a,b] \mid (a,b) \in \mathit{support}\ \delta_r^f, a \leq b\}$; multiplicity of $[a,b]$ = $\delta_r^f(a,b)$
- $\mathcal{B}_{r-1}^o(f) = \{(b,a) \mid (a,b) \in \mathit{support}\ \delta_r^f, a > b\}$; multiplicity of (b,a) =$\delta_r^f(a,b)$

We treat only the case of real valued map.

- For $a < b$ define

$$T_r^f(a,b) := \dim\ker(H_r(f^{-1}((-\infty a]) \to H_r(f^{-1}((-\infty, b])$$

For $a > b$ define

$$T_r^f(a,b) := \dim\ker(H_r(f^{-1}([a,\infty)]) \to H_r(f^{-1}([b,\infty)).$$

- For sets = **boxes above diagonal**, $B = (a' a] \times (b', b]$ i.e. $a' < a < b' < b$ define

$$T_r^f(B) = T_r^f(a,b) + T_r^f(a',b') - T_r^f(a',b) - T_r^f(a,b') \geq 0.$$

For set = **boxes below diagonal**, $B = [a, a'') \times [b, b'')$, i.e. $a < a'' < b < b''$ define

$$T_r^f(B) = T_r^f(a,b) + T_r^f(a'',b'') - T_r^f(a'',b) - T_r^f(a,b'') \geq 0.$$

- For $B = B' \sqcup B''$ with B, B', B'' all boxes above diagonal or boxes below diagonal one has

$$\boxed{T_r^f(B) = T_r^f(B') + T_r^f(B'')} \tag{2}$$

.

- For $a < b$, $\epsilon < (b - a)$ let $B(a, b; \epsilon) = (a - \epsilon, a] \times (b - \epsilon, b]$,

  For $a > b$ $\epsilon < (a - b))$ let $B(a, b; \epsilon) = [a, a + \epsilon) \times [b, b + \epsilon)$,

  $\epsilon' > \epsilon'' \Rightarrow T_r^f(B(a, b; \epsilon')) \geq T_r^f(B(a, b; \epsilon''))$.

- Define

$$\boxed{\gamma_r^f(a,b) := \lim_{\epsilon\to 0} T_r^f(B(a,b;\epsilon))}.$$

Since f is tame $\gamma_r^f(a,b) \neq 0 \Rightarrow a,b \in CR(f)$.

• $B \rightsquigarrow T_r(B)$ defines a $\mathbb{Z}-$valued measure on the sigma-algebra generated by boxes above and below diagonal , with density γ_r^f.

• When X is compact γ_r^f is a configuration of points in $\mathbb{R}^2 \setminus \Delta$.

Define

- $\mathcal{B}_r^{c,o}(f) = \{[a,b) \mid (a,b) \in \textit{support } \gamma_r^f, a \leq b\}$; multiplicity of $[a,b]$ = $\gamma_r^f(a,b)$
- $\mathcal{B}_r^{o,c}(f) = \{(b,a] \mid (a,b) \in \textit{support } \gamma_r^f, a > b\}$; multiplicity of (b,a) =$\gamma_r^f(a,b)$

Quiver representations approach

$$\mathcal{Z}: \ \cdots \xleftarrow{b_{i-1}} x_{2i-1} \xrightarrow{a_i} x_{2i} \xleftarrow{b_i} x_{2i+1} \xrightarrow{a_{i+1}} x_{2i+2} \xleftarrow{b_{i+1}} \cdots$$

$\mathcal{G}_{2m}$:

$$\begin{tikzcd}
 & x_2 & \\
x_3 \arrow[ur, "b_1"] \arrow[d, "a_2"] & & x_1 \arrow[ul, "a_1"'] \arrow[d, "b_m"'] \\
x_4 \arrow[d, dotted, leftrightarrow] & & x_{2m} \\
x_{2m-3} \arrow[dr, "a_{m-1}"] & & x_{2m-1} \arrow[u, "a_m"'] \arrow[dl, "b_{m-1}"'] \\
 & x_{2m-2}. &
\end{tikzcd}$$

$$\text{Representation } \rho \begin{cases} x_i \to V_i \\ a_i \to \alpha_i \\ b_i \to \beta_i \end{cases}$$

Representation $\rho_r(f)$

• for $f : X \to \mathbb{R}$ consider the critical values $c_1 < c_2 < \cdots c_r$,
• for $f : X \to \mathbb{S}^1$ consider the critical values
$0 \leq c_1 < c_2 < \cdots < c_m < 2\pi$,
• one chooses the regular values $t_0 < t_1 \cdots$ with $c_i < t_i < c_{i+1}$
(for angle-valued map $c_m < t_m < 2\pi$)

Define $\rho_r(f) =$

$$\begin{cases} V_{2i} := H_r(f^{-1}([t_{i-1}, t_i])) \\ V_{2i-1} := H_r(f^{-1}(t_i)) \\ \alpha_i : V_{2i-1} \to V_{2i} \text{ induced by } f^{-1}(t_i) \subset f^{-1}([t_i, t_{i+1}]) \\ \beta_i : V_{2i+1} \to V_{2i} \text{ induced by } f^{-1}(t_{i+1}) \subset f^{-1}([t_i, t_{i+1}]) \end{cases}$$

The indecomposable $\mathcal{Z}-$representations with finite support are in indexed by intervals $\{a, b\}$ $a, b \in \mathbb{Z}, a \leq b$ [1].

1. the interval $\{2i, 2j\}$ defines to the closed $r-$barcode $[c_i, c_j]$, regarded as the complex number $c_i + \sqrt{-1}c_j)$
2. the interval $\{2i+1, 2j+1\}$ defines to the open $r-$barcode (c_i, c_{j+1}) regarded as the complex number $c_{j+1} + \sqrt{-1}c_i$
3. the interval $\{2i, 2j+1\}$ defines to the closed-open $r-$barcode $[c_i, c_{j+1})$ regarded as the complex number $c_i + \sqrt{-1}c_{j+1}$
4. the interval $\{2i+1, 2j\}$ defines to the open-=closed $r-$barcode $(c_i, c_j]$ regarded as the complex number $c_j + \sqrt{-1}c_i)$

[1] The indecomposable representation indexed by $\{a, b\}$ has $V_i = V_{x_i} = \begin{cases} \kappa, \text{if a} \leq \text{i} \leq \text{b} \\ 0, \text{if i} < \text{a or i} > \text{b} \end{cases}$ and all linear maps between isomorphic vector spaces the identity

Indecomposable $\mathcal{G}_{2m}-$representations

The indecomposable $\mathcal{G}_{2m}-$representations are labelled by **equivalence classes up to translation by multiples of** $2m$ of intervals $\{a, b\}$, $a, b \in \mathbb{Z}$, $a \leq b$ and **conjugacy classes** of indecomposable invertible matrices with entries in κ. [2].

1. the interval $\{2i, 2j\}$ defines to the closed $r-$barcode $[c_i, c_j]$ regarded as the complex number $e^{\sqrt{-1}c_i+(c_j-c_i)}$
2. the interval $\{2i+1, 2j+1\}$ defines to the open $r-$barcode (c_i, c_{j+1}) regarded as the complex number $e^{\sqrt{-1}c_{j+1}+(c_i-c_{j+1})}$
3. the interval $\{2i, 2j+1\}$ defines to the closed-open $r-$barcode $[c_i, c_{j+1})$ regarded as the complex number $e^{\sqrt{-1}c_i+(c_{j+1}-c_i)}$
4. the interval $\{2i+1, 2j\}$ defines to open-closed] $r-$the barcode $(c_i, c_j]$ regarded as the complex number $e^{\sqrt{-1}c_j+(c_i-c_j)}$

[2] when κ is algebraically closed field such conjugacy class is determined by a pair (λ, n) $\lambda \in \kappa \setminus 0, n \in \mathbb{Z}_{\geq 1}$

Step 1. Pass from (X, f) when X is simplicial complex and f a simplicial map to the representation $\rho_r(f)$

Step 2 Pass from $\rho_r(f)$ to the indecomposable components (i.e. bar codes and Jordan cells) see [2] [1]for details.
(Explanations if the time permits)

relation with Data Analysis and Dynamics

Explanations if the time permits

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