WITTEN HELLFER SJÖSTRAND THEORY

1.DeRham Hodge Theorem

2.WHS-Theorem

3.Mathematics behind WHS-Theorem

4.WHS-Theorem in the presence of

symmetry

Notations

M **closed smooth manifold,**

$(V, < \ , \ >_V)$ **euclidean space**

$\rho : \pi_1(M) \to O(V)$ **orth. representation**

g **a Riemannian metric and** ρ

$\Rightarrow (\Omega^*(M,\rho) \ , < \ , \ >_g)$

$\rho \Rightarrow \ d_\rho : \Omega^*(M,\rho) \to \Omega^{*+1}(M,\rho)$

$\rho, g \Rightarrow \ d^\sharp_\rho : \Omega^{*+1}(M,\rho) \to \Omega^*(M,\rho)$

and $\Delta^q : \Omega^q(M,\rho) \to \Omega^q(M,\rho)$,

$\Delta = d_\rho \cdot d^\sharp_\rho + d^\sharp_\rho \cdot d_\rho$

$\mathcal{H}^q := \{\omega | \Delta^q \omega = 0\}$

τ **smooth triangulation**

$\tau \Rightarrow (C^*(\tau,\rho), \partial_\rho)$ **f.d. cochain complex**

$\Rightarrow H^q_\tau(M,\rho)$

ρ **and** $\tau \Rightarrow (C^*(\tau,\rho), < \ , \ >_\tau)$.

Theorem (deRham- Hodge).

g and τ induce an orthogonal decompositions

$\Omega^q(M,\rho)=\mathcal{H}^q\oplus{\Omega'}^q(M,\rho),\quad \mathcal{H}^q$ *f.d vect.space*

$({\Omega'}^q(M,\rho)=d(\Omega^{q-1}(M,\rho))\oplus d^\sharp(\Omega^{q-1}(M,\rho))$

and an isomorphism

$J^q: H^q_\tau(M,\rho)\to\mathcal{H}^q\quad\subset\Omega^q(M,\rho)$

Theorem (Witten Hellfer-Sjöstrand).

g and τ induce (for $t\in\mathbb{R}_+, t>>0,$) a decomposition

$$(\Omega^*(M,\rho),d_\rho(t))=(\Omega^*_0(M,\rho)(t),d_\rho(t))\oplus(\Omega^*_1(M,\rho)(t),d_\rho(t))$$

with $(\Omega^*_0(M,\rho)(t),d_\rho(t))$ is a f.d.cochain complex,

and an isomorphism

$J^q(t): C^*(\tau,\rho),\partial_\rho)\to(\Omega^*_0(M,\rho)(t),d_\rho(t))$

so that, for $S^q(t)=(\pi/t)^{(n-2q)/4}e^{-tq}\cdot Id$, the composition

$$(C^*(\tau,\rho),\partial_\rho(t))\to(C^*(\tau,\rho),\partial_\rho)\to(\Omega^*_0(M,\rho)(t),d_\rho(t))$$

is an $O(1/t)-$ isometry.

Both the decomposition and $J^*(t)$ are canonical.

TRIANGULATIONS

τ smooth triangulation

σ open simplex

x_σ baricenter of σ

$\mathcal{X}_q := \{\sigma | \dim(\sigma) = q\}$

pair $(g, h : M \to \mathbb{R})$

$X = -grad_g h$

$W_x^\pm = \{y \in M | \lim_{t \to \pm\infty} \gamma_y(t) = x\}$ for $x \in Cr(h)$.

Observation

Any smooth triangulation τ is given by a pair (g, h).

Theorem 1(Pozniak).

Given the Riemannian metric g and the smooth triangulation τ there exists $h : M \to \mathbb{R}$ a self indexing Morse function so that :

a) $Cr(h)_q = \{x_\sigma | \sigma \in \mathcal{X}_q\}$, *and* $h(x_\sigma) = \dim(\sigma)$

b) $x, y \in Cr(h) \Rightarrow W_x^- \pitchfork W_y^+$

c)$W_{x_\sigma}^- = \sigma$

Generalized triangulation $\tau = (g, h)$

a) $h : M \to \mathbb{R}$ a selfindexing Morse function

b) $x, y \in Cr(h) \Rightarrow W_x^- \pitchfork W_y^+$.

c) g is flat in the neighborhood of the critical points

The definition is justified because of the following

Theorem 2 (Morse theory).

If $\tau = (g, h)$ is a generalized triangulation, then:

1) $M = \cup_{x \in Crh} W_x^-$

2) W_x^- are submanifolds diffeomorphic to $\mathbb{R}^{index(x)}$

(i.e. open cells) which have canonical compactification

to compact smooth manifolds with corners $\hat{W}_x^-$.

3)The inclusion $i_x : W_x^- \to M$ has a smooth extension

$\hat{i}_x : \hat{W}_x^- \to M$, with $\hat{i}_x(\partial \hat{W}_x^-)$ a union of (open cells)

${W_y^-}'s$.

WITTEN DEFORMATION

Let $(M, \rho, \tau), \tau = (g, h)$

$(\Omega^*(M, \rho), d_\rho(t)),\ d_\rho(t) := e^{-th} \cdot d \cdot e^{th}$

$E^* := e^{th} \cdot Id : (\Omega^*(M, \rho), d_\rho(t)) \to (\Omega^*(M, \rho), d_\rho)$

$\rho,\ g$ induce the **Witten Laplacians** $\Delta^q(t)$

$\Delta^q(t) := d(t) \cdot d^\sharp(t) + d^\sharp(t) \cdot d(t)$

$\Delta^q(t) = \Delta^q + t(L_X + L_X^\sharp) + t^2 ||grad_g h||^2 Id$

Theorem 3 (Witten).

There exists positive constants C_1, C_2, C_3, T *so that:*

1) $Spec \Delta^q(t) \subset [0, C_1 e^{-C_2 t}] \cup [C_3 t, \infty),\ \ t > T$

2) $\sharp(Spect \Delta^q(t) \cap [0, C_1 e^{-C_2 t}] = \sharp(Cr_q(h)) \cdot \dim V$

Observation:

For $t >> 0$ Theorem 3 implies that $(\Omega(M, \rho), d_\rho(t)) =$

$(\Omega^*(M, \rho)_{sm}(t), d_\rho(t)) \oplus (\Omega(M, \rho)_{la}(t), d_\rho(t))$

$(\Omega^*(M,\rho)_{sm}(t), d_\rho(t)) \subset (\Omega^*(M,\rho), d_\rho(t)) \to$

$\to (C^*(\tau,\rho)\partial_\rho) \to (C^*(\tau,\rho)\partial_\rho)$

Theorem 4(Helffer- Sjöstrand).

1)For t large enough, $S^*(t) \cdot Int^* \cdot E^*(t)$ *restricted to* $(\Omega(M,\rho)_{sm}(t), d_\rho(t))$ *is an isomorphism of f.d. cochain complexes.*

2) $R^*(t) := (S^*(t) \cdot Int^* \cdot E^*(t))^{-1}$ *is an* $O(1/t)-$*isometry.*

3)If $\tau_1 = (g, h_1)$ *and* $\tau_2 = (g, h_2)$ *are two generalized trangulations with the same critical points and the same unstable sets then* $||R_1^*(t) - R_2^*(t)|| = O(1/t)$.

MATHEMATICS

Gap in the spectrum of $\Delta^q(t) : \Omega^*(M,\rho)_{L^2} \to \Omega^*(M,\rho)_{L^2}$

Lemma.

Suppose $A : H \to H$ *is a selfadjoint positive operator (not necessary bounded),* $W_i \subset H,\ i = 1, 2$ *closed subspaces and* $C_1 < C_2$ *two positive constants so that:*

i)$W_1 \cap W_2 = 0,$ *and* $W_1 \oplus W_2 = H,$

ii)$< A(x), x > \leq C_1 ||x||^2, x \in W_1$ *and*

iii)$< A(x), x > \geq C_2 ||x||^2, x \in W_2.$

Then, $SpecA \cap (C_1, C_2) = \emptyset.$

Construct W_1 and W_2

Harmonic Oscillator=

the operator $\frac{d^2}{dx^2} + \alpha x^2 + \beta,\ \alpha > 0$ on $L_2(R)$

Consider (R^n, g_0, h_k) with

$h_k(x_1 \cdots x_n) = k - 1/2(\sum_{i=1}^{k} x_i^2) + 1/2(\sum_{i=k+1}^{n} x_i^2)$

$\Delta_k^q(t) : \Omega_{L_2}^q(\mathbb{R}^n) \to \Omega_{L_2}^q(\mathbb{R}^n)$

the Witten Laplacians associated with h_k

Proposition (Harmonic oscillator).

1) $Spec\Delta_k^q(t) \subset 2t\mathbb{Z}$

2)$\ker \Delta_k^q(t) = 0$ *iff* $k \neq q$

3)$\ker \Delta_q^q(t) = \{\omega_q\}$ *with*

$\omega_q = (t/\pi)^{n/4} e^{-t||x||^2} dx_1 \wedge \cdots dx_q$

4)....Estimates about ω_q *multiplied but cutoff (away from zero) functions.*

G-symmetry

$G-$compact Lie group

ξ irreducible representation of G.

$\mu : G \times M \to M$, $G-$manifold

(compatible with ρ)

g invariant Riemannian metric

τ is a $G-$smooth triangulation

with $G-$ simplexes σ's,

$\sigma : G/H_\sigma \times \Delta(k(\sigma)) \to M$, G equivariant

$(C^*(\tau, \rho), \partial_\rho)$ is defined by:

$C^*(\tau, \rho) = \sum_\sigma (\Omega^{*+k(\sigma)}(G/H))$

$\partial_\rho = \partial'_\rho + \partial_\rho$"

∂'_ρ given by exterior differential in $\Omega^*(G/H)$

∂_ρ" given by ρ and the incidence of cells

$\Omega^*(M,\rho),d)$ and $(C^*(\tau,\rho)$ are

$G-$cochain complexes

$$Int^* : (\Omega^*(M,\rho),d) \to (C^*(\tau,\rho),\partial_\rho)$$

decomposes as $\bigoplus_\xi Int_\xi$ with

$$Int_\xi : (\Omega^*_\xi(M,\rho),d_{\rho,\xi}) \to (C^*_\xi(\tau,\rho),\partial_{\rho,\xi})$$

$(C^*(\tau,\rho)_\xi,\partial_{\rho,\xi})$ is f.d.

WHS Theorem holds for Int_ξ.

There are only finitely many $G-$simplexes σ.

The representation ξ and the subgroups H'_σs

provide a finite collection of explicitly computable

real real numbers say $\lambda^\xi_1,\lambda^\xi_2,\cdots,\lambda^\xi_N$

Theorem 3'.

There exists positive constants $C_1(\xi), C_2(\xi), C_3(\xi), T(\xi)$
so that:

1) $Spec\Delta^q(t) \subset \bigcup_{1\leq i\leq N}[\lambda_i - \ C_1 e^{-C_2 t}, \lambda_i + \ C_1 e^{-C_2 t}]$
$\cup [C_3 t, \infty)$ *if* $t > T(\xi)$

2) $\sharp(Spect\Delta^q(t) \cap [\lambda_i - \ C_1 e^{-C_2 t}, \lambda_i + \ C_1 e^{-C_2 t}]$
can be specified in terms of the number of G simplexes
in each dimension

What WHS-Theorem was good for?

Prove equality of invariants defined combinatorially

(i.e. using a smooth (generalized) triangulation and

analytically (i.e. using a Riemannian metric)

Theorem Witten Hellfer-Sjöstrand (reformulation).

g and τ induce an orthogonal decompositions

$$(\Omega^q(M,\rho), d_\rho(t) = (\Omega^q_{sm}(M,\rho)(t), d_\rho(t)) \oplus (\Omega^q_{la}(M,\rho), d_\rho(t))$$

and an isomorphism

$$R^*(t) \cdot S^*(t) : (C^*(\tau,\rho), \partial_\rho) \rightarrow (\Omega^q_{sm}(M,\rho)(t), d_\rho(t)) \ \subset (\Omega^q(M,\rho)(t), d_\rho(t))$$