WITTEN HELLFER SJÖSTRAND
THEORY

1. DeRham Hodge Theorem

2. WHS-Theorem

3. Mathematics behind WHS-Theorem

4. WHS-Theorem in the presence of

symmetry
Notations

$M$ closed smooth manifold,

$(V, <, >_V)$ euclidean space

$\rho : \pi_1(M) \to O(V)$ orth. representation

$g$ a Riemannian metric and $\rho$

$\Rightarrow (\Omega^*(M, \rho), <, >_g)$

$\rho \Rightarrow d_\rho : \Omega^*(M, \rho) \to \Omega^{*+1}(M, \rho)$

$\rho, g \Rightarrow d^\rho : \Omega^{*+1}(M, \rho) \to \Omega^*(M, \rho)$

and $\Delta^g : \Omega^q(M, \rho) \to \Omega^q(M, \rho)$,

$\Delta = d_\rho \cdot d^\rho + d^\rho \cdot d_\rho$

$H^q := \{\omega | \Delta^g \omega = 0\}$

$\tau$ smooth triangulation

$\tau \Rightarrow (C^*(\tau, \rho), \partial_\rho)$ f.d. cochain complex

$\Rightarrow H^q(\tau, \rho)$

$\rho$ and $\tau \Rightarrow (C^*(\tau, \rho), <, >_\tau)$. 
Theorem (deRham-Hodge).

$g$ and $\tau$ induce an orthogonal decompositions

$$\Omega^q(M, \rho) = \mathcal{H}^q \oplus \Omega'^q(M, \rho), \quad \mathcal{H}^q \text{ f.d vect. space}$$

$$(\Omega^q(M, \rho) = d(\Omega^{q-1}(M, \rho)) \oplus d^*(\Omega^{q-1}(M, \rho))$$

and an isomorphism

$$J^q : H^q(M, \rho) \to \mathcal{H}^q \subset \Omega^q(M, \rho)$$

Theorem (Witten Hellfer-Sjöstrand).

$g$ and $\tau$ induce (for $t \in \mathbb{R}^+$, $t >> 0$) a decomposition

$$(\Omega^*(M, \rho), d_\rho(t)) = (\Omega^0_0(M, \rho)(t), d_\rho(t)) \oplus (\Omega^*_q(M, \rho)(t), d_\rho(t))$$

with $(\Omega^*_q(M, \rho)(t), d_\rho(t))$ is a f.d.cochain complex,

and an isomorphism

$$J^q(t) : C^*(\tau, \rho) \to (\Omega^*_q(M, \rho)(t), d_\rho(t))$$

so that, for $S^q(t) = (\pi/t)^{(n-2q)/4} e^{-t^q} \cdot Id$, the composition

$$(C^*(\tau, \rho), \partial_\rho(t)) \to (C^*(\tau, \rho), \partial_\rho) \to (\Omega^*_q(M, \rho)(t), d_\rho(t))$$

is an $O(1/t)$-isometry.

Both the decomposition and $J^q(t)$ are canonical.
TRIANGULATIONS

\( \tau \) smooth triangulation
\( \sigma \) open simplex
\( x_\sigma \) baricenter of \( \sigma \)
\( X_q := \{ \sigma | \dim(\sigma) = q \} \)

pair \((g, h : M \to \mathbb{R})\)
\( X = -\text{grad}_gh \)
\( W_x^\pm = \{ y \in M | \lim_{t \to \pm\infty} \gamma_y(t) = x \} \) for \( x \in Cr(h) \).

Observation
Any smooth triangulation \( \tau \) is given by a pair \((g, h)\).

Theorem 1 (Pozniak).

Given the Riemannian metric \( g \) and the smooth triangulation \( \tau \) there exists \( h : M \to \mathbb{R} \) a self indexing Morse function so that:

a) \( Cr(h)_q = \{ x_\sigma | \sigma \in X_q \} \), and \( h(x_\sigma) = \dim(\sigma) \)
b) \( x, y \in Cr(h) \Rightarrow W_x^- \cap W_y^+ \)
c) \( W_x^- = \sigma \)
**Generalized triangulation** \( \tau = (g, h) \)

a) \( h : M \to \mathbb{R} \) a selfindexing Morse function
b) \( x, y \in \text{Cr}(h) \Rightarrow W^+_x \cap W^+_y \).
c) \( g \) is flat in the neighborhood of the critical points

The definition is justified because of the following

**Theorem 2 (Morse theory).**

If \( \tau = (g, h) \) is a generalized triangulation, then:

1) \( M = \bigcup_{x \in \text{Cr}h} W^-_x \)
2) \( W^-_x \) are submanifolds diffeomorphic to \( \mathbb{R}^{\text{index}(x)} \) (i.e. open cells) which have canonical compactification to compact smooth manifolds with corners \( \hat{W}^-_x \).
3) The inclusion \( i_x : W^-_x \to M \) has a smooth extension \( \hat{i}_x : \hat{W}^-_x \to M \), with \( \hat{i}_x(\partial \hat{W}^-_x) \) a union of (open cells) \( W^-_y \)'s.
WITTEN DEFORMATION

Let \((M, \rho, \tau) = (g, h)\)

\((\Omega^*(M, \rho), d_\rho(t))\), \(d_\rho(t) := e^{-th} \cdot d \cdot e^{th}\)

\(E^* := e^{th} \cdot Id : (\Omega^*(M, \rho), d_\rho(t)) \rightarrow (\Omega^*(M, \rho), d_\rho)\)

\(\rho, g\) induce the Witten Laplacians \(\Delta^q(t)\)

\(\Delta^q(t) := d(t) \cdot d^\sharp(t) + d^\sharp(t) \cdot d(t)\)

\(\Delta^q(t) = \Delta^q + t(L_X + L_X^2) + t^2||\text{grad}_g h||^2Id\)

**Theorem 3 (Witten).**

There exists positive constants \(C_1, C_2, C_3, T\) so that:

1) \(\text{Spec} \Delta^q(t) \subset [0, C_1 e^{-C_2 t}] \cup [C_3 t, \infty), \ t > T\)

2) \(\sharp (\text{Spec} \Delta^q(t) \cap [0, C_1 e^{-C_2 t}] = \sharp(Cr_q(h)) \cdot \text{dim} V\)

**Observation:**

For \(t >> 0\) Theorem 3 implies that \((\Omega(M, \rho), d_\rho(t)) = (\Omega^*(M, \rho)_{sm}(t), d_\rho(t)) \oplus (\Omega(M, \rho)_{la}(t), d_\rho(t))\)
\[(\Omega^*(M, \rho)_{sm}(t), d_\rho(t)) \subset (\Omega^*(M, \rho), d_\rho(t)) \to \]
\[\to (C^*(\tau, \rho)\partial_\rho) \to (C^*(\tau, \rho)\partial_\rho)\]

**Theorem 4 (Helffer- Sjöstrand).**

1) For \(t\) large enough, \(S^*(t) \cdot \text{Int}^* \cdot E^*(t)\) restricted to \((\Omega(M, \rho)_{sm}(t), d_\rho(t))\) is an isomorphism of f.d. cochain complexes.

2) \(R^*(t) := (S^*(t) \cdot \text{Int}^* \cdot E^*(t))^{-1}\) is an \(O(1/t)\)-isometry.

3) If \(\tau_1 = (g, h_1)\) and \(\tau_2 = (g, h_2)\) are two generalized triangulations with the same critical points and the same unstable sets then \(||R_1^*(t) - R_2^*(t)|| = O(1/t)\).
MATHEMATICS

Gap in the spectrum of $\Delta^q(t) : \Omega^*(M, \rho)_{L^2} \to \Omega^*(M, \rho)_{L^2}$

Lemma.

Suppose $A : H \to H$ is a selfadjoint positive operator (not necessarily bounded), $W_i \subset H_i$, $i = 1, 2$ closed subspaces and $C_1 < C_2$ two positive constants so that:

i) $W_1 \cap W_2 = 0$, and $W_1 \oplus W_2 = H$,

ii) $\langle A(x), x \rangle \leq C_1 ||x||^2, x \in W_1$ and

iii) $\langle A(x), x \rangle \geq C_2 ||x||^2, x \in W_2$.

Then, $SpecA \cap (C_1, C_2) = \emptyset$.

Construct $W_1$ and $W_2$

Harmonic Oscillator=

the operator $\frac{d^2}{dx^2} + \alpha x^2 + \beta$, $\alpha > 0$ on $L^2(R)$

Consider $(R^n, g_0, h_k)$ with

$h_k(x_1 \cdots x_n) = k - 1/2(\sum_{i=1}^k x_i^2) + 1/2(\sum_{i=k+1}^n x_i^2)$

$\Delta^q_k(t) : \Omega^q_{L^2}(R^n) \to \Omega^q_{L^2}(R^n)$

the Witten Laplacians associated with $h_k$
Proposition (Harmonic oscillator).

1) $\text{Spec} \Delta^q_k(t) \subset 2t\mathbb{Z}$

2) $\ker \Delta^q_k(t) = 0$ iff $k \neq q$

3) $\ker \Delta^q_q(t) = \{\omega_q\}$ with

$$\omega_q = (t/\pi)^{n/4}e^{-t||x||^2}dx_1 \wedge \cdots \wedge dx_q$$

4) Estimates about $\omega_q$ multiplied but cutoff (away from zero) functions.
**G-symmetry**

G-compact Lie group

ξ irreducible representation of G.

μ : G × M → M, G-manifold

(compatible with ρ)

$g$ invariant Riemannian metric

τ is a G-smooth triangulation

with G- simplexes σ’s,

σ : G/H × Δ(k(σ)) → M, G equivariant

$(C^*(τ, ρ), \partial_ρ)$ is defined by:

$C^*(τ, ρ) = \sum_σ (Ω^{*+k(σ)}(G/H))$

$\partial_ρ = \partial'_ρ + \partial''_ρ$

$\partial'_ρ$ given by exterior differential in $Ω^*(G/H)$

$\partial''_ρ$ given by ρ and the incidence of cells
\( \Omega^*(M, \rho, d) \) and \( C^*(\tau, \rho) \) are  
\( G \)-cochain complexes  

\[ \text{Int}^* : (\Omega^*(M, \rho, d), d) \rightarrow (C^*(\tau, \rho), \partial_{\rho}) \]

decomposes as \( \bigoplus \xi \text{Int}_\xi \) with  

\[ \text{Int}_\xi : (\Omega^*_\xi(M, \rho, d_{\rho, \xi}), d_{\rho, \xi}) \rightarrow (C^*_\xi(\tau, \rho), \partial_{\rho, \xi}) \]

\( (C^*(\tau, \rho)_{\xi}, \partial_{\rho, \xi}) \) is f.d.  
WHS Theorem holds for \( \text{Int}_\xi \).  

There are only finitely many \( G \)-simplexes \( \sigma \).  
The representation \( \xi \) and the subgroups \( H_{\sigma}'s \)

provide a finite collection of explicitly computable  
real real numbers say \( \lambda_{\xi}^1, \lambda_{\xi}^2, \cdots, \lambda_{\xi}^N \)
Theorem 3’.

There exists positive constants $C_1(\xi), C_2(\xi), C_3(\xi), T(\xi)$ so that:

1) $\text{Spec} \Delta^q(t) \subset \bigcup_{1 \leq i \leq N} [\lambda_i - C_1 e^{-C_2t}, \lambda_i + C_1 e^{-C_2t}]$

$\cup [C_3 t, \infty)$ if $t > T(\xi)$

2) $\sharp (\text{Spect} \Delta^q(t) \cap [\lambda_i - C_1 e^{-C_2t}, \lambda_i + C_1 e^{-C_2t}])$

can be specified in terms of the number of $G$ simplexes in each dimension.

What WHS-Theorem was good for?

Prove equality of invariants defined combinatorially

(i.e. using a smooth (generalized) triangulation and

analytically (i.e. using a Riemannian metric)
Theorem Witten Hellfer-Sjöstrand (reformulation).

$g$ and $\tau$ induce an orthogonal decompositions

$$(\Omega^q(M,\rho),d\rho(t) = (\Omega^q_{sm}(M,\rho)(t), d\rho(t)) \oplus (\Omega^q_{la}(M,\rho), d\rho(t))$$

and an isomorphism

$$R^*(t) \cdot S^*(t) : (C^*(\tau,\rho), \partial \rho) \to (\Omega^q_{sm}(M,\rho)(t), d\rho(t)) \subset (\Omega^q(M,\rho)(t), d\rho(t))$$