

# Alternative to Morse-Novikov Theory for a closed 1-form (II)

Dan Burghelea \*

## Abstract

This paper is a continuation of [1] and establishes:

- a) a refinement of Poincaré duality to an equality between the configurations  ${}^{BM}\delta^\omega$  and  $\delta^\omega$  resp.  ${}^{BM}\gamma^\omega$  and  $\gamma^\omega$  in complementary dimensions,
- b) the stability property for the configurations  $\delta_r^\omega$ ,
- c) a collection of additional results needed for the proof of Theorems 1.2 and 1.3 in [1].

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Linear algebra preliminary</b>	<b>3</b>
<b>3</b>	<b>Notations and definitions</b>	<b>10</b>
<b>4</b>	<b>Proof of Poincaré duality, Theorem 1.1</b>	<b>19</b>
<b>5</b>	<b>Proof of stability property, Theorem 1.2</b>	<b>25</b>
<b>6</b>	<b>Standard homology versus Borel-Moore homology (Proposition 1.4)</b>	<b>27</b>

## 1 Introduction

This paper is a continuation of [1] and establishes:

a) a refinement of Poincaré duality to an equality between the configurations  ${}^{BM}\delta^\omega$  and  $\delta^\omega$  resp.  ${}^{BM}\gamma^\omega$  and  $\gamma^\omega$  in complementary dimensions

b) the stability property for the configurations  $\delta_r^\omega$ ,

c) a collection of additional results needed for the proof of Theorems 1.2 and 1.3 announced in [1]. The reader is supposed to be familiar with the notations and the main definitions in [1].

Since we are interested in Poincaré duality which has to be considered for non compact manifolds, it is necessary to involve the functor  ${}^{BM}H_r$ , the Borel-Moore homology, with coefficients in a fixed field  $\kappa$ , abbreviated BM-homology and the natural transformation  $\theta_r : H_r \rightarrow {}^{BM}H_r$ . We refer to [7] chapter 5 for reference to Borel-Moore homology. Recall that Borel-Moore homology with coefficients in a field  $\kappa$  is a collection  ${}^{BM}H_r$  of  $\kappa$ -vector space valued functors defined on the category of pairs  $(X, Y)$ ,  $X$  and  $Y$  locally compact spaces, with  $Y \subseteq X$  closed subset of  $X$ , and proper continuous maps of pairs, which are homotopic functors (i.e. proper homotopic maps induce equal linear maps) and have excision

---

\*Department of Mathematics, The Ohio State University, Columbus, OH 43210,USA. Email: burghele@math.ohio-state.edu

and the long exact sequence property for any such pair  $(X, Y)$ . When restricted to the subcategory of pairs of compact spaces they coincide with the standard homology. The reader unfamiliar with Borel-Moore homology should be aware that for  $(M, N)$  a pair of f.d. manifolds with  $N$  a closed subset,  ${}^{BM}H_r(M, N) = H_r(\hat{M}, \hat{N})$  and  ${}^{BM}H_r(M) = H_r(\hat{M}, *)$  where  $\hat{X}$  denotes the one point compactification of the locally compact space  $X$ . This explains the natural maps  $H_r(\dots) \rightarrow {}^{BM}H_r(\dots)$  from the standard homology (= singular homology) to the Borel-Moore homology as well as the contravariant behavior of Borel-Moore homology when restricted to open sets of a locally compact space.

The Borel-Moore homology is not a priori defined for a pair  $(X, Y)$  when both are locally compact but  $Y$  is not closed, however a vector space  ${}^{BM}\mathcal{H}_r^f(X, Y)$  can be introduced in the case  $Y$  is open in  $X$  and appears as an open sub-level of a continuous real-valued map  $f$ , as described in section 3.

Note that all definitions and concepts derived using standard homology in [1] can be "word by word" repeated for Borel-Moore homology. In particular one can produce the Borel-Moore version of the vector spaces  $\mathbb{I}_a^f(r)$ ,  $\mathbb{I}_f^b(r)$ ,  $\mathbb{F}_r^f(a, b)$ ,  $\mathbb{T}_r^f(a, b)$ ,  $\hat{\delta}_r^f(a, b)$ ,  $\hat{\gamma}_r^f(a, b)$  and ultimately of the numbers  $\delta_r^f(a, b)$ ,  $\gamma_r^f(a, b)$ ,  $\delta_r^\omega(r)$ ,  $\gamma_r^\omega(t)$ , denoted by the same symbols with the addition of the left-side exponent  $\{BM\}$ . In general these spaces / numbers are different from the ones defined using standard homology. This is reviewed in section 3. The natural transformations  $\theta_r : H_r(\dots) \rightsquigarrow {}^{BM}H_r(\dots)$  induce the natural linear maps  $\theta_r(a) : H_r(\tilde{X}_a, \tilde{X}_{<a}) \rightarrow {}^{BM}\mathcal{H}_r(\tilde{X}_a, \tilde{X}_{<a})$ ,  $\theta_r^\delta(a, b) : \hat{\delta}_r^f(a, b) \rightarrow {}^{BM}\hat{\delta}_r^f(a, b)$  and  $\theta_r^\gamma(a, b) : \hat{\gamma}_r^f(a, b) \rightarrow {}^{BM}\hat{\gamma}_r^f(a, b)$ .

Recall that  $\mathcal{Z}_t^1(X; \xi) \subset \mathcal{Z}^1(X; \xi)$  denotes the space of tame topological closed one forms with the topology induced from the compact open topology on  $\mathcal{Z}^1(X; \xi)$ , as defined in Section 2 of [1]. The topology on  $Conf_{\beta_r^N(X; \xi)}(\mathbb{R})$ , the set of configurations of points with multiplicity located in  $\mathbb{R}$  of total cardinality  $\beta_r^N(X, \xi)^1$ , is the *collision topology* cf. [1]. Precisely, the space of configurations of points in  $Y$  of total cardinality  $N$  can be identified to the quotient space  $Y^N / \Sigma_N$ , where  $\Sigma_N$  denotes the group of permutations of  $N$  elements acting on  $Y^N$  by permutations.

The main results verified in this paper are the following:

### Theorem 1.1

Suppose  $M$  is a closed topological manifold and  $\omega \in \mathcal{Z}_t^1(X; \xi)$ . Then the following holds true.

1.  ${}^{BM}\delta_r^\omega(t) = \delta_{n-r}^\omega(-t)$ ,
2.  ${}^{BM}\gamma_r^\omega(t) = \gamma_{n-r-1}^{-\omega}(t)$ .

### Theorem 1.2

The assignments  $\omega \rightsquigarrow \delta_r^\omega$  from  $\mathcal{Z}_t^1(X; \xi)$  to  $Conf_{\beta_r^N(X; \xi)}(\mathbb{R})$  is continuous when the source  $\mathcal{Z}_t^1(X; \xi)$  is equipped with the compact-open topology and the target  $Conf_{\beta_r^N(X; \xi)}(\mathbb{R})$  with the collision topology.

We will also establish the following results essential for the proofs in Theorems 1.2 and 1.3 in [1].

If  $f : \tilde{X} \rightarrow \mathbb{R}$  is a lift of a tame TC1-form  $\omega$  on a closed compact ANR then for both the standard and the Borel-Moore homology one has the following equalities.

### Proposition 1.3

1.  $\dim(H_r(\tilde{X}_a, \tilde{X}_{<a})) = \sum_{t \in \mathbb{R}} \delta_r^f(a, t) + \sum_{t > a} \gamma_r^f(a, t) + \sum \gamma_{r-1}^f(t, a)$ .
2.  $\dim({}^{BM}\mathcal{H}_r(\tilde{X}_a, \tilde{X}_{<a})) = \sum_{t \in \mathbb{R}} {}^{BM}\delta_r^f(a, t) + \sum_{t > a} {}^{BM}\gamma_r^f(a, t) + \sum_{t < a} {}^{BM}\gamma_{r-1}^f(t, a)$ .

Formula 1. was established in [1]

---

<sup>1</sup>the Novikov-Betti number of  $(X, \xi)$ ,  $\xi \in H^1(X : \kappa)$

**Proposition 1.4** *If  $f : \tilde{X} \rightarrow \mathbb{R}$  is a lift of a tame TC1-form  $\omega$  on a closed topological manifold  $X$  then the linear maps  $\theta_r(a) : H_r(\tilde{X}_a, \tilde{X}_{<a}) \rightarrow^{BM} \mathcal{H}_r(\tilde{X}_a, \tilde{X}_{<a})$  are isomorphisms.*

One expects that both  $\theta_r^\delta(a, b) : \hat{\delta}_r^f(a, b) \rightarrow^{BM} \delta_r^f(a, b)$  and  $\theta_r^\gamma(a, b) : \hat{\gamma}_r^f(a, b) \rightarrow^{BM} \gamma_r^f(a, b)$  are isomorphisms (see Corollary (6.2) in section 6). This is indeed the case when  $\omega$  is of degree of irrationality 1.

Theorem 1.2 verifies Theorem 3 item 1 in [1]. Item 2 will be treated in part 3 of this work.

Theorem 1.1 verifies Theorem T2 announced in [1] provided  $\theta_r^\delta$  and  $\theta_r^\gamma$  are isomorphisms, not yet established in full generality and possibly not true.

The paper requires some basic linear algebra, most likely familiar to most of the readers but for convenience of the reader reviewed in Section 2. The reader can skip this section at the first reading and return to it when necessary.

## 2 Linear algebra preliminary

### 2.1 Linear algebra

In this section ”=” designates equality or canonical isomorphism and  $\simeq$  indicates the existence of an isomorphism.

Let  $\kappa$  be a fixed field and  $\mathcal{Vect}_\kappa$  the category of  $\kappa$ -vector spaces; the objects are  $\kappa$ -vector spaces and the morphisms are linear maps. Recall that for a linear map  $\alpha : A \rightarrow B$  the *canonical exact sequence associated to  $\alpha$*

$$0 \longrightarrow \ker \alpha \xrightarrow{i_\alpha} A \xrightarrow{\alpha} B \xrightarrow{\pi_\alpha} \text{coker } \alpha \longrightarrow 0.$$

By passing to ”duals” one obtains the exact sequence

$$0 \longleftarrow (\ker \alpha)^* \xleftarrow{i_\alpha^*} A^* \xleftarrow{\alpha^*} B^* \xleftarrow{\pi_\alpha^*} (\text{coker } \alpha)^* \longleftarrow 0,$$

canonically isomorphic to the canonical exact sequence of  $\alpha^*$ ,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker(\alpha^*) & \xrightarrow{i_{\alpha^*}} & B^* & \xrightarrow{\alpha^*} & A^* & \xrightarrow{\pi_{\alpha^*}} & (\text{coker } \alpha^*) & \longrightarrow & 0 \\ & & \uparrow \theta & & \uparrow = & & \uparrow = & & \uparrow \theta & & \\ 0 & \longrightarrow & (\text{coker } \alpha)^* & \xrightarrow{\pi_\alpha^*} & B^* & \xrightarrow{\alpha^*} & A^* & \xrightarrow{i_\alpha^*} & (\ker \alpha)^* & \longrightarrow & 0. \end{array} \quad (1)$$

For a diagram

$$D := \begin{array}{ccc} A & \xrightarrow{\alpha_1} & B_1 \\ \alpha_2 \downarrow & & \downarrow \beta_1 \\ B_2 & \xrightarrow{\beta_2} & C \end{array}$$

let  $\underline{\alpha} : A \rightarrow \alpha_2 \cup_A \alpha_1$  be the push-forward of  $\alpha_2$  and  $\alpha_1$ , and  $\overline{\beta} : \beta_2 \times_C \beta_1$  be the pullback of  $\beta_2$  and  $\beta_1$ , with

$$\begin{aligned} \alpha_2 \cup_A \alpha_1 &:= B_2 \oplus B_1 / \{\alpha_2(a) - \alpha_1(a) \mid a \in A\}, \\ \underline{\alpha}(a) &= \langle \alpha_2(a) \oplus \alpha_1(a) \rangle, \\ \beta_1 \times_C \beta_2 &:= \{(b_1, b_2) \in B_1 \times B_2 \mid \beta_1(b_1) = \beta_2(b_2)\}, \\ \overline{\beta}(b_1, b_2) &= \beta_1((b_1) = \beta_2(b_2)). \end{aligned}$$

with  $\langle b_2 \oplus b_1 \rangle$  denoting the image of  $b_1 \oplus b_2$  in  $\alpha_2 \cup_A \alpha_1$ .

Let  $\alpha : A \rightarrow \alpha_2 \cup_A \alpha_1$  and  $\beta : \beta_2 \times_C \beta_1 \rightarrow C$  be given by

$$\begin{aligned}\alpha(a) &= (\alpha_2(a), \alpha_1(a)) \\ \beta((b_2 \oplus b_1)) &= \beta_2(b_1) + \beta_2(b_2).\end{aligned}$$

Define

$$\begin{aligned}\ker(D) &:= \ker \alpha \\ \operatorname{coker}(D) &:= \operatorname{coker} \beta.\end{aligned}$$

**Observation 2.1** *The canonical isomorphisms  $\theta$  extend to the canonical isomorphisms*

$$\theta(D) : (\operatorname{coker} D)^* \rightarrow \ker(D^*)$$

and

$$\theta(D) : (\ker D)^* \rightarrow \operatorname{coker}(D^*).$$

*Proof:* To check the statements observe that the diagram  $D$  can be completed to the diagrams  $\underline{D}$  and  $\overline{D}$

$$\begin{array}{ccc} \underline{D} := & \begin{array}{ccc} A & \xrightarrow{\alpha_1} & B_1 \\ \alpha \searrow & & \nearrow i_1 \\ & \alpha_2 \cup_A \alpha_1 & \\ i_2 \nearrow & & \searrow \beta \\ B_2 & \xrightarrow{\beta_2} & C \end{array} & \overline{D} := & \begin{array}{ccc} A & \xrightarrow{\alpha_1} & B_1 \\ \alpha \searrow & & \nearrow p_1 \\ & \beta_2 \times_C \beta_1 & \\ p_2 \nearrow & & \searrow \bar{\beta} \\ B_2 & \xrightarrow{\beta_2} & C \end{array} \end{array}$$

and notice that  $(\underline{D})^*$  identifies to  $(\overline{D^*})$  and  $(\overline{D})^*$  identifies to  $(D^*)$  which imply the statements. q.e.d.

Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$ ,  $\gamma : C \rightarrow D$  be linear maps. To these three maps we associate the diagrams,  $\widehat{D}$  and  $\underline{D}$ :

$$\widehat{D}(\alpha, \beta, \gamma) \equiv \begin{cases} \ker(\gamma\beta\alpha) \xrightarrow{j_2} \ker(\gamma\beta) \\ \uparrow i_1 \quad \nearrow k \quad \uparrow i_2 \\ \ker(\beta\alpha) \xrightarrow{j_1} \ker(\beta) \end{cases} \quad \underline{D}(\alpha, \beta, \gamma) \equiv \begin{cases} \operatorname{coker}(\gamma\beta\alpha) \xrightarrow{j'_2} \operatorname{coker}(\gamma\beta) \\ \uparrow i'_1 \quad \nearrow k' \quad \uparrow i'_2 \\ \operatorname{coker}(\beta\alpha) \xrightarrow{j'_1} \operatorname{coker}(\beta) \end{cases} .$$

In the diagram  $\widehat{D}$

- (a)  $i_1$  and  $i_2$  are injective,
- (b)  $i_1 : \ker j_1 \rightarrow \ker j_2$  is an isomorphism,
- (c)  $\operatorname{img} k = \operatorname{img} j_2 \cap \operatorname{img} i_2$

and in the diagram  $\underline{D}$

- (a)  $j'_1$  and  $j'_2$  are surjective,
- (b)  $\operatorname{coker} i'_1 \rightarrow \operatorname{coker} i'_2$  is an isomorphism,
- (c)  $\ker(j'_2 \cdot i'_1 = i'_2 \cdot j'_1) = \ker i'_1 + \ker j'_1$ .

In view of Observation 2.1 one has

**Observation 2.2**  $(\hat{D}(\alpha, \beta, \gamma))^* = \underline{D}(\gamma^*, \beta^*, \alpha^*)$  and  $(\underline{D}(\alpha, \beta, \gamma))^* = \hat{D}(\gamma^*, \beta^*, \alpha^*)$ .

One defines the vector space

$$\begin{aligned} \hat{\omega}(\alpha, \beta, \gamma) &= \text{coker} \hat{D}(\alpha, \beta, \gamma) := \text{coker}(j_1 \cup_{\ker(\beta\alpha)} i_1 \rightarrow \ker(\gamma\beta)) \\ &= \boxed{\ker(\gamma\beta) / (j_2(\ker(\gamma\beta\alpha)) + i_2(\ker \beta))}, \end{aligned} \quad (2)$$

a quotient space of  $\ker(\gamma\beta)$ , and the vector space

$$\underline{\omega}(\alpha, \beta, \gamma) = \ker \underline{D}(\alpha, \beta, \gamma) := \boxed{\ker(\text{coker}(\beta\alpha) \rightarrow i'_2 \times_{\text{coker}(\gamma\beta)} j'_2)},$$

a subspace of  $\text{coker}(\beta\alpha)$ .

Note that the assignments  $(\alpha, \beta, \gamma) \rightsquigarrow \hat{\omega}(\alpha, \beta, \gamma)$  and  $(\alpha, \beta, \gamma) \rightsquigarrow \underline{\omega}(\alpha, \beta, \gamma)$  are functorial and in view of the definitions above if  $\alpha$  is surjective or if  $\gamma$  is injective then  $\hat{\omega}(\alpha, \beta, \gamma) = 0$ .

**Theorem 2.3**

1. For  $\alpha, \beta, \gamma$  linear maps the isomorphisms  $\theta$  extend to the canonical isomorphisms

$$\theta : \hat{\omega}(\alpha, \beta, \gamma)^* \rightarrow \underline{\omega}(\gamma^*, \beta^*, \alpha^*)$$

$$\theta : \underline{\omega}(\alpha, \beta, \gamma)^* \rightarrow \hat{\omega}(\gamma^*, \beta^*, \alpha^*).$$

2. For  $\alpha, \beta, \gamma$  and  $\alpha', \beta', \gamma'$  linear maps consider the diagram

$$\begin{array}{ccccccc} M & \xrightarrow{=} & M & \xrightarrow{=} & M & \xrightarrow{=} & M \\ \lambda_A \uparrow & & \lambda_B \uparrow & & \lambda_C \uparrow & & d \uparrow \\ A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & D \\ a \uparrow & & b \uparrow & & c \uparrow & & d \uparrow \\ A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \xrightarrow{\gamma'} & D' \\ \theta_A \uparrow & & \theta_B \uparrow & & \theta_C \uparrow & & \theta_D \uparrow \\ N & \xrightarrow{=} & N & \xrightarrow{=} & N & \xrightarrow{=} & N \end{array} \quad (3)$$

If the columns are exact sequences then the linear maps

$$\hat{\omega}(\alpha', \beta', \gamma') \rightarrow \hat{\omega}(\alpha, \beta, \gamma)$$

and

$$\underline{\omega}(\alpha, \beta, \gamma) \rightarrow \underline{\omega}(\alpha', \beta', \gamma')$$

are isomorphisms.

3. For  $\alpha, \beta, \gamma$  linear maps the following holds true.

(i) A factorization  $\alpha = \alpha_2 \cdot \alpha_1$ , with  $\alpha_1 : A \rightarrow A'$  and  $\alpha_2 : A' \rightarrow B$  linear maps, induces the short exact sequences

$$\begin{aligned} 0 \rightarrow \hat{\omega}(\alpha_1, \beta\alpha_2, \gamma) \rightarrow \hat{\omega}(\alpha, \beta, \gamma) \rightarrow \hat{\omega}(\alpha_2, \beta, \gamma) \rightarrow 0, \\ 0 \rightarrow \underline{\omega}(\alpha_1, \beta\alpha_2, \gamma) \rightarrow \underline{\omega}(\alpha, \beta, \gamma) \rightarrow \underline{\omega}(\alpha_2, \beta, \gamma) \rightarrow 0. \end{aligned} \quad (4)$$

(ii) A factorization  $\beta = \beta_2 \cdot \beta_1$ , with  $\beta_1 : B \rightarrow B'$  and  $\beta_2 : B' \rightarrow B$  linear maps, induces the short exact sequences

$$\begin{aligned} 0 \rightarrow \widehat{\omega}(\alpha, \beta_1, \beta_2) \rightarrow \widehat{\omega}(\alpha, \beta_1, \gamma\beta_2) \rightarrow \widehat{\omega}(\alpha, \beta, \gamma) \rightarrow 0, \\ 0 \rightarrow \underline{\omega}(\alpha, \beta_1, \beta_2) \rightarrow \underline{\omega}(\alpha, \beta_1, \gamma\beta_2) \rightarrow \underline{\omega}(\alpha, \beta, \gamma) \rightarrow 0. \end{aligned} \quad (5)$$

(iii) A factorization  $\gamma = \gamma_2 \cdot \gamma_1$ , with  $\gamma_1 : C \rightarrow C'$  and  $\gamma_2 : C' \rightarrow D$  maps, induces the short exact sequences

$$\begin{aligned} 0 \rightarrow \widehat{\omega}(\alpha, \beta, \gamma_1) \rightarrow \widehat{\omega}(\alpha, \beta, \gamma) \rightarrow \widehat{\omega}(\alpha, \gamma_1\beta, \gamma_2) \rightarrow 0, \\ 0 \rightarrow \underline{\omega}(\alpha, \beta, \gamma_1) \rightarrow \underline{\omega}(\alpha, \beta, \gamma) \rightarrow \underline{\omega}(\alpha, \gamma_1\beta, \gamma_2) \rightarrow 0. \end{aligned} \quad (6)$$

*Proof:* Item 1. follows from (2.1) and (2.2).

To prove Item 2. notice the following.

i. If  $N = 0$ , then

$$\begin{aligned} \ker(\beta) &\simeq \ker(\beta'), \\ \ker(\beta\gamma) &\simeq \ker(\beta'\gamma') \\ \ker(\alpha\beta) &\simeq \ker(\alpha'\beta'), \\ \ker(\alpha\beta\gamma) &\simeq \ker(\alpha'\beta'\gamma'). \end{aligned}$$

hence  $\widehat{D}(\alpha', \beta', \gamma') = \widehat{D}(\alpha, \beta, \gamma)$  and then  $\widehat{\omega}(\alpha', \beta', \gamma') = \widehat{\omega}(\alpha, \beta, \gamma)$ .

ii. If  $M = 0$  then the obvious induced maps

$$\begin{aligned} \ker(\beta'\alpha') &\rightarrow \ker(\beta\alpha) \\ \ker(\gamma'\beta'\alpha') &\rightarrow \ker(\gamma\beta\alpha) \\ \ker(\gamma'\beta') &\rightarrow \ker(\gamma\beta) \\ \ker(\beta') &\rightarrow \ker(\beta) \end{aligned}$$

are surjective and then in view of the definition of  $\widehat{\omega}$  the induced linear map

$$\ker(\beta', \gamma') / (i_2(\ker(\gamma\beta, \alpha) + \ker(\beta))) \rightarrow \ker \beta, \gamma / (i_2(\ker(\gamma\beta, \alpha) + \ker(\beta)))$$

is surjective.

To show that this induced map is equally injective one has to verify that any  $x' \in \ker(\gamma'\beta')$  with  $b(x') = \alpha(z) + y$ ,  $z \in \ker(\gamma\beta\alpha)$  and  $y \in \ker(\beta)$ , can be written as  $x' = \alpha(z') + y'$  with  $z' \in \ker(\gamma'\beta'\alpha')$  and  $y' \in \ker(\beta')$ . In view of the above mentioned surjectivity one chooses  $\underline{z}' \in \ker(\gamma'\beta'\alpha')$  and  $y' \in \ker(\beta')$ , with  $a(\underline{z}') = z$  and  $b(y') = y$ , and one considers  $x' - \alpha'(\underline{z}') - y' \in \ker(\gamma'\beta')$  which by  $b$  is sent to zero, hence is of the form  $\theta_B(u)$ ,  $u \in N$ . Clearly  $\gamma'\beta'(\theta_B(u)) = 0$ , therefore one can correct  $\underline{z}'$  into  $z' = \underline{z}' + \theta_A(u)$ , and one obtains  $x' = \alpha'(z') + y'$  with  $z' \in \ker(\gamma'\beta'\alpha')$ .

Consequently, the induced map  $\widehat{\omega}(\alpha', \beta', \gamma') \rightarrow \widehat{\omega}(\alpha, \beta, \gamma)$  is an isomorphism.

iii. To prove the result for  $M$  and  $N$  arbitrary consider the diagram

$$\begin{array}{ccccccc} & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & D \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \ker a & \xrightarrow{\alpha''} & \ker b & \xrightarrow{\beta''} & \ker c & \xrightarrow{\gamma''} & \ker d & \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ & A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \xrightarrow{\gamma'} & D' \end{array}$$

and the linear maps

$$\widehat{\omega}(\alpha', \beta', \gamma') \rightarrow \widehat{\omega}(\alpha'', \beta'', \gamma'') \rightarrow \widehat{\omega}(\alpha, \beta, \gamma).$$

The first arrow is an isomorphism by (ii) above and the second by (i) above. This establishes Item 2.

To prove Item 3 consider the diagram

$$\begin{array}{ccccc}
 & & \overset{i_2}{\curvearrowright} & & \\
 & & \xrightarrow{i_2^A} & & \xrightarrow{i_2^B} \\
 A_2 & \xrightarrow{\quad} & B_2 & \xrightarrow{\quad} & C_2 \\
 \uparrow j_A & \nearrow & \uparrow j_B & \nearrow & \uparrow j_C \\
 A_1 & \xrightarrow{\quad} & B_1 & \xrightarrow{\quad} & C_1 \\
 & & \underset{i_1}{\curvearrowleft} & & \\
 & & \xrightarrow{i_1^A} & & \xrightarrow{i_1^B}
 \end{array} \tag{7}$$

and make the following observation.

**Observation 2.4** Suppose that each of the three diagrams  $\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}$ , associated with (7),  $\mathbb{B}_1$  with vertices  $A_1, A_2, B_1, B_2$ ,  $\mathbb{B}_2$  with vertices  $B_1, B_2, C_1, C_2$  and  $\mathbb{B}$  with vertices  $A_1, A_2, C_1, C_2$  satisfy the properties (a) (b) (c) of the diagram  $\hat{D}$ . Then (7) induces the exact sequence

$$\begin{array}{ccccc}
 B_2/(i_2^A(A_2) + j_B(B_1)) & & & & \\
 \uparrow & \searrow i & & & \\
 0 & C_2/(i_2(A_2) + j_C(C_1)) & & 0 & \\
 & \searrow p & & \uparrow & \\
 & C_2/(i_2^B(B_2) + j_C(C_1)) & & & 
 \end{array}$$

with  $i$  induced by  $i_2^B$ , well defined because  $\text{img}(i_2^B \cdot j_B) \subseteq \text{img}j_C$ , and  $p$  the projection induced by the inclusion  $(i_2(A_2) + j_C(C_1)) \subseteq (i_2^B(B_2) + j_C(C_1))$ .

Clearly  $p$  is surjective and  $p \cdot i = 0$ . Property (c) implies that  $i$  is injective. Properties (a), (b) (c) imply that the sequence is exact.

A similar observation holds for the diagram

$$\begin{array}{ccccc}
 & & \overset{i_3}{\curvearrowright} & & \\
 & & \xrightarrow{i_3} & & \\
 A_3 & \xrightarrow{\quad} & B_3 & & \\
 \uparrow j_2^A & \nearrow & \uparrow j_2^B & \nearrow & \\
 A_2 & \xrightarrow{\quad} & B_2 & & \\
 \uparrow j_1^A & \nearrow & \uparrow j_1^B & \nearrow & \\
 A_1 & \xrightarrow{\quad} & B_1 & & \\
 & & \underset{i_1}{\curvearrowleft} & & \\
 & & \xrightarrow{i_1} & & 
 \end{array} \tag{8}$$

**Observation 2.5** Suppose that each of the three diagrams  $\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}$ , associated with (8),  $\mathbb{B}_1$  with vertices  $A_2, A_3, B_2, B_3$ ,  $\mathbb{B}_2$  with vertices  $A_1, A_2, B_1, B_2$  and  $\mathbb{B}$  with vertices  $A_1, A_3, B_1, B_2$

satisfy the properties (a) (b) (c) of the diagram  $\hat{D}$  Then (8) induces the exact sequence

$$\begin{array}{ccccc}
 B_2/(i_2(A_2) + j_1^B(B_1)) & & & & \\
 \uparrow & \searrow j & & & \\
 0 & & B_3/(i_3(A_3) + j^B(B_1)) & & 0 \\
 & & \searrow p & & \uparrow \\
 & & & & B_3/(i_3(A_3) + j_2^B(B_2))
 \end{array}$$

Observation 2.4 applied to diagram (7) with

$$\begin{aligned}
 A_1 &= \ker(\beta\alpha), & A_2 &= \ker(\gamma\beta\alpha), \\
 B_1 &= \ker(\beta\alpha_2), & B_2 &= \ker(\gamma\beta\alpha_2), \\
 C_1 &= \ker(\beta), & C_2 &= \ker(\gamma\beta)
 \end{aligned}$$

verifies Item 3. (i).

Observation 2.5 applied to diagram (8) with

$$\begin{aligned}
 A_1 &= \ker(\beta_1\alpha), & B_1 &= \ker(\beta_1), \\
 A_2 &= \ker \beta\alpha, & B_2 &= \ker(\beta), \\
 A_3 &= \ker(\gamma\beta\alpha), & B_3 &= \ker(\gamma\beta)
 \end{aligned}$$

verifies Item 3. (ii) and applied to diagram (8) with

$$\begin{aligned}
 A_1 &= \ker(\beta\alpha), & B_1 &= \ker(\beta), \\
 A_2 &= \ker(\gamma_1\beta\alpha), & B_2 &= \ker(\gamma_1\beta), \\
 A_3 &= \ker(\gamma\beta\alpha), & B_3 &= \ker(\gamma\beta)
 \end{aligned}$$

verifies Item 3. (iii). ■

The above considerations/proofs were already contained in [2] but under the additional hypothesis that all the linear maps were Fredholm.

As in [8] for a direct system of vector spaces

$$\{A_i, u_i\} := A_1 \xrightarrow{u_1} A_2 \xrightarrow{u_2} A_3 \xrightarrow{u_3} \cdots \xrightarrow{u_{i-1}} A_i \xrightarrow{u_i} A_{i+1} \xrightarrow{u_{i+1}} \cdots$$

consider  $\Sigma : \bigoplus_i A_i \rightarrow \bigoplus A_i$  defined by

$$\Sigma(a_1, a_2, \cdots, a_i, \cdots) = (a_1, a_2 - u_1(a_1), \cdots, a_i - u_{i-1}(a_{i-1}), \cdots).$$

Recall the definition of the direct limit

$$\varinjlim_i \{A_i, u_i\} := \text{coker} \Sigma.$$

For an inverse system of vector spaces

$$\{A_i, p_i\} := A_1 \xleftarrow{p_2} A_2 \xleftarrow{p_3} A_3 \xleftarrow{p_3} \cdots \xleftarrow{p_i} A_i \xleftarrow{p_{i+1}} A_{i+1} \xleftarrow{p_{i+1}} \cdots$$

consider  $\Pi : \otimes_i A_i \rightarrow \otimes A_i$  defined by

$$\Pi(a_1, a_2, \dots, a_i, \dots) = (a_1, a_2 - u_1(a_1), \dots, a_i - u_{i-1}(a_{i-1}), \dots).$$

Recall the definitions of the *inverse limit* and the *derived inverse limit*

$$\varprojlim_i \{A_i, p_i\} := \ker \Pi, \quad \varprojlim'_i \{A_i, p_i\} := \text{coker} \Pi.$$

For the proof of Theorem 1.1 one needs some additional definitions.

**Definition 2.6** A sub-surjection  $\tilde{\pi} : A \rightsquigarrow A'$  is a pair  $\tilde{\pi} := \{\pi : A \rightarrow P, P \supseteq A'\}$  consisting of a surjective linear map  $\pi$  and a subspace  $A'$  of  $P$ .

To a directed system of sub-surjections

$$A_1 \xrightarrow{\tilde{\pi}_1} A_2 \xrightarrow{\tilde{\pi}_2} A_3 \xrightarrow{\tilde{\pi}_3} \dots \xrightarrow{\tilde{\pi}_{k-1}} A_k \xrightarrow{\tilde{\pi}_k} A_{k+1} \xrightarrow{\tilde{\pi}_{k+1}} \dots$$

one provides a unique maximal directed system of surjections

$$\begin{array}{ccccccc} A_1 & & A_2 & & A_3 & & A_k & & A_{k+1} \\ \subseteq \uparrow & & \subseteq \uparrow & & \subseteq \uparrow & & \subseteq \uparrow & & \subseteq \uparrow \\ A_1^\infty & \xrightarrow{\pi_1^\infty} & A_2^\infty & \xrightarrow{\pi_2^\infty} & A_3^\infty & \xrightarrow{\pi_3^\infty} & \dots & \xrightarrow{\pi_{k-1}^\infty} & A_k^\infty & \xrightarrow{\pi_k^\infty} & A_{k+1}^\infty & \xrightarrow{\pi_{k+1}^\infty} & \dots \end{array}$$

The construction of this sequence is rather straightforward and it goes as follows:

Starting with the right side of the diagram (10) (i.e. the collections of linear maps  $\{\pi_i, \supseteq\}$ ), one inductively, from right to left and from up to down (i.e from the lower index  $i$  to lower index  $(i - 1)$  and from the upper index  $k$  to the upper index  $(k + 1)$ ) one produces the subspace  $A_i^{k+1} \subset A_i^k$  and the surjective linear maps  $\pi_i^k : A_i^k \rightarrow A_{i+1}^k$  defined by:

$$\begin{aligned} A_{k+1}^{k+1} &= (\pi^k)^{-1}(A_{k+1}), \\ A_k^{k+1} &= (\pi^k)^{-1}(A_{k+1}) \text{ and} \\ \pi_i^{k+1} &= \text{restriction of } \pi_i^k. \end{aligned}$$

Take  $A_k^\infty = \bigcap_{i>k} A_i^k$  and  $\pi_i^\infty$  the restriction of  $\pi_i^k$  and define

$$\varprojlim_{i \rightarrow \infty} \tilde{\pi}_i := \varprojlim_{i \rightarrow \infty} \pi_i^\infty. \tag{9}$$



By passing to direct limit when  $\epsilon \rightarrow 0$  the long exact sequence for the pair  $(\tilde{X}_a, \tilde{X}_{a-\epsilon})$  leads to the long exact sequence

$$\cdots \longrightarrow H_r(\tilde{X}_{<a}) \longrightarrow H_r(\tilde{X}_a) \longrightarrow \mathcal{H}_r(\tilde{X}_a, \tilde{X}_{<a}) \longrightarrow H_{r-1}(\tilde{X}_{<a}) \longrightarrow \cdots \quad (11)$$

Given  $f : Y \rightarrow \mathbb{R}$  a continuous map

- (a)  $t \in \mathbb{R}$  is a *regular value* iff for any  $r \geq 0$  and any open set  $U \subseteq Y$  one has  $h_r(U_t, U_{<t}) = h_r(U^t, U^{>t}) = 0$  with  $h_r$  denoting the singular homology. If  $Y$  is a smooth manifold, possibly with boundary  $\partial Y$ , and  $f$  is a smooth map with  $t$  a regular value in the sense of differential calculus for both  $Y$  and  $\partial Y$ , then  $t$  is a regular value in the sense mentioned above.
- (b)  $t \in \mathbb{R}$  is a *critical value* if not regular. Denote by  $CR(f) \subset \mathbb{R}$  the set of all critical values.
- (c)  $x \in X$  is a *critical point* if for some small enough neighborhood  $U$  of  $x$ ,  $f(x)$  is a critical value for the restriction of  $f$  to  $U$ . Denote by  $Cr(f) \subset X$  the set of critical points.

The continuous map  $f : Y \rightarrow \mathbb{R}$  is *tame* if the following holds true:

- (i)  $Y$  is a locally compact ANR and  $f^{-1}(I)$  is an ANR for any closed interval  $I \subset \mathbb{R}$ ,
- (ii)  $CR(f)$  is countable, hence the set of regular values is dense in  $\mathbb{R}$ ,
- (iii) for any  $t \in CR(f)$  the set  $Cr_t(f) = f^{-1}(t) \cap Cr(f)$  is a compact ANR.

A TC1-form  $\omega$  on the compact ANR  $X$  is *tame* if one lift  $f : \tilde{X} \rightarrow \mathbb{R}$ , and then any other lift, is a tame map and in addition the set of orbits of the free action of  $\Gamma$  on  $CR(f)$  is finite.

A smooth closed one form on a closed smooth manifold with the property that for any zero one can find coordinates s.t. its components in these coordinates are polynomials, in particular a Morse closed one form, is tame.

As in [1], for  $f : \tilde{X} \rightarrow \mathbb{R}$  a lift of  $\omega$  one denotes by

$$\begin{aligned} \tilde{X}_a^f &:= f^{-1}((-\infty, a]) \text{ sublevel at } a, & \tilde{X}_{<a}^f &:= f^{-1}((-\infty, a)) \text{ open - sublevel at } a, \\ \tilde{X}_f^a &:= f^{-1}([a, \infty)) \text{ overlevel at } a, & \tilde{X}_f^{>a} &:= f^{-1}((a, \infty)) \text{ open - overlevel at } a, \\ \tilde{X}(a) &:= f^{-1}(a) \text{ level.} \end{aligned}$$

and one defines

- $\mathbb{I}_a^f(r) := \text{img}(H_r(\tilde{X}_a) \rightarrow H_r(\tilde{X}))$ ,
- $\mathbb{I}_{<a}^f(r) := \text{img}(H_r(\tilde{X}_{<a}) \rightarrow H_r(\tilde{X})) = \varinjlim_{\epsilon \rightarrow 0} \text{img}(H_r(\tilde{X}_{a-\epsilon}) \rightarrow H_r(\tilde{X}))$ ,
- $\mathbb{I}_f^a(r) := \text{img}(H_r(\tilde{X}^a) \rightarrow H_r(\tilde{X}))$ ,
- $\mathbb{I}_f^{>a}(r) := \text{img}(H_r(\tilde{X}^{>a}) \rightarrow H_r(\tilde{X})) = \varinjlim_{\epsilon \rightarrow 0} \text{img}(H_r(\tilde{X}^{a+\epsilon}) \rightarrow H_r(\tilde{X}))$ ,
- $\mathbb{I}_r^f(a', a) := \mathbb{I}_a^f(r) / \mathbb{I}_{a'}^f(r)$  for  $a' < a$ ,  $\mathbb{I}_r^f(<a', a) := \mathbb{I}_a^f(r) / \mathbb{I}_{<a'}^f(r)$  for  $a' \leq a$ ,
- $\mathbb{F}_r^f(a, b) := \mathbb{I}_a^f(r) \cap \mathbb{I}_f^b(r)$ ,  $\mathbb{F}_r^f(<a, b) := \mathbb{I}_{<a}^f(r) \cap \mathbb{I}_f^b(r)$ ,  $\mathbb{F}_r^f(a, >b) := \mathbb{I}_a^f(r) \cap \mathbb{I}_f^{>b}(r)$ ,

- $\mathbb{G}_r^f(a, b) := H_r(\tilde{X})/(\mathbb{I}_a^f(r) + \mathbb{I}_f^b(r))$ ,  $\mathbb{G}_r^f(< a, b) := H_r(\tilde{X})/(\mathbb{I}_{<a}^f(r) + \mathbb{I}_f^b(r))$ ,  $\mathbb{G}_r^f(a, > b) := H_r(\tilde{X})/(\mathbb{I}_a^f(r) + \mathbb{I}_f^{>b}(r))$ ,

In order to lighten the notation, when implicit from the context,  $f$  might be dropped off the notation.

For a **box**  $B \subset \mathbb{R}^2$ ,  $B = (a', a] \times [b, b')$ ,  $a' < a, b < b'$ , consider the diagrams  $\mathcal{F}_r(B)$  and  $\mathcal{G}_r(B)$

$$\mathcal{F}_r(B) := \begin{cases} \mathbb{F}_r(a', b') \xrightarrow{\subseteq} \mathbb{F}_r(a, b') \\ \downarrow \subseteq \\ \mathbb{F}_r(a', b) \xrightarrow{\subseteq} \mathbb{F}_r(a, b) \end{cases}, \quad \mathcal{G}_r(B) := \begin{cases} \mathbb{G}_r(a', b') \twoheadrightarrow \mathbb{G}_r(a, b') \\ \downarrow \\ \mathbb{G}_r(a', b) \twoheadrightarrow \mathbb{G}_r(a, b) \end{cases}$$

whose arrows are the obviously induced linear maps. In the diagram  $\mathcal{F}_r(B)$  all these maps are injective and in the diagram  $\mathcal{G}_r(B)$  all are surjective.

As in [1] one defines

- $\mathbb{F}_r(B) := \text{coker}(\mathcal{F}(B)) = \frac{\mathbb{F}_r(a, b)}{\mathbb{F}_r(a', b) + \mathbb{F}_r(a, b')}$ ,
- $\mathbb{G}_r(B) := \ker \mathcal{G}_r(B) = \ker(G_r(\alpha, \beta) \rightarrow G_r(\alpha, \beta') \times_{G_r(\alpha', \beta')} G_r(\alpha', \beta))$ .

The inclusion  $\mathbb{I}_a(r) \cap \mathbb{I}^b(r) \subset (\mathbb{I}_{a'}(r) + \mathbb{I}^b(r)) \cap (\mathbb{I}^{b'}(r) + \mathbb{I}_a(r))$  induces a canonical isomorphism

$$\boxed{\theta_r(B) : \mathbb{F}_r(B) \rightarrow \mathbb{G}_r(B)}. \quad (12)$$

For  $a'' < a' < a$  and  $b < b' < b''$  consider the boxes  $B_1, B_2, B$  with  $B = (a'', a] \times [b, b'')$  and either  $B_1 = (a'', a'] \times [b, b'')$ ,  $B_2 = (a', a] \times [b, b'')$  or  $B_1 = (a'', a] \times [b', b'')$ ,  $B_2 = (a'', a] \times [b, b']$ . In both cases  $B = B_1 \sqcup B_2$ . As in [1] or [2] one has the following.

**Proposition 3.1** (cf. [1]) *The boxes  $B_1, B_2, B$  described above induce the commutative diagram whose rows are exact sequences and vertical arrows are isomorphisms.*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{F}_r(B_1) & \xrightarrow{i_{B_1}^{B_2}} & \mathbb{F}_r(B) & \xrightarrow{\pi_{B_1}^{B_2}} & \mathbb{F}_r(B_2) \longrightarrow 0 \\ & & \downarrow \theta_r(B_1) & & \downarrow \theta_r(B) & & \downarrow \theta_r(B_2) \\ 0 & \longrightarrow & \mathbb{G}_r(B_1) & \xrightarrow{i_{B_1}^{B_2}} & \mathbb{G}_r(B) & \xrightarrow{\pi_{B_1}^{B_2}} & \mathbb{G}_r(B_2) \longrightarrow 0. \end{array}$$

**Observation 3.2** *As a consequence if  $B_1 \subset B$  are boxes, with  $B_1$  located in the upper-left corner of  $B$ . then the induced linear map  $i_{B_1}^{B_2} : \mathbb{F}_r(B_1) \rightarrow \mathbb{F}_r(B)$  is injective and, if  $B_2$  is located in the down-right corner of  $B$ , then the induced linear map  $\pi_{B_1}^{B_2} : \mathbb{F}_r(B) \rightarrow \mathbb{F}_r(B_2)$  is surjective.*

In Figure 1 below  $B_{11}$  is located in the upper-left corner of  $B$  and  $B_{22}$  in the lower-right corner of  $B$ .

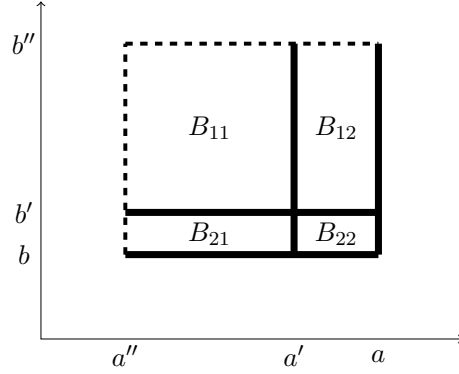


Figure 1

Define

$$\boxed{{}^F\hat{\delta}_r(a, b) := \lim_{\epsilon, \epsilon' \rightarrow 0} \mathbb{F}_r((a - \epsilon, a] \times [b, b + \epsilon'))} \quad \text{and} \quad \boxed{{}^G\hat{\delta}_r(a, b) := \lim_{\epsilon, \epsilon' \rightarrow 0} \mathbb{G}_r((a - \epsilon, a] \times [b, b + \epsilon'))}.$$

In view of (12) one has

$${}^F\hat{\delta}_r^f(a, b) = {}^G\hat{\delta}_r^f(a, b).$$

Denote by

$$\boxed{\hat{\delta}_r^f(a, b) := {}^F\hat{\delta}_r^f(a, b) = {}^G\hat{\delta}_r^f(a, b), \quad \delta_r^f(a, b) := \dim \hat{\delta}_r^f(a, b)}.$$

Note also that

$$\hat{\delta}_r(a, b) = \frac{\mathbb{F}_r(a, b)}{\mathbb{F}_r(< a, b) + \mathbb{F}_r(a, > b)} \quad (13)$$

and in view of (13) one has the obvious surjective linear map

$$\boxed{\pi_{a,b}^\delta(r) : \mathbb{F}_r(a, b) \rightarrow \hat{\delta}_r^f(a, b)}.$$

As in [1], for  $f : \tilde{X} \rightarrow \mathbb{R}$  a tame map and  $a, b \in \mathbb{R}$  with  $a < b$ , denote by  $i_a^b(r) : H_r(\tilde{X}_a) \rightarrow H_r(\tilde{X}_b)$  the inclusion induced linear map.

Define by

- $\mathbb{T}_r(a, b) := \ker(H_r(\tilde{X}_a) \rightarrow H_r(\tilde{X}_b)),$
- $\mathbb{C}_r(a, b) := \text{coker}(H_r(\tilde{X}_a) \rightarrow H_r(\tilde{X}_b)),$

and for  $a' < b \leq b' < b$  denote by  $i_{a'b'}^{a,b}(r) : \mathbb{T}_r(a', b') \rightarrow \mathbb{T}_r(a, b)$  the induced linear map.

For a **box above diagonal**  $B = (a', a] \times (b', b]$  with  $a' < a \leq b' < b$  observe that  $\mathbb{T}_r(a, b') \subseteq \mathbb{T}_r(a, b)$  and define

$$\mathbb{T}_r(B) := \frac{\mathbb{T}_r(a, b)}{i_{a'b'}^{a,b}(r)(\mathbb{T}_r(a', b')) + \mathbb{T}_r(a, b')}. \quad (14)$$

In view of formula (2) (in subsection 2.1)

$$\mathbb{T}_r(B) = \hat{\omega}(i_{a'}^a(r), i_a^{b'}(r), i_{b'}^b(r)) \quad (15)$$

with  $i_a^{a'}(r), i_a^{b'}(r), i_{b'}^b(r)$  the linear maps in the sequence

$$H_r(X_{a'}) \xrightarrow{i_a^{a'}} H_r(\tilde{X}_a) \xrightarrow{i_a^{b'}} H_r(\tilde{X}_{b'}) \xrightarrow{i_{b'}^b} H_r(\tilde{X}_b) .$$

For  $a'' < a' < a$  and  $b > b' > b''$  consider boxes above diagonal  $B_1, B_2, B$  with  $B = (a'', a] \times (b'', b]$  and either  $B_1 = (a'', a'] \times (b'', b]$ ,  $B_2 = (a', a] \times (b'', b]$  or  $B_1 = (a'', a] \times (b', b']$ ,  $B_2 = (a'', a] \times (b', b]$ . IN both cases  $B = B_1 \sqcup B_2$ . As in [1] or [2] one has the following proposition.

**Proposition 3.3** (cf. [1]) *The boxes  $B_1, B_2, B$  as above induce the linear maps  $i_{B_1}^B(r)$  and  $\pi_B^{B_2}(r)$  which make the following sequence exact.*

$$0 \longrightarrow \mathbb{T}_r(B_1) \xrightarrow{i_{B_1}^B(r)} \mathbb{T}_r(B) \xrightarrow{\pi_B^{B_2}(r)} \mathbb{T}_r(B_2) \longrightarrow 0$$

**Observation 3.4** *As a consequence, if  $B_1 \subset B$  is located in the down-left corner of  $B$  then the induced linear map  $i_{B_1}^B(r) : \mathbb{T}_r(B_1) \rightarrow \mathbb{T}_r(B)$  is injective and, if  $B_2 \subset B$  is located in the upper-right corner of  $B$  then the induced linear map  $\pi_B^{B_2}(r) : \mathbb{T}_r(B) \rightarrow \mathbb{T}_r(B_2)$  is surjective.*

In Figure 2 below  $B_{11}$  is located in the down-left corner of  $B$  and  $B_{22}$  in the upper-right corner of  $B$ .

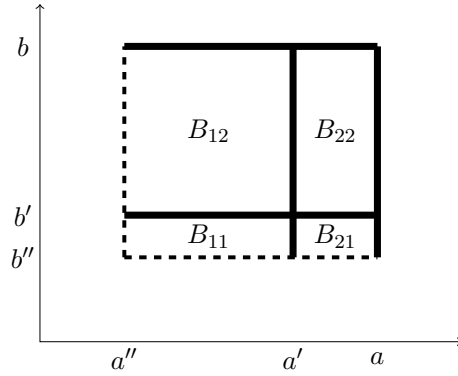


Figure 2

2

Define

$$\hat{\gamma}_r^f(a, b) := \lim_{\epsilon, \epsilon' \rightarrow 0} \mathbb{T}_r((a - \epsilon, a] \times (b - \epsilon', b]), \quad \gamma_r^f(a, b) = \dim \hat{\gamma}_r^f(a, b)$$

and observe that

$$\hat{\gamma}_r^f(a, b) = \lim_{\epsilon, \epsilon' \rightarrow 0} \frac{\mathbb{T}_r(a, b)}{i_{a-\epsilon, b}^{a, b}(\mathbb{T}_r(a - \epsilon, b)) + \mathbb{T}_r(a, b - \epsilon')} = \frac{\mathbb{T}_r(a, b)}{(i_{<a, b}^{a, b} \mathbb{T}_r(< a, b)) + \mathbb{T}_r(a, < b)} \quad (16)$$

In view of (16) there is the obvious surjective linear map

$$\pi_{a, b}^\delta(r) : \mathbb{T}_r(a, b) \rightarrow \hat{\gamma}_r^f(a, b) .$$

<sup>2</sup>For reasons of space Figure 2 is misleading. All boxes are supposed to be above diagonal and not necessary in the first quadrant

Suppose that  $f : \tilde{X} \rightarrow \mathbb{R}$  is a lift of a tame TC1-form  $\omega$ . Suppose  $H_r(\dots)$  is a homology theory s.t.  $\mathcal{H}_r(\tilde{X}_a, \tilde{X}_{<a})$  is finite dimensional for any  $a \in \mathbb{R}$ .

As shown in section 5 of [1] for the standard homology one has the following result, valid for any homology theory.

**Proposition 3.5**

The supports of  $\delta_r^f$  and  $\gamma_r^f$  are subsets of  $CR(f) \times CR(f)$  with the following properties:

1. If  $(a, b) \in \text{supp } \delta_r^f$  resp.  $(a, b) \in \text{supp } \gamma_r^f$  then for any  $g \in \Gamma$  one has  $(a + g, b + g) \in \text{supp } \delta_r^f$  resp.  $(a + g, b + g) \in \text{supp } \gamma_r^f$ .
2. For any  $a \in \mathbb{R}$   $\text{supp } \delta_r^f \cap \mathbb{R} \times a$ ,  $\text{supp } \delta_r^f \cap a \times \mathbb{R}$ ,  $\text{supp } \gamma_r^f \cap \mathbb{R} \times a$ ,  $\text{supp } \gamma_r^f \cap a \times \mathbb{R}$  are finite sets, empty if  $a \in \mathbb{R} \setminus CR(f)$ .
3. There exists a finite set of lines  $\Delta_{t_i^\delta}(r)$  resp.  $\Delta_{t_i^\gamma}(r)$ , in the plane  $\mathbb{R}^2$  given by the equations  $y = x + t_i^\delta$  resp.  $y = x + t_i^\gamma$ ,  $i = 1, 2, \dots, N_r^\delta$  resp.  $i = 1, 2, \dots, N_r^\gamma$  s.t.  $\text{supp } \delta_r^f \subset \cup_{i=1, \dots, N_r^\delta} \Delta_{t_i^\delta}$  resp.  $\text{supp } \gamma_r^f \subset \cup_{i=1, \dots, N_r^\gamma} \Delta_{t_i^\gamma}$ .

As a consequence define the maps  $\delta_r^\omega : \mathbb{R} \rightarrow \mathbb{Z}_{\geq 0}$  and  $\gamma_r^\omega : \mathbb{R}_{>0} \rightarrow \mathbb{Z}_{\geq 0}$

$$\delta_r^\omega(t) := \begin{cases} \delta_r^f(a, b), t = b - a, (a, b) \in \text{supp } \delta_r^f \\ 0, (a, b) \notin \text{supp } \delta_r^f \end{cases}$$

$$\gamma_r^\omega(t) := \begin{cases} \gamma_r^f(a, b), t = b - a, (a, b) \in \text{supp } \gamma_r^f \\ 0, (a, b) \notin \text{supp } \gamma_r^f \end{cases}$$

which are configurations (i.e. maps with finite support) with  $\text{supp } \delta_r^\omega := \{t_1^\delta, t_2^\delta, \dots, t_{N_r^\delta}^\delta\}$  resp.  $\text{supp } \gamma_r^\omega := \{t_1^\gamma, t_2^\gamma, \dots, t_{N_r^\gamma}^\gamma\}$  with  $-\infty < t_1^\delta < t_2^\delta, \dots, t_{N_r^\delta}^\delta < \infty$  resp.  $0 < t_1^\gamma < t_2^\gamma, \dots, t_{N_r^\gamma}^\gamma < \infty$ .

*Proof:*(sketch)

Item (a) follows from the  $\Gamma$ -equivariance of the lift  $f$ .

As in [1] for  $a' < \alpha < a \leq b$  one defines:

$$\mathbb{T}_r((a', \alpha] \times b) := \varinjlim_{\epsilon \rightarrow 0} \mathbb{T}_r(a', \alpha] \times (b - \epsilon, b], \quad \epsilon < b - \alpha$$

and then

$$\mathbb{T}_r((a', a) \times b) := \varinjlim_{\epsilon \rightarrow 0} \mathbb{T}_r(a', a - \epsilon] \times b, \quad \epsilon < a - a'$$

By passing to limit when  $\epsilon \rightarrow 0$  the exact sequence

$$0 \longrightarrow \mathbb{T}_r((a', \alpha] \times (b - \epsilon, b]) \longrightarrow \mathbb{T}_r((a', a] \times (b - \epsilon, b]) \longrightarrow \mathbb{T}_r((\alpha, a] \times (b - \epsilon, b]) \longrightarrow 0$$

(cf. Proposition 3.3) one obtains the short exact sequence

$$0 \longrightarrow \mathbb{T}_r((a', \alpha] \times b) \longrightarrow \mathbb{T}_r((a', a] \times b) \longrightarrow \mathbb{T}_r((\alpha, a] \times b) \longrightarrow 0 \quad (17)$$

and by passing to limit when  $\epsilon' \rightarrow 0$ , for  $a$  replaced by  $a - \epsilon'$ , the sequence (17) gives the short exact sequence

$$0 \longrightarrow \mathbb{T}_r((a', \alpha] \times b) \longrightarrow \mathbb{T}_r((a', a) \times b) \longrightarrow \mathbb{T}_r((\alpha, a) \times b) \longrightarrow 0. \quad (18)$$

By passing to limit when  $\epsilon \rightarrow 0$ , the surjective maps  $\mathbb{T}_r(b - 2\epsilon, b) \rightarrow \mathbb{T}_r((a', b - 2\epsilon] \times (b - \epsilon, b])$  and  $\mathbb{C}_r(a - \epsilon, b) \rightarrow \mathbb{I}_a(r)/\mathbb{I}_{a-\epsilon}(r)$  give the surjective maps

$$\mathbb{T}_r(< b, b) \rightarrow \mathbb{T}_r((a', b) \times b) \text{ and } \mathbb{C}_r(< a, b) \rightarrow \mathbb{I}_a(r)/\mathbb{I}_{<a}(r) \quad (19)$$

In view of in view of (17), (18) and (19) the finite dimensionality of  $\mathcal{H}_{r+1}(\tilde{X}_b, \tilde{X}_{<b})$  implies the finite dimensionality of  $\mathbb{T}_r(< b, b) := \ker H_r(\tilde{X}_{<b}) \rightarrow H_r(\tilde{X}_b)$  and then of  $\mathbb{T}_r((a', b) \times b)$  and  $\mathbb{T}_r((a', a] \times b)$  and of  $\mathbb{I}_a(r)/\mathbb{I}_{<a}(r)$  for any  $a' < a \leq b$  resp. any  $a$ . In the case  $b$  resp.  $a$  is a regular value these spaces are trivial. This verifies item (b).

This implies that the assignments  $t \rightarrow \mathbb{T}_r((-\infty, t] \times b)$   $t \in (-\infty, b)$ , resp.  $t \rightsquigarrow \mathbb{I}_a(r) \cap \mathbb{I}^t(r)/\mathbb{I}_{<a}(r) \cap \mathbb{I}^t(r)$  are integer-valued step functions with finitely many jumps located at  $-\infty < t_1 < t_2 < \dots < t_k < b$ , resp.  $-\infty < t_1 < t_2 < \dots < t_k < b$ . The jump at  $t = t_i$  for the first assignment at is given by

$$\lim_{\epsilon \rightarrow 0} \mathbb{T}_r((a, t_i] \times b)/\mathbb{T}_r((a', t_i - \epsilon] \times b) = \hat{\gamma}_r(t_i, b)$$

$-\infty \leq a < t_i - \epsilon$  and for the second by  $\delta_r^f(a, t_i)$  hence

$$\mathbb{T}_r((a', a] \times b) = \bigoplus_{a' < t \leq a} \hat{\gamma}_r(t, b) \text{ and } \mathbb{I}_a(r)/\mathbb{I}_{<a}(r) = \bigoplus_{-\infty < t < \infty} \hat{\delta}_r(a, t).$$

By the same arguments as in [1] Theorem 3.2 one verifies that the subspace  $\subset \bigcap_{-\infty < T < a} i_{T,b}^{a,b}(\mathbb{T}_r(T, b))$  of  $\mathbb{C}_r(a, b)$  is trivial and then

$$\varinjlim_{a \rightarrow b} \varprojlim_{a' \rightarrow -\infty} \mathbb{T}_r((a', a] \times b) = \varprojlim_{a' \rightarrow -\infty} \varinjlim_{a \rightarrow b} \mathbb{T}_r((a', a] \times b) = \mathbb{T}_r((-\infty, b) \times b) = \bigoplus_{t < a} \hat{\gamma}_r(t, b)$$

which finalizes the proof of item (c).

q.e.d.

We also have

### Proposition 3.6

For any  $a, b \in \mathbb{R}$ ,  $a < b$  (in both cases for standard and Borel-Moore homology)

$$\mathbb{T}_r((-\infty, a] \times b) = \mathbb{T}_r(a, b)/\mathbb{T}_r(a, < b)$$

$$\mathbb{T}_r((-\infty, b) \times b) = \mathbb{T}_r(< b, b)$$

*Proof:* As pointed out above  $\bigcap_{-\infty < T < a} i_{T,b}^{a,b}(\mathbb{T}_r(T, b))$  is trivial. Note that for  $a \leq b' < b$   $\mathbb{T}_r((-\infty, a] \times (b', b]) = \mathbb{T}_r(a, b)/(\bigcap_{-\infty < T < a} i_{T,b}^{a,b}(\mathbb{T}_r(T, b)) + \mathbb{T}_r(a, b')) = \mathbb{T}_r(a, b)/\mathbb{T}_r(a, b')$  and therefore

$$\mathbb{T}_r((-\infty, a] \times b) = \mathbb{T}_r(a, b)/\mathbb{T}_r(a, < b)$$

and therefore

$$\mathbb{T}_r((-\infty, b) \times b) = \mathbb{T}_r(< b, b)/\mathbb{T}_r(< b, < b) = \mathbb{T}_r(< b, b). \quad \blacksquare$$

Some additional notations, not existing in [1], are be of help.

$$\begin{aligned} \mathbb{T}_r(a', a : t) &:= \ker(i_{a',t}^{a,t}), a' < a < t \leq \infty, \\ \mathbb{C}_r(a', a; t) &:= \text{coker}(i_{a',t}^{a,t}), a' < a < t \leq \infty, \\ \mathbb{I}_r(a', a; t) &:= (\mathbb{I}_a^f(r) \cap \mathbb{I}_{a'}^f(r))/(\mathbb{I}_{a'}^f(r) \cap \mathbb{I}_f^t(r)) \subseteq \mathbb{I}_a^f(r)/(\mathbb{I}_{a'}^f(r)), a' < a, \end{aligned} \quad (20)$$

With these notations one has the diagram (21) below whose rows and columns are exact sequences

$$\begin{array}{ccccccc}
 & & 0 & & & & (21) \\
 & & \downarrow & & & & \\
 & & \mathbb{C}_r(a - \epsilon, a; \infty) & & & & \\
 & & \downarrow & & & & \\
 0 & \longrightarrow & \mathbb{C}_r(a - \epsilon, a) & \longrightarrow & H_r(\tilde{X}_a, \tilde{X}_{a-\epsilon}) & \longrightarrow & \mathbb{T}_r(a - \epsilon, a) \longrightarrow 0 \\
 & & \downarrow & & & & \\
 & & \mathbb{I}_r(a - \epsilon, a) & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & & 
 \end{array}$$

and the following filtrations:

1.  $\{\mathbb{I}_r(a - \epsilon, a) \cdots \supseteq \mathbb{I}_r(a - \epsilon, a; t) \supseteq \mathbb{I}_r(a - \epsilon, a; t + \epsilon') \supseteq \cdots\}$  indexed by  $t \in \mathbb{R}$  with the property
 
$$\boxed{\mathbb{I}_r(a - \epsilon, a; t) / \mathbb{I}_r(a - \epsilon, a; t + \epsilon) = \mathbb{F}_r((a - \epsilon, a] \times [t, t + \epsilon))},$$
2.  $\{\mathbb{C}_r(a - \epsilon, a) \cdots \supseteq \mathbb{C}_r(a - \epsilon, a; t) \supseteq \mathbb{C}_r(a - \epsilon, a; t - \epsilon) \supseteq \cdots\}$ ,  $t - \epsilon > a$  indexed by  $t \in (a, \infty)$ , which in view of (14) satisfies
 
$$\boxed{\mathbb{C}_r(a - \epsilon, a; t) / \mathbb{C}_r(a - \epsilon, a; t - \epsilon) = \mathbb{T}_r(a - \epsilon, a] \times (t - \epsilon, t]}.$$

Recall / observe that:

- (i)  $\mathcal{H}_r(\tilde{X}_a, \tilde{X}_{<a}) = \varinjlim_{\epsilon \rightarrow 0} H_r(\tilde{X}_a, \tilde{X}_{a-\epsilon})$ ,  
 $\mathbb{T}_r(<a, a) := \varinjlim_{\epsilon \rightarrow 0} \mathbb{T}_r(a - \epsilon, a)$ ,  
 $\mathbb{C}_r(<a, a) := \varinjlim_{\epsilon \rightarrow 0} \mathbb{C}_r(a - \epsilon, a)$ ,  
 $\mathbb{I}_r(<a, a) := \varinjlim_{\epsilon \rightarrow 0} \mathbb{I}_r(a - \epsilon, a) = \mathbb{I}_r(a)(r) / \mathbb{I}_{<a}(r)$ .
- (ii)  $\mathbb{I}_r(<a, a; t) := \varinjlim_{\epsilon \rightarrow 0} \mathbb{I}_r(a - \epsilon, a; t)$ , and  
 $\hat{\delta}_r^f(a, t) := \varinjlim_{\epsilon \rightarrow 0} \mathbb{I}_r(<a, a; t) / \mathbb{I}_r(<a, a; t + \epsilon)$
- (iii)  $\mathbb{C}_r(<a, a; t) := \varinjlim_{\epsilon \rightarrow 0} \mathbb{C}_r(a - \epsilon, a; t)$ ,  $a < t$  and  
 $\hat{\gamma}_r^f(a, t) := \varinjlim_{\epsilon \rightarrow 0} \mathbb{C}_r(<a, a; t) / \mathbb{C}_r(<a, a; t - \epsilon)$ ,  $a < t - \epsilon$ .
- (iv)  $\mathbb{T}_r(<a, a; t) := \varinjlim_{\epsilon \rightarrow 0} \mathbb{T}_r(a - \epsilon, a; t)$ ,  $a < t$  and  
 $\hat{\gamma}_r^f(a, t) := \varinjlim_{\epsilon \rightarrow 0} \mathbb{K}_r(<a, a; t) / \mathbb{K}_r(<a, a; t - \epsilon)$ ,  $\epsilon < t - a$ .

By passing to limit when  $\epsilon \rightarrow 0$  the diagram (21) becomes



One extends these splittings to any  $(\alpha, \beta) \in B$  as being the linear injective maps  $i_{\alpha, \beta}^B(r) : \hat{\delta}_r^f(a, b) \rightarrow \mathbb{F}_r(B)$  resp.  $i_{\alpha, \beta}^B(r) : \hat{\gamma}_r^f(a, b) \rightarrow \mathbb{T}_r(B)$  defined by the composition  $i_{B'}^B(r) \cdot i_{\alpha, \beta}^{B'}(r)$  where  $B' = (a', \alpha] \times [\beta, b')$ ,  $a' < \alpha \leq a, b \leq \beta < b'$  resp.  $B' = (a', \alpha] \times (b', \beta]$ ,  $a' < \alpha \leq a, b' < \beta \leq b$ .

For  $f : \tilde{X} \rightarrow \mathbb{R}$  a lift of a tame TC1-form  $\omega$ , a collection  $S^\delta$  resp.  $S^\gamma$  of splittings  $S^\delta = \{i_{\alpha, b}^\delta(r)\}$  resp.  $S^\gamma = \{i_{\alpha, b}^\gamma(r)\}$  is called  $\Gamma$ -compatible if for any  $(a, b) \in \text{supp} \delta_r^f$  resp.  $(a, b) \in \text{supp} \gamma_r^f$  the diagrams

$$\begin{array}{ccc} \mathbb{F}_r^f(a, b) \xrightarrow{\langle g \rangle} \mathbb{F}_r(g+a, g+b) & \text{resp.} & \mathbb{T}_r^f(a, b) \xrightarrow{\langle g \rangle} \mathbb{T}_r(g+a, g+b) \\ \uparrow i_{\alpha, b}^\delta(r) & & \uparrow i_{\alpha, b}^\gamma(r) \\ \hat{\delta}_r^f(a, b) \xrightarrow{\langle g \rangle} \hat{\delta}_r^f(a+g, b+g) & & \hat{\gamma}_r^f(a, b) \xrightarrow{\langle g \rangle} \hat{\gamma}_r^f(a+g, b+g) \end{array} \quad (23)$$

remain commutative. Note that collections of  $\Gamma$ -compatible splittings  $S^\delta$  resp.  $S^\gamma$  exist. Indeed, if one chooses one point  $(a_i, b_i)$  in each  $\Delta_{t_i}^\delta(r) \cap \text{supp} \delta_r^f$  resp.  $\Delta_{t_i}^\gamma(r) \cap \text{supp} \gamma_r^f$  and a splitting  $i_{\alpha_i, b_i}^\delta(r)$  resp.  $i_{\alpha_i, b_i}^\gamma(r)$  and one defines

$$i_{\alpha_i+g, b_i+g}^\delta(r) = \langle g \rangle \cdot i_{\alpha_i, b_i}^\delta(r) \cdot \langle g \rangle^{-1}$$

then, since  $(a_i + g, b_i + g)$  exhaust all points in the support of  $\hat{\delta}_r^f$  resp.  $\hat{\gamma}_r^\gamma$ , one obtains a collection of  $\Gamma$ -compatible splittings  $S^\delta$  resp.  $S^\gamma$ .

Given a collection of splittings  $S^\delta$  and a set  $A \subset \text{supp} \delta_r^f$ , as in [1] Proposition 4, one shows that the linear map

$$S^\delta I_A(r) = (\oplus_{(\alpha, \beta) \in A} i_{\alpha, \beta}^\delta(r)) : \oplus \hat{\delta}_r^f(\alpha, \beta) \rightarrow H_r(\tilde{X})$$

is injective, has the image contained in  $\mathbb{F}_r(a, b)$  provided  $A \subset (-\infty, a] \times [b, \infty)$ , and remains injective when composed by  $\pi_{a, b}^B : \mathbb{F}_r((a', a] \times [b, b'))$  provided that  $A \subset \text{supp} \delta_r \cap B$ ,  $B = (a', a] \times [b, b')$ .

The same remains true for a collection of splittings  $S^\gamma$ ,  $A \subset \{\alpha, \beta \in \text{supp} \gamma_r^f \mid \alpha \leq a\}$  and the map

$$S^\gamma I_A(r) = (\oplus_{(\alpha, \beta) \in A} i_{\alpha, \beta}^\gamma(r)) : \oplus \hat{\gamma}_r^f(\alpha, \beta) \rightarrow \mathbb{T}_r(a, \infty).$$

## 4 Proof of Poincaré duality, Theorem 1.1

Suppose  $a, b$  regular values for  $f : \tilde{M} \rightarrow \mathbb{R}$  a lift of a tame TC1-form  $\omega$  on a topological closed manifold  $M$ , hence the levels  $\tilde{M}(a), \tilde{M}(b)$  are codimension one submanifolds of  $\tilde{M}$ . Poincaré duality for the topological manifolds with boundary  $\tilde{M}_a$  and  $\tilde{M}^b$  combined with the excision property for standard cohomology provide the canonical isomorphisms "PD<sub>r</sub>" from the Borel-Moore homology  ${}^{BM}H_r$  to the standard cohomology  $H^{n-r}$  and then the commutative diagrams below.

$$\begin{array}{ccccc} H_r^{BM}(\tilde{M}_a) & \xrightarrow{i_a(r)} & H_r^{BM}(\tilde{M}) & \xrightarrow{j_a(r)} & H_r^{BM}(\tilde{M}, \tilde{M}_a) \\ \downarrow \text{PD}_a^1 & & \downarrow \text{PD} & & \downarrow \text{PD}_a^2 \\ H^{n-r}(\tilde{M}, \tilde{M}^a) & \xrightarrow{s^a(n-r)} & H^{n-r}(\tilde{M}) & \xrightarrow{r^a(n-r)} & H^{n-r}(\tilde{M}^a) \\ \downarrow & & \downarrow & & \downarrow \\ (H_{n-r}(\tilde{M}, \tilde{M}^a))^* & \xrightarrow{(j^a(n-r))^*} & (H_{n-r}(\tilde{M}))^* & \xrightarrow{(i^a(n-r))^*} & (H_{n-r}(\tilde{M}^a))^* \end{array} \quad (24)$$

$$\begin{array}{ccccc}
H_r^{\text{BM}}(\widetilde{M}^b) & \xrightarrow{i^b(r)} & H_r^{\text{BM}}(\widetilde{M}) & \xrightarrow{j^b(r)} & H_r^{\text{BM}}(\widetilde{M}, \widetilde{M}^b) \\
\downarrow \text{PD}_1^b & & \downarrow \text{PD} & & \downarrow \text{PD}_2^b \\
H^{n-r}(\widetilde{M}, \widetilde{M}^b) & \xrightarrow{s_b(n-r)} & H^{n-r}(\widetilde{M}) & \xrightarrow{r_b(n-r)} & H^{n-r}(\widetilde{M}^b) \\
\downarrow & & \downarrow & & \downarrow \\
(H_{n-r}(\widetilde{M}, \widetilde{M}^b))^* & \xrightarrow{(j_b(n-r))^*} & (H_{n-r}(\widetilde{M}))^* & \xrightarrow{(i_b(n-r))^*} & (H_{n-r}(\widetilde{M}^b))^*
\end{array} \tag{25}$$

In view of (24) and (25) one has

$$\begin{aligned}
{}^{\text{BM}}\mathbb{F}_r(a, b) &:= \text{img } i_a(r) \cap \text{img } i^b(r) = \ker j_a(r) \cap \ker j^b(r) \\
\ker(i^a(n-r))^* \cap \ker(i_b(n-r))^* &\simeq (\text{coker}(i_b(n-r) \oplus i^a(n-r)))^* =: \mathbb{G}_{n-r}^f(b, a)^*.
\end{aligned}$$

The first equality holds by exactness of the first row in these diagrams, the second by the equality of the top- and bottom-right horizontal arrows, the third by the linear algebra duality and the fourth by the definition of  $\mathbb{G}_{n-r}^f$ .

This in turn implies that for  $a' < a, b < b'$  regular values with the boxes  $B$  and  $B'$  given by  $B = (a', a] \times [b, b']$ ,  $B' = (b, b'] \times [a', a]$  the diagram below is commutative with the horizontal arrows all isomorphisms in particular

$$\text{PD}_r(B) : {}^{\text{BM}}\mathbb{F}_r^f((a', a] \times [b, b']) \rightarrow (\mathbb{G}_{n-r}^f((b, b'] \times [a', a]))^* \tag{26}$$

$$\begin{array}{ccccc}
{}^{\text{BM}}\mathbb{F}_r^f(a', b') & \xrightarrow{\text{PD}_r(a', b')} & (\mathbb{G}_{n-r}^f(b', a'))^* & & \\
\downarrow & & \downarrow & & \\
{}^{\text{BM}}\mathbb{F}_r^f(a, b) & \xrightarrow{\text{PD}_r(a, b)} & (\mathbb{G}_{n-r}^{\tilde{f}}(b, a))^* & & \\
\downarrow & \searrow \text{BM } \pi_{ab, r}^B & \downarrow \text{PD}_r(B) & \swarrow (u_{n-r}(B'))^* & \\
{}^{\text{BM}}\mathbb{F}_r^f(B) & & (\mathbb{G}_{n-r}^{\tilde{f}}(B))^* & & \\
\downarrow & & \downarrow (p^{ba}(r))^* & & \downarrow (\theta_{n-r}(B'))^* \\
H_r^{\text{BM}}(\widetilde{M}) & \xrightarrow{\text{PD}_r} & (H_{n-r}(\widetilde{M}))^* & & (\mathbb{F}_{n-r}^{\tilde{f}}(B'))^*
\end{array} \tag{27}$$

Suppose  $a < b < c < d$  with all  $a, b, c, d$  regular values and consider the box above diagonal  $B = (a, b] \times (c, d]$ . Poincaré duality and excision property provide the following commutative diagram

$$\begin{array}{ccccccc}
(1) & BM H_r(\tilde{M}_a^f) & \xrightarrow{\alpha} & BM H_r(\tilde{M}_b^f) & \xrightarrow{\beta} & BM H_r(\tilde{M}_c^f) & \xrightarrow{\gamma} & BM H_r(\tilde{M}_d^f) & (28) \\
& \downarrow \text{PD} & & \downarrow \text{PD} & & \downarrow \text{PD} & & \downarrow \text{PD} & \\
(2) & H^{n-r}(\tilde{M}, \tilde{M}_f^a) & \longrightarrow & H^{n-r}(\tilde{M}, \tilde{M}_f^b) & \longrightarrow & H^{n-r}(\tilde{M}, \tilde{M}_f^c) & \longrightarrow & H^{n-r}(\tilde{M}, \tilde{M}_f^d) & \\
& \uparrow \partial & & \uparrow \partial & & \uparrow \partial & & \uparrow \partial & \\
(3) & H^{n-r-1}(\tilde{M}_f^a) & \longrightarrow & H^{n-r-1}(\tilde{M}_f^b) & \longrightarrow & H^{n-r-1}(\tilde{M}_f^c) & \longrightarrow & H^{n-r-1}(\tilde{M}_f^d) & \\
& \uparrow = & & \uparrow = & & \uparrow = & & \uparrow = & \\
(4) & (H_{n-r-1}(\tilde{M}_f^a))^* & \longrightarrow & (H_{n-r-1}(\tilde{M}_f^b))^* & \longrightarrow & (H_{n-r-1}(\tilde{M}_f^c))^* & \longrightarrow & (H_{n-r-1}(\tilde{M}_f^d))^* & \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
(5) & (H_{n-r-1}(\tilde{M}_{-a}^-))^* & \xrightarrow{\gamma'^*} & (H_{n-r-1}(\tilde{M}_{-b}^-))^* & \xrightarrow{\beta'^*} & (H_{n-r-1}(\tilde{M}_{-c}^-))^* & \xrightarrow{\alpha'^*} & (H_{n-r-1}(\tilde{M}_{-d}^-))^* &
\end{array}$$

In view of Theorem 2.3 item 2 this diagram implies that  $\omega(\alpha, \beta, \gamma) = \omega(\gamma'^*, \beta'^*, \alpha'^*)$  (cf subsection 2.1 for definitions) which by Theorem 2.3 item 1 is canonically isomorphic to  $\omega(\gamma', \beta', \alpha')^*$ . In particular one has the isomorphism

$${}^{BM}PD_r(B) : {}^{BM}\mathbb{T}_r^f(a, b] \times (c, d] \rightarrow (\mathbb{T}_{n-r-1}^{-f}((-d, -c] \times (-b, -a)))^*. \quad (29)$$

For  $a, b \in \mathbb{R}$  and  $\epsilon_1 > \epsilon_2$  one considers two sub-surjections

$${}^1\tilde{\pi} : A = {}^{BM}\mathbb{F}_r(B_1) \rightsquigarrow {}^{BM}\mathbb{F}_r(B_2) = A'$$

and

$${}^2\tilde{\pi} : A = (\mathbb{G}_{n-r}(\underline{B}_1))^* \rightsquigarrow (G_{n-r}(\underline{B}_2))^* = A'$$

defined based on the boxes

$$(a) \quad B_1 = (a - \epsilon_1, a + \epsilon_1] \times [b - \epsilon_1, b + \epsilon_1]$$

$$B_2 = (a - \epsilon_2, a + \epsilon_2] \times [b - \epsilon_2, b + \epsilon_2]$$

$$C = (a - \epsilon_2, a + \epsilon_1] \times [b - \epsilon_1, b + \epsilon_2]$$

with  $C$  as a lower-right corner of  $B_1$ ,  $B_2$  as an upper-left corner of  $C$

and on the boxes

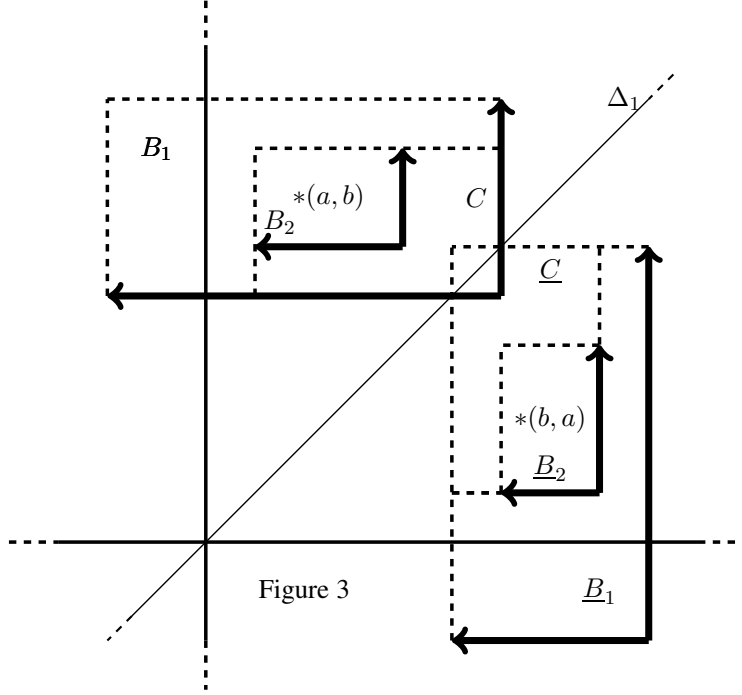
$$(b) \quad \underline{B}_1 = (b - \epsilon_1, b + \epsilon_1] \times [a - \epsilon_1, a + \epsilon_1]$$

$$\underline{B}_2 = (b - \epsilon_2, b + \epsilon_2] \times [a - \epsilon_2, a + \epsilon_2]$$

$$\underline{C} = (b - \epsilon_1, b + \epsilon_2] \times [a - \epsilon_2, a + \epsilon_1]$$

with  $\underline{C}$  as an upper left corner of  $\underline{B}_1$ ,  $\underline{B}_2$  as a down right corner of  $\underline{C}$ .

The boxes  $\underline{B}_1, \underline{C}, \underline{B}_2$  are the symmetric of the boxes  $B_1, C, B_2$  w.r. to the first diagonal  $\Delta_1 := \{(x, y) \in \mathbb{R}^2 \mid x - y = 0\}$ . See Figure 3 below.



In view of Observation 3.2

$\pi_{B_1}^C : {}^{BM}\mathbb{F}_r(B_1) \rightarrow {}^{BM}\mathbb{F}_r(C)$  is surjective and

$i_{B_2}^C : {}^{BM}\mathbb{F}_r(B_2) \rightarrow {}^{BM}\mathbb{F}_r(C)$  is injective and identifies  ${}^{BM}\mathbb{F}_r(B_2)$  to a subspace of  ${}^{BM}\mathbb{F}_r(C)$ .

Define the sub-surjection  ${}^1\tilde{\pi}$  by:  $A := {}^{BM}\mathbb{F}_r(B_1)$ ,  $P = {}^{BM}\mathbb{F}_r(C)$ ,  $A' = {}^{BM}\mathbb{F}_r(B_2)$  and  $\pi := \pi_{B_1}^C$ .

In view of Observation 3.2

$(i_{\underline{C}}^{\underline{B}_1})^* : (\mathbb{G}_{n-r}(\underline{B}_1))^* \rightarrow (\mathbb{G}_{n-r}(\underline{C}))^*$  is surjective and

$(\pi_{\underline{C}}^{\underline{B}_2})^* : (\mathbb{G}_{n-r}(\underline{B}_2))^* \rightarrow (\mathbb{G}_{n-r}(\underline{C}))^*$  is injective<sup>3</sup>.

Define the second sub-surjection  ${}^2\tilde{\pi}$  by:  $A := (\mathbb{G}_{n-r}(\underline{B}_1))^*$ ,  $P = (\mathbb{G}_{n-r}(\underline{C}))^*$ ,  $A' = (\mathbb{G}_{n-r}(\underline{B}_2))^*$  and  $\pi := (i_{\underline{C}}^{\underline{B}_1})^*$ . In view of (26) Poincaré duality identifies the two sub-surjections.

For  $a < b$ ,  $a, b \in \mathbb{R}$  and  $\epsilon_1 > \epsilon_2$  with  $a < 2\epsilon_1 < b$  one considers two sub-surjections  ${}^3\tilde{\pi} : A = {}^{BM}\mathbb{T}_r^f(B_1) \rightsquigarrow {}^{BMT}\mathbb{T}^f(B_2)$  and  ${}^4\tilde{\pi} : (\mathbb{T}_{n-r-1}^{-f}(\underline{B}_1))^* \rightsquigarrow (\mathbb{T}_{n-r-1}^{-f}(\underline{B}_2))^*$  based on the boxes above diagonal.

$$(a) \quad B_1 = (a - \epsilon_1, a + \epsilon_1] \times (b - \epsilon_1, b + \epsilon_1]$$

$$B_2 = (a - \epsilon_2, a + \epsilon_2] \times (b - \epsilon_2, b + \epsilon_2]$$

$$C = (a - \epsilon_2, a + \epsilon_1] \times [b - \epsilon_2, b + \epsilon_1]$$

with  $C$  as a upper-right corner of  $B_1$  and  $B_2$  as the lower-left corner of  $C$  and on the boxes above diagonal

$$(b) \quad \underline{B}_1 = (-b - \epsilon_1, -b + \epsilon_1] \times (-a - \epsilon_1, -a + \epsilon_1]$$

$$\underline{B}_2 = (-b - \epsilon_2, -b + \epsilon_2] \times (-a - \epsilon_2, -a + \epsilon_2]$$

$$\underline{C} = (-b - \epsilon_1, -b + \epsilon_2] \times (-a - \epsilon_1, -a + \epsilon_2]$$

with  $\underline{C}$  as a lower left corner of  $\underline{B}_1$ ,  $\underline{B}_2$  as an upper-right corner of  $\underline{C}$

<sup>3</sup>since  $i_{\underline{C}}^{\underline{B}_1} : \mathbb{G}(\underline{C}) \rightarrow \mathbb{G}(\underline{B}_1)$  is injective and  $\pi_{\underline{C}}^{\underline{B}_2} : \mathbb{G}(\underline{C}) \rightarrow \mathbb{G}(\underline{B}_2)$  is surjective



In view of the finite dimensionality of  $\hat{\delta}_{n-r}^f(b, a) =^F \hat{\delta}_{n-r}^f(b, a) =^G \hat{\delta}_{n-r}^f(b, a)$

$$\dim ({}^G \hat{\delta}_{n-r}^f(b, a))^* = \dim {}^G \hat{\delta}_{n-r}^f(b, a)$$

hence  ${}^{BM} \delta_r^f(a, b) = \delta_{n-r}^f(b, a)$  hence

$${}^{BM} \delta_r^\omega(t) = \delta_{n-r}^\omega(-t).$$

This establishes Item 1 in Theorem (1.1).

Similarly, given  $a < b$  consider the collection of sub-surjections  ${}^3 \tilde{\pi}_i^f : A_i \rightsquigarrow A_{i+1}$  with  $A_i := {}^{BM} \mathbb{T}_r^f(B_i)$ ,  $B_i$  the box above diagonal  $B_i := (a - \epsilon_i, a + \epsilon_i] \times (b - \epsilon_i, b + \epsilon_i]$  which identifies by Poincaré duality to the collection of sub-surjections  ${}^4 \tilde{\pi}_i^{-f} : A_i \rightsquigarrow A_{i+1}$  with  $A_i := (\mathbb{T}_{n-r-1}^{-f}(\underline{B}_i))^*$  with  $\underline{B}_i$  the box above diagonal  $\underline{B}_i = (-b - \epsilon_i, -b + \epsilon_i] \times (-a - \epsilon_i, a + \epsilon_i]$ .

In view of (29) one has  $\lim_{i \rightarrow \infty} {}^3 \tilde{\pi}_i^f = \lim_{i \rightarrow \infty} {}^4 \tilde{\pi}_i^{-f}$ .

Proposition 4.1 below implies

$${}^{BM} \hat{\gamma}_r^f(a, b) = (\hat{\gamma}_{n-r-1}^{-f}(-b, -a))^*$$

and in view of the finite dimensionality of  $\hat{\gamma}_{n-r-1}^{-f}(-b, -a)$  one has

$\dim \hat{\gamma}_{n-r-1}^{-f}(-b, -a) = (\dim \hat{\gamma}_{n-r-1}^{-f}(-b, -a))^*$  hence  ${}^{BM} \gamma_r^f(a, b) = \gamma_{n-r-1}^{-f}(-b, -a)$  hence

$${}^{BM} \gamma_r^\omega(t) = \gamma_{n-r-1}^{-\omega}(t).$$

This establishes Item 2 in Theorem 1.1.

#### Proposition 4.1

1.  $\lim_{i \rightarrow \infty} {}^1 \tilde{\pi}_i = {}^{BM} \hat{\delta}_r^f(a, b)$
2.  $\lim_{i \rightarrow \infty} {}^2 \tilde{\pi}_i = (\hat{\delta}_{n-r}^f(b, a))^*$
3.  $\lim_{i \rightarrow \infty} {}^3 \tilde{\pi}_i^f = {}^{BM} \hat{\gamma}_r^f(a, b)$
4.  $\lim_{i \rightarrow \infty} {}^4 \tilde{\pi}_i^{-f} = (\hat{\gamma}_{n-r-1}^{-f}(-b, -a))^*$

Proof:

Using the description of  $A_i^\infty$  (at the end of section 1) Observations 3.2 and Observation 3.4 one concludes that for each directed system  ${}^1 \tilde{\pi}_i, {}^2 \tilde{\pi}_i, {}^3 \tilde{\pi}_i, {}^4 \tilde{\pi}_i$  the corresponding  $A_i^\infty$  are given by

$$A_i^\infty = \begin{cases} {}^{BM} \mathbb{F}_r((a - \epsilon_i, a] \times [b, b + \epsilon_i)) \\ (\mathbb{G}_{n-r}((-b - \epsilon_i, b] \times [a, a + \epsilon_i]))^* \\ {}^{BM} \mathbb{T}_r^t((a - \epsilon_i, a] \times (b - \epsilon_i, b]) \\ (\mathbb{T}_{n-r-1}^{-f}((-b - \epsilon_i, -b] \times (-a - \epsilon_i, -a]))^*. \end{cases}$$

In view of the Definitions 2.6 and of the definitions of  ${}^{BM} \hat{\delta}_r^f, {}^G \hat{\delta}_r^f, {}^{BM} \hat{\gamma}_r^f$  one has

$$\lim_{i \rightarrow \infty} (\pi_i' : A_i^\infty \rightarrow A_{i+1}^\infty) = \begin{cases} {}^{BM} \hat{\delta}_r^f(a, b) \\ ({}^G \hat{\delta}_{n-r}^f(b, a))^* \\ {}^{BM} \hat{\gamma}_r^f(a, b) \\ (\hat{\gamma}_{n-r-1}^{-f}(-b, -a))^* \end{cases}$$

q.e.d

## 5 Proof of stability property, Theorem 1.2

In view of Proposition 3.5 Note that for each diagonal  $\Delta_t := \{(x, y) \in \mathbb{R}^2, y - x = t\}$  the set  $\Delta_t \cap \text{supp} \delta_r^f$  if not empty is  $\Gamma$ -invariant w.r. to the diagonal action and the set  $(\Delta_t \cap \text{supp} \delta_r^f)/\Gamma$  consists of finitely many orbits say  $o_1, o_2, \dots, o_k$ . Choose in any such orbit one point  $(a_i, b_i)$ , and let  $\delta_r^\omega(t) = \sum_{i=1,2,\dots,k} \delta_r^f(a_i, b_i)$ . Clearly  $\delta_r^\omega(t)$  does not depend on the choice of  $(a_i, b_i)$ .

Denote by

$$\begin{aligned} \check{\mathbb{F}}_r^f(\{a, b\}) &:= \bigoplus_{g \in \Gamma} \delta_r^f(a + g, b + g), \\ \check{\mathbb{F}}_r^\omega(t) &:= \bigoplus_{\{(a,b) \in \text{supp} \delta_r^f, b-a=t\}} \delta_r^f(a, b) = \bigoplus_{i=1,2,\dots,k} \check{\mathbb{F}}_r^f(\{a_i, b_i\}), \\ \check{\mathbb{F}}_r^\omega(\leq t) &:= \bigoplus_{\{(a,b) \in \text{supp} \delta_r^f, b-a \leq t\}} \delta_r^f(a, b) = \bigoplus_{s \leq t} \check{\mathbb{F}}_r^\omega(s), \\ \check{\mathbb{F}}_r^\omega(< t) &:= \bigoplus_{\{(a,b) \in \text{supp} \delta_r^f, b-a < t\}} \delta_r^f(a, b) = \bigoplus_{s < t} \check{\mathbb{F}}_r^\omega(s), \\ \check{\mathbb{F}}_r^\omega &:= \bigoplus_{\{(a,b) \in \text{supp} \delta_r^f, b-a=t\}} \delta_r^f(a, b) = \bigoplus_{s \in \text{supp} \delta_r^\omega} \check{\mathbb{F}}_r^\omega(s). \end{aligned}$$

Let  $\langle g \rangle_{a,b} : \delta_r^f(a, b) \rightarrow \delta_r^f(a + g, b + g)$  be the linear isomorphism derived from the linear isomorphism  $\langle g \rangle : H_r(\tilde{X}) \rightarrow H_r(\tilde{X})$  end equip  $\check{\mathbb{F}}_r^\omega$  with a structure of  $\kappa[\Gamma]$ -module given by

$$\langle g \rangle := \bigoplus_{(a,b)} \langle g \rangle_{a,b} \delta_r^f(a, b) \rightarrow \delta_r^f(a + g, b + g).$$

The  $\kappa[\Gamma]$ -modules  $\check{\mathbb{F}}_r^f(\{a, b\})$ ,  $\check{\mathbb{F}}_r^\omega(t)$ ,  $\check{\mathbb{F}}_r^\omega(\leq t)$ ,  $\check{\mathbb{F}}_r^\omega(< t)$  and  $\check{\mathbb{F}}_r^\omega$ , are all free f.g.  $\kappa[\Gamma]$ -modules and in view of Proposition (3.5) of finite rank

$$\begin{aligned} &\delta_r^f(a, b), \\ &\delta_r^\omega(t) = \sum_{i=1,2,\dots,k, b_i-a_i=t} \delta_r^f(a_i, b_i), \\ &\sum_{s \in \text{supp} \delta_r^\omega, s \leq t} \delta_r^\omega(s), \\ &\sum_{s \in \text{supp} \delta_r^\omega, s < t} \delta_r^\omega(s), \\ &\sum_{s \in \text{supp} \delta_r^\omega} \delta_r^\omega(s). \end{aligned}$$

Denote by

$$\begin{aligned} \overline{\mathbb{F}}_r^f(\{a, b\}) &:= \sum_{g \in \Gamma} (F_r^f(a + g, b + g)/\text{Tor} H_r(\tilde{X})) \subseteq \overline{\mathbb{F}}_r^f, \\ \overline{\mathbb{F}}_r^\omega(t) &:= \sum_{\{(a,b) \in \text{supp} \delta_r^f, b-a=t\}} (F_r^f(a, b)/\text{Tor} H_r(\tilde{X})) \subseteq \overline{\mathbb{F}}_r^\omega, \\ \overline{\mathbb{F}}_r^\omega(\leq t) &:= \sum_{(a,b) \in \text{supp} \delta_r^f, b-a \leq t} (F_r^f(a, b)/\text{Tor} H_r(\tilde{X})) \subseteq \overline{\mathbb{F}}_r^\omega, \\ \overline{\mathbb{F}}_r^\omega(< t) &:= \sum_{\{(a,b) \in \text{supp} \delta_r^f, b-a < t\}} (F_r^f(a, b)/\text{Tor} H_r(\tilde{X})) \subseteq \overline{\mathbb{F}}_r^\omega, \\ \overline{\mathbb{F}}_r^\omega &:= H_r(\tilde{X})/\text{Tor} H_r(\tilde{X}), \end{aligned}$$

Since  $\kappa[\Gamma]$  is Noetherian and  $H_r(\tilde{X})$  is f.g. module all these  $\kappa$ -vector spaces are f.g.  $\kappa[\Gamma]$ -modules. Consequently there are finitely many  $t \in \mathbb{R}$ ,  $t_1 < t_2 < \dots < t_k$  s.t

1.  $\overline{\mathbb{F}}_r^\omega(t) = 0$  for  $t < t_1$ ,
2.  $\overline{\mathbb{F}}_r^\omega(t) = \overline{\mathbb{F}}_r^\omega(t_i)$  for  $t_i \leq t < t_{i+1}$ ,
3.  $\overline{\mathbb{F}}_r^\omega(t) = H_r(\tilde{X})/\text{Tor} H_r(\tilde{X})$  for  $t_k \leq t$ .

Choose a collection of  $\Gamma$  compatible splittings  $S^\delta := \{i_{a,b}(r) : \delta_r^f(a, b) \rightarrow \mathbb{F}_r(a, b) \mid (a, b) \in \text{supp} \delta_r^f\}$  (like in subsection 2.4) and consider

$${}^S I(r) : \check{\mathbb{F}}_r^\omega \rightarrow \overline{\mathbb{F}}_r^f = H_r(\tilde{X})/\text{Tor} H_r(\tilde{X}).$$

The map  ${}^S I(r)$  is  $\kappa[\Gamma]$ -linear, in view of Proposition 4 in [1] is injective and sends  $\check{\mathbb{F}}_r^\omega$  into  $\overline{\mathbb{F}}_r^f$ , and restricts to the maps

$$\begin{aligned}
SI_{\{a,b\}}(r) &: \check{\mathbb{F}}^f(\{a,b\}) \rightarrow \overline{\mathbb{F}}_r^f(\{a,b\}), \\
SI(r)_t &: \check{\mathbb{F}}_r^f(t) \rightarrow \overline{\mathbb{F}}_r^\omega(t), \\
SI(r)_{\leq t} &: \check{\mathbb{F}}_r^f(\leq t) \rightarrow \overline{\mathbb{F}}_r^\omega(\leq t), \\
SI(r)_{< t} &: \check{\mathbb{F}}_r^f(< t) \rightarrow \overline{\mathbb{F}}_r^\omega(< t)
\end{aligned}$$

all injective  $\kappa[\Gamma]$ -linear maps.

**Proposition 5.1** *The above maps are all bijective, hence  $\overline{\mathbb{F}}_r^\omega(t)$ ,  $\overline{\mathbb{F}}_r^\omega(\leq t)$ ,  $\overline{\mathbb{F}}_r^\omega(< t)$ ,  $\overline{\mathbb{F}}_r^\omega$  are all f.g. free modules of rank*

1.  $\delta_r^\omega(t)$ ,
2.  $\sum_{\{s \in \text{supp } \delta_r^\omega | s \leq t\}} \delta_r^\omega(s)$ ,
3.  $\sum_{\{s \in \text{supp } \delta_r^\omega | s < t\}} \delta_r^\omega(s)$ .
4.  $\sum_{\{s \in \text{supp } \delta_r^\omega\}} \delta_r^\omega(s)$ ,

In view of the above it suffices to check the surjectivity of  $SI(r)$ .

Indeed for  $x \in H_r(\tilde{X})$  define:

- $\alpha(x) := \inf\{a \mid x \in \mathbb{I}_a^f(r)\}$ ,  $\alpha(x) \in [-\infty, \infty)$   
with  $\alpha(x) = -\infty$  if  $x \in \cap_{a \in \mathbb{R}} \mathbb{I}_a^f(r)$ ,
- $\beta(x) := \sup\{b \mid x \in \mathbb{I}_b^f(r)\}$ ,  $\beta(x) \in (-\infty, \infty]$   
with  $\beta(x) = \infty$  if  $x \in \cap_{b \in \mathbb{R}} \mathbb{I}_b^f(r)$ ,
- $t^\delta(x) = \beta(x) - \alpha(x)$ ,  $t^\delta(x) \in (-\infty, \infty]$ .

In view of Proposition 3.1 in [1]  $x \in \text{Tor}H_k(\tilde{X})$  iff  $t(x) = \infty$ . Observe that if  $x \in H_r(\tilde{X})$  with  $\alpha(x) = a, \beta(x) = b, t = t(x) = b - a$ , hence  $x \notin \text{Tor}H_r(\tilde{X})$ , then it can be written (in the presence of a collection of splittings) as  $x = i_{a,b}(\hat{x}) + x'$  with  $\hat{x} = \pi_{a,b}(x) \in \hat{\delta}_r(a, b)$  and  $x' \in H_r(\tilde{X})$  with  $t(x') < t$ . This implies that in view of finite generation of  $H_r(\tilde{X})$  there exists a finite sequence  $t_0 > t_1 > t_2 \cdots > t_k$  and  $x_i$  with  $\alpha(x_i) = a_i, \beta(x_i) = b_i, t(x_i) = t_i$  s.t.  $x = \sum_{0 \leq i \leq k} i_{a_i, b_i}(\hat{x}_i)$ ,  $\hat{x}_i \neq 0$ ; otherwise  $H_r(\tilde{X})$  contains an infinite sequence of submodules  $\cdots \overline{\mathbb{F}}_r(t_i) \supset \overline{\mathbb{F}}_r(t_{i+1}) \supset \cdots$  which contradicts the f.g property of  $H_r(\tilde{X})$ .  
q.e.d.

To finalize the proof of Theorem 1.2 one follows the arguments in [4] or in [2] section 5 . Observe that for any two TC 1-forms  $\omega_1, \omega_2$  in the same cohomology class and two lifts  $f_1 : \tilde{X} \rightarrow \mathbb{R}$  of  $\omega_1$  and  $f_2 : \tilde{X} \rightarrow \mathbb{R}$  of  $\omega_2$  with  $\|f_1 - f_2\|_\infty < \epsilon$  and  $a, b \in \mathbb{R}$  one has:

$$\begin{aligned}
\mathbb{I}_{a-\epsilon}^{f_1}(r) &\subseteq \mathbb{I}_a^{f_2}(r) \subseteq \mathbb{I}_{a+\epsilon}^{f_1}(r) \\
\mathbb{I}_{f_1}^{b-\epsilon}(r) &\supseteq \mathbb{I}_{f_2}^b(r) \supseteq \mathbb{I}_{f_1}^{b+\epsilon}(r)
\end{aligned} \tag{30}$$

and therefore

$$\mathbb{F}_r^{f_1}(a - \epsilon, b + \epsilon) \subseteq \mathbb{F}_r^{f_2}(a, b) \subseteq \mathbb{F}_r^{f_1}(a + \epsilon, b - \epsilon)$$

and therefore for any  $t \in \text{supp} \delta_r^\omega$

$$\begin{aligned}
\overline{\mathbb{F}}_r^{\omega_1}(t - 2\epsilon) &\subseteq \overline{\mathbb{F}}_r^{\omega_2}(t) \subseteq \overline{\mathbb{F}}_r^{\omega_1}(t + 2\epsilon) \\
\overline{\mathbb{F}}_r^{\omega_1}(< (t - 2\epsilon)) &\subseteq \overline{\mathbb{F}}_r^{\omega_2}(< t) \subseteq \overline{\mathbb{F}}_r^{\omega_1}(< (t + 2\epsilon)).
\end{aligned} \tag{31}$$

Clearly, if  $\overline{\mathbb{F}}_r^{\omega_1}(t - \epsilon) = \overline{\mathbb{F}}_r^{\omega_1}(t + \epsilon)$  resp.  $\overline{\mathbb{F}}_r^{\omega_1}(< t - \epsilon) = \overline{\mathbb{F}}_r^{\omega_1}(< t + \epsilon)$  then  $\overline{\mathbb{F}}_r^{\omega_2}(t) = \overline{\mathbb{F}}_r^{\omega_1}(t - \epsilon) = \overline{\mathbb{F}}_r^{\omega_1}(t)$  resp.  $\overline{\mathbb{F}}_r^{\omega_2}(< t) = \overline{\mathbb{F}}_r^{\omega_1}(< t - \epsilon) = \overline{\mathbb{F}}_r^{\omega_1}(< t)$ .

For a tame TC1-form  $\omega$  denote by  $\sigma_r(\omega) := \inf |t_i - t_j|, t_i \neq t_j$  with  $t_i, t_j \in \text{supp} \delta_r^\omega$ . As an immediate consequence of Proposition (5.1) one has Proposition (5.2) which, as in [4] or [2] section 5, implies the continuity of the assignment  $\omega \rightsquigarrow \delta_r^\omega$ .

**Proposition 5.2** *For any  $\epsilon < \sigma_f(\omega)$ ,  $\omega, \omega'$  two tame TC1-forms in the same cohomology class with  $|\omega - \omega'| < \epsilon/3$  and  $t_i \in \text{supp} \hat{\delta}_r^\omega$  one has*

$$\delta_r^\omega(t_i) = \sum_{t_i - \epsilon < s < t_i + \epsilon} \delta_r^{\omega'}(s) \quad (32)$$

$$\text{supp} \delta_r^{\omega'} \subset \cup_{t_i \in \text{supp} \delta_r^\omega} \delta_r^\omega(t_i - \epsilon, t_i + \epsilon). \quad (33)$$

In fact

$$\hat{\delta}_r^\omega(t_i) \simeq \bigoplus_{t_i - \epsilon < s < t_i + \epsilon} \hat{\delta}_r^{\omega'}(s).$$

## 6 Standard homology versus Borel-Moore homology (Proposition 1.4)

In this section  $H_r$  denotes the standard (singular) homology,  ${}^{BM}H_r$  the Borel-Moore homology and the linear maps  $\theta_r^{X,Y} : H_r(X, Y) \rightarrow {}^{BM}H_r(X, Y)$  define the natural transformation from the standard to the Borel-Moore homology. In general the linear maps  $\theta_r^{X,Y} : H_r(X, Y) \rightarrow {}^{BM}H_r(X, Y)$  are not isomorphisms.

Consider pairs  $(U, V)$  of locally compact ANRs,  $V$  closed subset of  $U$ . Recall that  ${}^{BM}H_r(U, V) = H_r(\hat{U}, \hat{V})$ , with  $\hat{U}$  resp.  $\hat{V}$  the one-point compactification of  $U$  resp.  $V$  and  $\theta_r^{U,V} : H_r(U, V) \rightarrow {}^{BM}H_r(U, V)$  identifies to the inclusion induced linear map  $H_r(U, V) \rightarrow H_r(\hat{U}, \hat{V})$ .

For an inclusion of such pairs  $(U_1, V_1) \subset (U_2, V_2)$ , s.t.  $U_1$  open subset set in  $U_2$  and  $V_1$  open set of  $V_2$  consider the commutative diagram

$$\begin{array}{ccc} (U_1, V_1) & \xrightarrow{\subset} & (\hat{U}_1, \hat{V}_1) \\ \downarrow \subset & & \uparrow p \\ (U_2, V_2) & \xrightarrow{\subset} & (\hat{U}_2, \hat{V}_2) \end{array}$$

with  $p$  the map which restricts to the identity on  $U_1$  resp.  $V_1$  and sends  $U_2 \setminus U_1$  resp.  $V_2 \setminus V_1$  to the added point of the compactification  $\hat{U}_2$  resp.  $\hat{V}_2$ . All other three arrows are inclusions of pairs. This diagram induces the linear maps  $i_{U_1, V_1}^{U_2, V_2} : H_r(U_1, V_1) \rightarrow H_r(U_2, V_2)$  and  $p_{U_2, V_2}^{U_1, V_1} : {}^{BM}H_r(U_2, V_2) \rightarrow {}^{BM}H_r(U_1, V_1)$  which together with  $\theta_r^{U_1, V_1}$  and  $\theta_r^{U_2, V_2}$  make the diagram

$$\begin{array}{ccc} H_r(U_1, V_1) & \xrightarrow{\theta_r^{U_1, V_1}} & {}^{BM}H_r(U_1, V_1) \\ i_{U_1, V_1}^{U_2, V_2} \downarrow & & \uparrow p_{U_2, V_2}^{U_1, V_1} \\ H_r(U_2, V_2) & \xrightarrow{\theta_r^{U_2, V_2}} & {}^{BM}H_r(U_2, V_2) \end{array} \quad (34)$$

commutative. When  $V_1$  and  $V_2$  are the empty sets one has the commutative diagram

$$\begin{array}{ccc} H_r(U_1) & \xrightarrow{\theta_r^{U_1}} & {}^{BM}H_r(U_1) \\ \downarrow i_{U_1}^{U_2} & & \uparrow p_{U_2}^{U_1} \\ H_r(U_2) & \xrightarrow{\theta_r^{U_2}} & {}^{BM}H_r(U_2). \end{array}$$

By similar arguments as in [8], when regarded as a contravariant functor on the category whose objects are open subsets  $U$  of a locally compact ANR  $X$  and morphisms are the inclusions, the assignment  $U \rightsquigarrow {}^{BM}H_r(U)$  induces the exact sequence

$$0 \longrightarrow \varprojlim'_{i \rightarrow \infty} {}^{BM}\mathcal{H}_{r-1}(U(i)) \longrightarrow {}^{BM}\mathcal{H}_r(X) \longrightarrow \varprojlim_{i \rightarrow \infty} {}^{BM}\mathcal{H}_r(U(i)) \longrightarrow 0 \quad (35)$$

for any open filtration of  $X$  by open sets  $U(i)$ ,

$$U(1) \subset U(2) \subset \cdots \subset U(i) \subset U(i+1) \subset \cdots \subset X,$$

$X = \cup_i U(i)$ . Here  $\varprojlim$  and  $\varprojlim'$  denotes *inverse limit* and the *derived inverse limit* of the inverse system of vector spaces given by the linear maps

$${}^{BM}H_r(U(i)) = H_r(\hat{U}(i), *) \leftarrow {}^{BM}H_r(U(i+1)) = H_r(\hat{U}(i+1), *).$$

For a pair  $(X, Y)$  as above equipped with a filtration by open pairs,

$$(U(1), V(1)) \subset (U(2), V(2)) \subset \cdots \subset (U(i), V(i)) \subset (U(i+1), V(i+1)) \subset (X, Y),$$

i.e.  $U(i)$  open subset of  $X$ ,  $V(i)$  open subset of  $Y$ ,  $X = \cup_i U(i)$ ,  $Y = \cup_i V(i)$ , the exact sequence (35) implies the short exact sequence

$$0 \longrightarrow \varprojlim'_{i \rightarrow \infty} {}^{BM}\mathcal{H}_{r-1}(U(i), V(i)) \longrightarrow {}^{BM}\mathcal{H}_r(X, Y) \longrightarrow \varprojlim_{i \rightarrow \infty} {}^{BM}\mathcal{H}_r(U(i), V(i)) \longrightarrow 0 \quad (36)$$

with the *derived inverse limit* and the *inverse limit* considered for of the inverse system defined by linear the maps

$${}^{BM}H_r(U(i), V(i)) \leftarrow {}^{BM}H_r(U(i+1), V(i+1)).$$

Consider  $f : \tilde{X} \rightarrow \mathbb{R}$  a lift of a tame TC1-form  $\omega$  and

$$U(1) \subset U(2) \subset \cdots \subset U(i) \subset U(i+1) \subset \cdots \subset U(\infty) = \tilde{X}$$

a filtration by open sets of  $\tilde{X}$ . By applying the exact sequence (36) to  $X = \tilde{X}_a$ ,  $Y = \tilde{X}_{a-\epsilon}$  and the filtration by the pairs  $(U(i)_a, U(i)_{a-\epsilon})$  of the pair  $(\tilde{X}_a, \tilde{X}_{a-\epsilon})$  and by passing to direct limit when  $\epsilon \rightarrow 0$  one obtains the exact sequence

$$0 \longrightarrow \varprojlim'_{i \rightarrow \infty} {}^{BM}\mathcal{H}_{r-1}(U(i)_a, U(i)_{<a}) \longrightarrow {}^{BM}\mathcal{H}_r(\tilde{X}_a, \tilde{X}_{<a}) \longrightarrow \varprojlim_{i \rightarrow \infty} {}^{BM}\mathcal{H}_r(U(i)_a, U(i)_{<a}) \longrightarrow 0 \quad (37)$$

For the proof of Proposition 1.4 we will also need the following Proposition (6.1).

**Proposition 6.1** *Suppose  $M$  is a compact manifold with boundary  $\partial M$  and interior  $\text{Int}M = U$ , and  $f : M \rightarrow \mathbb{R}$  a tame map with  $\text{Cr}(f) \subset U$ . Suppose  $t$  is a regular value for  $f|_{\partial M}$  the restriction of  $f$  to the boundary, but possibly critical value for  $f$ . Then for  $\epsilon > 0$  small enough the inclusion induced linear map  $i_r(t) := i_{U_t, U_{t-\epsilon}}^{M_t, M_{t-\epsilon}}$  and the  $\theta_r(t) := \theta_r^{U_t, U_{t-\epsilon}}$  in the sequence (38) below are isomorphisms.*

$$H_r(M_t, M_{t-\epsilon}) \xleftarrow{i_r(t)} H_r(U_t, U_{t-\epsilon}) \xrightarrow{\theta_r(t)} {}^{BM}H_r(U_t, U_{t-\epsilon}) \quad (38)$$

Proof: We leave to the reader to check by himself that the inclusion  $(U_t, U_{t-\epsilon}) \subset (M_t, M_{t-\epsilon})$  is a homotopy equivalence of pairs which implies  $i_r(t)$  is an isomorphism.

Since  $t$  is regular value for  $f|_{\partial M}$  there exists  $\epsilon > 0$  small enough s.t. the inclusion  $\partial M(t-\epsilon) \subset \partial M_{[t-\epsilon, t]}$ <sup>5</sup> is a homotopy equivalence, hence the inclusion  $(M_t, M_{t-\epsilon}) \subset (M_t, M_{t-\epsilon} \cup \partial M_{[t-\epsilon, t]})$  is a homotopy equivalence of compact pairs, hence

$$H_r(M_t, M_{t-\epsilon}) = H_r(M_t, M_{t-\epsilon} \cup \partial M_{[t-\epsilon, t]}) = {}^{BM}H_r(M_t, M_{t-\epsilon} \cup \partial M_{[t-\epsilon, t]}). \quad (39)$$

Since  $M_t \setminus (M_{t-\epsilon} \cup \partial M_{[t-\epsilon, t]}) = (U_t \setminus U_{t-\epsilon})$  one has

$${}^{BM}H_r(M_t, M_{t-\epsilon} \cup \partial M_{[t-\epsilon, t]}) = {}^{BM}H_r(U_t, U_{t-\epsilon}). \quad (40)$$

Combining the two isomorphisms one concludes that  $\theta_r^{U_t, U_{t-\epsilon}}(t) \cdot i_r(t)^{-1}$  is an isomorphism, hence so is  $\theta_r^{U_t, U_{t-\epsilon}}(t)$ .

q.e.d.

#### Proof of Proposition (1.4)

For a fix  $t \in \mathbb{R}$  choose a filtration of  $\tilde{X}$  by open sets

$$U(1) \subset U(2) \subset \cdots \subset U(k) \subset U(k+1) \subset \cdots \subset \tilde{X},$$

$\bigcup_i U(i) = \tilde{X}$ , with the properties:

- (a)  $\overline{U}(i)$ <sup>6</sup> compact submanifold with boundary s.t.  $\overline{U}(i) \subset U(i+1)$ ,
- (b) the restrictions of  $f$  to  $\overline{U}(i)$  are tame maps,
- (c)  $t \in \text{Cr}(f)$  is a regular value for the restriction of  $f$  to  $\partial(\overline{U}(i))$

In view of tameness of  $f$

(d): for any  $t \in \text{Cr}(f)$  there exists  $N(t) \in \mathbb{Z}_{\geq 0}$  s.t. the compact set  $f^{-1}(t) \cap \text{Cr}(f)$  is contained in  $U(i)$  for  $i > N(t)$ .

Clearly such filtrations exists. One can choose  $\overline{U}(i)$  to be the closure of a finite union of translates of a properly chosen fundamental domain of the free action of  $\Gamma$  on  $\tilde{X}$  (i.e. whose interior is a manifold). It satisfies (a) and (b). One can increase (arbitrary little  $\overline{U}(i)$  inside  $U(i+1)$ ) to make (c) also satisfied.

Observe that diagram (34) implies the commutative diagram

---

<sup>5</sup> $\partial M_{[t-\epsilon, t]} = f_{\partial M}^{-1}([t-\epsilon, t])$   
<sup>6</sup> $\overline{U}$  denotes the closure of  $U$  in  $\tilde{X}$

$$\begin{array}{ccc}
H_r(U(i)_t, U(i)_{<t}) & \xrightarrow{\theta_r^i(t)} & {}^{BM}H_r(U(i)_t, U(i)_{<t}) \\
\downarrow i_i^{i+k} & & \uparrow p_{i+k}^i \\
H_r(U(i+k)_t, U(i+k)_{<t}) & \xrightarrow{\theta_r^{i+k}(t)} & {}^{BM}H_r(U(i+k)_t, U(i+k)_{<t}) \\
\downarrow i_i^\infty & & \uparrow p_{i+k}^\infty \\
H_r(\tilde{X}_t, \tilde{X}_{<t}) & \xrightarrow{\theta_r^\infty(t)} & {}^{BM}H_r(\tilde{X}_t, \tilde{X}_{<t})
\end{array}
\quad \begin{array}{l} 7 \\ (41) \end{array}$$

s.t.  $i_j^{j+k+r} = i_{j+k}^{j+k+r} \cdot i_j^{j+k}$  and  $p_{i+j+k}^j = p_{j+k}^j \cdot p_{j+k+r}^{j+k}$ . Inspection of this diagram combined with Propositions 6.1 and the exact sequence (37) lead to the result as follows.

F1. In view of Property (d) of the filtration, Proposition (6.1) implies that for  $i > N(t)$  and  $\epsilon$  small enough  $\theta_r^{U(i)_t, U(i)_{t-\epsilon}} : H_r(U(i)_t, U(i)_{t-\epsilon}) \rightarrow {}^{BM}H_r(U(i)_t, U(i)_{t-\epsilon})$  is an isomorphism. Passing to limit when  $\epsilon \rightarrow 0$  one obtains

$$\theta_r^i(t) : H_r(U(i)_t, U(i)_{<t}) \rightarrow {}^{BM}H_r(U(i)_t, U(i)_{<t})$$

is an isomorphism.

F2. For  $j > N(t)$  the linear map  $i_j^{j+k}$  is an isomorphism by excision property of the standard homology. Combined with the commutativity of the diagram (41) and F1. one obtains  $p_{j+k}^j$  is an isomorphism for any  $j > N(t)$  and  $k$ .

F3. The isomorphisms  $i_j^{j+k}$  stated in F2. imply  $i_j^\infty$  is isomorphism for  $j > N(t)$ , and the isomorphisms  $p_{j+k}^j$  for  $j > N(t)$  stated in F2., in view of the exact sequence (E25) imply  $p_\infty^j$  is isomorphism for  $j > N(t)$ . Hence in view of diagram (41),  $\theta_r(t) = \theta_r^\infty(t)$  is an isomorphism. This finalizes the proof of Proposition (1.4).

### Corollary 6.2

The induced linear maps,

1.  $\theta_r(a) : \mathbb{I}_a(r)/\mathbb{I}_{<a}(r) \rightarrow {}^{BM}\mathbb{I}_a(r)/{}^{BM}\mathbb{I}_{<a}(r)$
2.  $\theta_r(a, b) : \mathbb{F}_r(a, b)/\mathbb{F}_r(<a, b) \rightarrow {}^{BM}\mathbb{F}_r(a, b)/{}^{BM}\mathbb{F}_r(<a, b)$
3.  $\theta_r(a, b) : \mathbb{T}_r(a, b)/i_{<a, b}^{a, b}\mathbb{T}_r(<a, b) \rightarrow {}^{BM}\mathbb{T}_r(a, b)/i_{<a, b}^{a, b} {}^{BM}\mathbb{T}_r(<a, b)$

are injective

*Proof:* Consider the diagram

$$\begin{array}{ccc}
{}^{BM}H_r(\tilde{X}_a) & \longrightarrow & \varinjlim_{\epsilon \rightarrow 0} {}^{BM}H_r(\tilde{X}_a, \tilde{X}_{a-\epsilon}) = \mathcal{H}_r(\tilde{X}_a, \tilde{X}_{<a}) \\
\theta_r^{\tilde{X}_a} \uparrow & & \uparrow \theta_r(a) \\
H_r(\tilde{X}_a) & \xrightarrow{j} & \varinjlim_{\epsilon \rightarrow 0} H_r(\tilde{X}_a, \tilde{X}_{a-\epsilon}) = \mathcal{H}_r(\tilde{X}_a, \tilde{X}_{<a})
\end{array}
\quad (42)$$

A nonzero element  $\underline{x} \in \mathbb{I}_a(r)/\mathbb{I}_{<a}(r)$  is represented by an element  $x \in H_r(\tilde{X}_a)$  s.t  $j(x) \neq 0$ , hence  $\theta_r(a)(j(x)) \neq 0$ . This implies  $\theta_r^{\tilde{X}_a}(x)$  which represents  $\theta_r(a)(\underline{x})$  and whose image in  ${}^{BM}\mathcal{H}_r(\tilde{X}_a, \tilde{X}_{<a})$  is

<sup>7</sup>recall  $\varinjlim_{\epsilon \rightarrow 0} {}^{BM}H_r(U(i)_t, U(i)_{t-\epsilon}) = {}^{BM}\mathcal{H}_r(U(i)_t, U(i)_{<t})$

$\theta_r(a)(j(x))$  is nontrivial. This establishes items 1. and 2.. Similarly a nonzero element  $\underline{y} \in \mathbb{T}(a, b)/\mathbb{T}_r(< a, b)$  is represented by an element  $y \in H_r(\tilde{X}_a)$  whose image in  $H_r(\tilde{X}_b)$  vanishes s.t  $j(x) \neq 0$  hence  $\theta_r(a)(j(y)) \neq 0$  which implies  $\theta_r^{\tilde{X}_a}(y)$  represents  $\theta_r(a, b)(\underline{y})$  which by the same arguments should be nontrivial. This establishes items 3. ■

## References

- [1] Dan Burghelea, Alternative to Morse-Novikov Theory for a closed one form (I), European Journal of Mathematics <https://doi.org/10.1007/s40879-019-00368-x> arXiv 1919 , arXiv:1903.00240
- [2] Dan Burghelea, *New topological invariants for real- and angle-valued maps; an alternative to Morse-Novikov theory* Word scientific Publishing Co. Pte. Ltd, 2017
- [3] D. Burghelea and T. K. Dey, *Persistence for circle valued maps. Discrete Comput. Geom.* **50** 2013 pp69-98 ; arXiv:1104.5646
- [4] Dan Burghelea, Stefan Haller, *Topology of angle valued maps, bar codes and Jordan blocks*, J. Appl. and Comput. Topology, vol 1, pp 121-197, 2017; arXiv:1303.4328; Max Plank preprints
- [5] D.Burghelea, A refinement of Betti numbers and homology in the presence of a continuous function I, *Algebr. Geom. Topol.* 17. pp 2051-2080, 2017, arXiv 1501.02486
- [6] D.Burghelea A refinement of Betti numbers and homology in the presence of a continuous function II, (the case of an angle-valued map), arXiv 160301861
- [7] G Bredon Sheaf theory, second edition Sheaf theory Graduate text in Mathematics 170, Springer
- [8] J.Milnor On axiomatic homology theory, *Pacific Journal of Mathematics* Vol 12, No 1, pp 337- 341, 1962,
- [9] Michael Usher and Juo Zhang, *Persistent homology and Floer-Novikov theory*, *Geometry and Topology*. Volume 20, Number 6 pp 3333-3430 (2016).