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RESEARCH ARTICLE

Alternative to Morse–Novikov theory for a closed 1-form. I

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Dedicated to the memory of Ştefan Papadima

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Abstract

This paper extends the Alternative to Morse–Novikov theory we have proposed in Burghelea (New topological invariants for real- and angle-valued maps, Word Scientific, Hackensack, 2018) from real- and angle-valued maps to closed 1-forms. For a topological closed 1-form ω on a compact ANR (= absolute neighborhood retract) X, a concept generalizing closed differential 1-form on a compact manifold, under the mild hypothesis of tameness, a field κ and a non-negative integer r, we propose two configurations $\boldsymbol{\delta}^\omega_r\colon \mathbb{R}\to\mathbb{Z}_{\geqslant 0}$ and $\boldsymbol{\gamma}^\omega_r\colon \mathbb{R}_{>0}\to\mathbb{Z}_{\geqslant 0}$ which recover Novikov–Betti numbers and the Novikov complex associated with a Morse closed 1-form with non-degenerated zeros. Precisely, the sum of the multiplicities of the points in the support of $\boldsymbol{\delta}^\omega_r$ equals the rth Novikov–Betti number and that of the points in the support of $\boldsymbol{\gamma}^\omega_r$ equals the rank of the boundary map in the Novikov complex. We formulate the basic properties of these configurations, the stability and the Poincaré duality when X is a κ-orientable closed topological manifold, which in full generality will be proven in the second part of this work.

Keywords Morse–Novikov theory · Closed one form · Configurations of points on real line · Novikov–Betti numbers

Mathematics Subject Classification 55N35 · 46M20 · 57R19

1 Introduction

In this paper we extend the configurations δ^f_r and γ^f_r, previously defined in [5] for a tame real- or angle-valued map f, to a tame topological closed 1-form ω (cf. Sect. 2 for definition). As a consequence we extend our Alternative to Morse–Novikov theory

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(cf. [3]) to topological closed 1-forms on a compact ANR (abbreviation of absolute neighborhood retract). The concept *topological closed* 1*-form*, abbreviated TC1-form, is a substantial generalization of closed differential 1-form on a smooth manifold. As in the case of real- or angle-valued maps (which corresponds to the case of closed 1-forms of degree of irrationality zero or one) we will analyze the real-valued map $f^\omega : \tilde{X} \to \mathbb{R}$, a lift of ω, defined on the total space of the associated principal $\Gamma = \mathbb{Z}^k$-covering $\pi : \tilde{X} \to X$ associated to the cohomology class $[\omega] \in H^1(X; \mathbb{R})$ with k the *degree of irrationality* of ω.

It might appear that this should be a routine extension of the case $k = 1$ but this is not quite so because:

1. the map f^ω is never proper when degree of irrationality k is greater than 1, so the homology vector spaces of the levels sets are not of finite dimension in general,
2. the set of critical values of f^ω is not discrete when $k > 1$ but the opposite, always dense if not empty; the approach of Zig-Zag persistence based on graph representations, cf. [4], is apparently not applicable.

The tameness of ω, which means the tameness of the lift f^ω of omega, as described in Sect. 2, and the fact that the group Γ defined by the form ω appears as a subgroup of $\mathbb{R}$ make the approach described in [3, Sections 6 and 7][1] applicable. Ultimately this leads to the finite configurations $\boldsymbol{\delta}^\omega_r$ and $\boldsymbol{\gamma}^\omega_r$ of points in $\mathbb{R}$ and $\mathbb{R}_+ := \mathbb{R}_{>0}$ rather than $\mathbb{C}$ and $\mathbb{C} \setminus 0$. These configurations refine and implicitly recover the Novikov complex (associated to a Morse closed differential 1-form) up to isomorphism. To prove our result we consider in Sect. 4 an apparently new definition of Novikov–Betti numbers based on the lifts of ω and verify in Sect. 5.2 that this definition is equivalent to the standard ones (cf. [6] for definitions). We are unaware if it already exists in literature.

For the configurations $\boldsymbol{\delta}^\omega_r$ and $\boldsymbol{\gamma}^\omega_r$ one can prove a stability property and a Poincaré duality property similar to [5, Theorems 1.3 and 1.5] and, in view of the stability property of the assignment $\omega \rightsquigarrow \boldsymbol{\gamma}^\omega_r$ (cf. Theorem 1.3), show that the configurations $\boldsymbol{\delta}^\omega_r$ can be actually defined for any TC1-form, not necessary tame. It also can be refined in the spirit of [1,2] to an assignment with values $\kappa[\Gamma]$-modules but we are not interested in this aspect nor in its implications at this time.

The main results about the configurations $\boldsymbol{\delta}^\omega_r$ and $\boldsymbol{\gamma}^\omega_r$ are stated in Theorems 1.1, 1.2, and 1.3. In this paper (part I) only Theorem 1.1 is proven entirely, the other two will be established in part II of this work.

To formulate them, for a fixed field κ, X a compact ANR, $\xi \in H^1(X; \mathbb{R})$, denote by $\beta^N_r(X; \xi)$ the rth Novikov–Betti number, $r = 0, 1, 2, \ldots$, and by $\mathcal{Z}^1(X; \xi)$ the set of topological closed 1-forms on X in the cohomology class ξ (cf. definition Sect. 2) equipped with the compact open topology. For a space Y and closed subset $K \subset Y$ one denotes by $\mathrm{Conf}_N(Y)$ the space of configurations of total cardinality N equipped with the collision topology and by $\mathrm{Conf}\,(Y \setminus K)$ the space of configurations of points in $Y \setminus K$ equipped with the *bottleneck topology*, all these topologies are described in Sect. 3.1. In this paper $Y = \mathbb{R}$ and $K = (-\infty, 0]$.

[1] initiated in [5]

Theorem 1.1 (i) *If ω is a tame topological closed 1-form on X (cf. Sect. 2 for definition) then*

$$\sum_{t\in\mathbb{R}} \boldsymbol{\delta}^{\omega}_r(t) = \beta^N_r(X;\xi(\omega))$$

where $\xi(\omega)=[\omega]$ denotes the cohomology class determined by ω.
The supports of $\boldsymbol{\delta}^{\omega}_r$ and $\boldsymbol{\gamma}^{\omega}_r$ are real numbers of the form $c'-c''$ with c' and c'' critical values of any lift f of the TC1-form ω (cf. Sect. 2 for definitions).

(ii) *If $X=M^n$ is a closed smooth manifold of dimension n, ω is a closed differential 1-form with all zeros of Morse type, and $c_r(\omega)$ denotes the number of zeros of Morse index r, then*

$$c_r(\omega)=\sum_{t\in\mathbb{R}}\boldsymbol{\delta}^{\omega}_r(t)+\sum_{t\in\mathbb{R}_+}\boldsymbol{\gamma}^{\omega}_r(t)+\sum_{t\in\mathbb{R}_+}\boldsymbol{\gamma}^{\omega}_{r-1}(t).$$

Item (i) explains in what sense $\boldsymbol{\delta}^{\omega}_r$ refines the Novikov–Betti numbers.

Since a chain complex of finite dimensional vector spaces is, up to a non-canonical isomorphism, completely determined by the dimensions of its homology vector spaces and the dimensions of its components, Item (ii) explains to what extent the configurations $\boldsymbol{\delta}^{\omega}_r$ and $\boldsymbol{\gamma}^{\omega}_r$ provide together a refinement of the Novikov complex when considered over the Novikov field. Note that the formula in Item (ii) implicitly gives the rank of $d_r:\ C_r\to C_{r-1}$, the boundary map in the Novikov complex. Precisely,

$$\text{rank}\, d_r=\sum_{t\in\mathbb{R}_{>0}}\boldsymbol{\gamma}^{\omega}_r(t).$$

The tameness of a TC1-form should be regarded as a considerable weakening of the hypothesis "all zeros of ω are of Morse type" in order to still make possible the construction of a Novikov complex.

Theorem 1.2 *Suppose M is a closed topological manifold and $\omega\in\mathcal{Z}^1_t(X;\xi)$, then*

(i) $\boldsymbol{\delta}^{\omega}_r(t)=\boldsymbol{\delta}^{\omega}_{n-r}(-t)$,
(ii) $\boldsymbol{\gamma}^{\omega}_r(t)=\boldsymbol{\gamma}^{-\omega}_{n-r-1}(t)$.

Let $\mathcal{Z}^1_t(X;\xi)\subset\mathcal{Z}(X;\xi)$ denote the space of tame topological closed 1-forms with the topology induced from the compact open topology on $\mathcal{Z}^1(X;\xi)$, defined in Sect. 2. The topology on $\text{Conf}_{\beta^N_r(X;\xi)}(\mathbb{R})$ is the collision topology and the topology on $\text{Conf}(\mathbb{R}_+)$, with $\mathbb{R}_+$ viewed as $Y\setminus K$ for $Y=\mathbb{R}$ and $K=(-\infty,0]$, is the bottleneck topology described in Sect. 3.1.

Theorem 1.3 (i) *The assignment $\boldsymbol{\delta}_r:\ \mathcal{Z}^1_t(X;\xi)\rightsquigarrow\text{Conf}_{\beta^N_r(X;\xi)}(\mathbb{R})$ is continuous and extends to a continuous assignment on the entire $\mathcal{Z}^1(X;\xi)$.*

(ii) *The assignment $\boldsymbol{\gamma}_r:\ \mathcal{Z}^1_t(X;\xi)\rightsquigarrow\text{Conf}(\mathbb{R}_+)$ is continuous.*

To understand the relations between this paper and the previous works [3–5] the following observations are useful:

1. When X is connected, a topological closed 1-form ω can be represented by a real-valued map $f: \widetilde{X} \to \mathbb{R}$, called lift of ω, $\widetilde{X}$ the total space of the principal covering associated to ω. This lift is determined up to an additive constant, cf. Sect. 2 (i.e., f_1 and f_2 are lifts of the form ω implies $f_1 = f_2 + t$, $t \in \mathbb{R}$). The configurations $\boldsymbol{\delta}^\omega_r$ and $\boldsymbol{\gamma}^\omega_r$ are derived from the integer-valued maps δ^f_r, respectively γ^f_r, which have as supports points $(a, b) \in \mathbb{R}^2$, respectively $\mathbb{R}^2_+ = \{(x, y) : x < y\}$, with a, b critical values of f. Since the support of δ^{f+t}, respectively γ^{f+t}, is the t-diagonal translate of the support δ^f, respectively γ^f, in order to get "independence on the representative f" one passes to the quotient spaces $\mathbb{R}^2/\mathbb{R} = \mathbb{R}$, respectively $\mathbb{R}^2_+/\mathbb{R} = \mathbb{R}_+$, where the quotient is taken w.r. to the action $\mu(t, (a, b)) = (a + t, b+t)$. The supports of $\boldsymbol{\delta}^\omega_r$ and $\boldsymbol{\gamma}^\omega_r$ are the images by $p: \mathbb{R}^2 \to \mathbb{R}$, $p(a, b) = b-a$, of the supports of δ^f_r and γ^f_r with $\boldsymbol{\delta}^\omega_r(t) = \sum_{s\in\mathbb{R}} \delta^f_r(s, s + t)$ and $\boldsymbol{\gamma}^\omega_r(t) = \sum_{s\in\mathbb{R}} \gamma^f_r(s, s + t)$.
2. In this paper the notation γ^f_r refers to the restriction to $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y-x > 0\}$ of the map denoted by the same letter γ^f_r in [3]. Note that such restriction to $\mathbb{R}^2_+$ collects information on the so-called closed-open bar codes of f, the ones of relevance in the Morse–Novikov theory, while the restriction to $\mathbb{R}^2_- = \{(x, y) \in \mathbb{R}^2 : y - x < 0\}$ collects information on the open-closed bar codes of f. Note also that the open-closed bar codes of f correspond to the closed-open bar codes of $-f$ via the correspondence $(a, b) \to (-b, -a)$.
3. An interesting example of a tame closed topological 1-form is provided by a simplicial 1-cocycle on a finite simplicial complex. An algorithm to derive the bar codes (i.e., points in the supports of $\boldsymbol{\delta}^\omega$ and $\boldsymbol{\gamma}^\omega$) with their multiplicity is desirable. This is possible and will be the topic of a subsequent work.

In this paper we write "=" for equality or canonical isomorphism and "$\simeq$" for isomorphism, not necessary canonical.

An alternative treatment via persistent homology of Floer–Novikov theory was proposed by Usher and Zhang, cf. [8]. Their work has challenged us to extend the results presented in [3] from angle-valued maps to topological closed 1-forms.

2 Topological closed 1-forms and tameness

2.1 Topological closed 1-form

A topological closed 1-form, abbreviated TC1-form, extends the concept of closed differential 1-form on a smooth manifold M to an arbitrary topological space X. One way to obtain this is to view it as an equivalence class of multi-valued maps (first definition), another way is to view it as an equivalence class of equivariant maps on the associated principal $\mathbb{Z}^k$-covering (second definition).

First definition

1. A *multi-valued map* is a system $\{U_\alpha, f_\alpha : U_\alpha \to \mathbb{R}, \alpha \in A\}$ such that

(a) U_α are open sets with $X = \bigcup U_\alpha$,
(b) f_α are continuous maps such that $f_\alpha - f_\beta \colon U_\alpha \cap U_\beta \to \mathbb{R}$ is locally constant.

2. Two multi-valued maps are *equivalent* if put together remain a multi-valued map.

Definition 2.1 A *TC1-form* is an equivalence class of multi-valued maps.

A TC1-form ω determines a cohomology class $\xi(\omega) \in H^1(X;\mathbb{R})$. It suffices to show that a representative $\{U_\alpha, f_\alpha \colon U_\alpha \to \mathbb{R}\}$ of ω defines for any continuous path $\gamma \colon [a,b] \to X$ the number $\int_\gamma \omega \in \mathbb{R}$, independent of the homotopy class rel. boundary of γ and additive w.r. to juxtaposition of paths. Indeed, if $\gamma[a,b] \subset U_\alpha$ for some α, then $\int_\gamma \omega = f_\alpha(b) - f_\alpha(a)$; if not, one chooses a subdivision of $[a,b]$, $a = t_0 < t_1 < \cdots < t_r = b$, such that $\gamma_i := \gamma|_{[t_i,t_{i+1}]}$ lie in some open set U_α, and defines $\int_\gamma \omega := \sum \int_{\gamma_i} \omega$. This assignment defines a homomorphism $\xi(\omega) \colon H_1(X;\mathbb{Z}) \to \mathbb{R}$, equivalently a cohomology class $[\omega] = \xi(\omega)$.

One denotes by $\mathcal{Z}^1(X)$ the set of all TC1-forms and by $\mathcal{Z}^1(X;\xi) := \{\omega \in \mathcal{Z}^1(X) \colon \xi(\omega) = \xi\}$. Clearly $\mathcal{Z}^1(X)$ is an $\mathbb{R}$-vector space and

$$\mathcal{Z}^1(X) = \bigsqcup_{\xi \in H^1(X;\mathbb{R})} \mathcal{Z}^1(X;\xi).$$

In view of this definition, for any X compact ANR one can find open covers of X, $\{U_\alpha, \alpha \in A\}$ with the properties that A is a finite set and $\overline{U_\alpha}$ is compact, connected and simply-connected. Such cover is called a *good cover*. A choice $x_\alpha \in U_\alpha$ makes ω uniquely represented by a multi-valued map $\{f_\alpha^\omega \colon U_\alpha \to \mathbb{R}\}$ with $f_\alpha^\omega(x_\alpha) = 0$. One calls the system $\mathcal{U} := \{U_\alpha, x_\alpha, \alpha \in A\}$ with $\{U_\alpha, \alpha \in A\}$ good cover a *base-pointed good cover* of X.

The choice of a base pointed good cover $\mathcal{U}$ defines for the vector space $\mathcal{Z}^1(X)$ a complete norm

$$\|\omega\|_{\mathcal{U}} := \sup_{\alpha \in A} \sup_{x \in \overline{U}_\alpha} |f_\alpha^\omega(x)|,$$

and then a distance in $\mathcal{Z}^1(X)$ and implicitly in $\mathcal{Z}^1(X;\xi)$,

$$D(\omega_1, \omega_2)_{\mathcal{U}} := \|\omega_1 - \omega_2\|_{\mathcal{U}}.$$

Different base-pointed good covers lead to equivalent norms. The induced topology on $\mathcal{Z}^1(X)$ is referred to as the *compact open topology*. The subsets $\mathcal{Z}^1(X;\xi)$ are the connected components of $\mathcal{Z}^1(X)$.

Examples 1. A closed differential 1-form, $\omega \in \Omega^1(M)$, $d\omega = 0$, defines a TC1-form.
Indeed, in view of the Poincaré Lemma, for any $x \in M$ one chooses an open neighborhood $U_x \ni x$ and $f_x \colon U_x \to \mathbb{R}$ a smooth map such that $\omega_x|_{U_x} = df_x$. The system $\{U_x, f_x \colon U_x \to \mathbb{R}\}$ provides a representative of a TC1-form.

2. A simplicial 1-cocycle on the simplicial complex X defines a TC1-form.
   If X is a simplicial complex, $\mathcal{X}_0$ the collection of vertices, and $S \subset \mathcal{X}_0 \times \mathcal{X}_0$ the collection of pairs (x, y), $x, y \in \mathcal{X}_0$, such that x, y are boundaries of a 1-simplex, then a simplicial 1-cocycle is a map $\delta\colon S \to \mathbb{R}$ with the properties $\delta(x, y) = -\delta(y, x)$ and for any three vertices x, y, z with $(x, y), (y, z), (x, z) \in S$ one has $\delta(x, y) + \delta(y, z) + \delta(z, x) = 0$. The collection of open sets U_x, U_x the *open star* of the vertex $x \in \mathcal{X}_0$, and the maps $f_x\colon U_x \to \mathbb{R}$, the linear extensions to open simplexes of U_x of the map given on the vertices in U_x by $f_x(y) := \delta(x, y)$ and $f_x(x) = 0$, provides a representative of a TC1-form.

Second definition

Let $\xi \in H^1(X; \mathbb{R}) = \mathrm{Hom}(H_1(X; \mathbb{Z}), \mathbb{R})$ and $\Gamma = \Gamma(\xi) := \mathrm{img}(\xi) \subset \mathbb{R}$. If X is a compact ANR then $\Gamma \simeq \mathbb{Z}^k$ with k called the *degree of irrationality* of ξ.

The surjective homomorphism $\xi\colon H_1(X; \mathbb{Z}) \to \Gamma$ defines the associated Γ-principal covering, $\pi\colon \widetilde{X} \to X$, i.e., a free action $\mu\colon \Gamma \times \widetilde{X} \to \widetilde{X}$ with π the quotient map $\widetilde{X} \to \widetilde{X}/\Gamma = X$. This principal covering is unique up to isomorphism. When X is connected so is $\widetilde{X}$.

A continuous map $f\colon \widetilde{X} \to \mathbb{R}$ is Γ-equivariant if $f(\mu(g, x)) = f(x) + g$.

Definition 2.2 A *TC1-form* ω of cohomology class ξ is an equivalence class of continuous Γ-equivariant real-valued maps $f\colon \widetilde{X} \to \mathbb{R}$ where f_1 is equivalent to f_2 iff $f_1 - f_2$ is locally constant.

One refers to any representative f in this class as a lift of ω. Clearly any Γ-equivariant map on a Γ-principal covering $\widetilde{X} \to X$ defines a cohomology class in $H^1(X; \mathbb{R}) = \mathrm{Hom}(H_1(X; \mathbb{Z}), \mathbb{R})$, the same for equivalent equivariant maps. This because for any continuous path $\gamma\colon [0, 1] \to X$ and $\tilde{x} \in \widetilde{X}$ there is a unique $\tilde{\gamma}\colon [0, 1] \to \widetilde{X}$ with $\tilde{\gamma}(0) = \tilde{x}$, $\gamma = \pi \cdot \tilde{\gamma}$ and, by taking $\int_\gamma \omega := f(\tilde{\gamma}(1)) - f(\tilde{x})$, one obtains a homomorphism $H_1(X; \mathbb{Z}) \to \mathbb{R}$. Denote by $\mathcal{Z}^1(X; \xi)$ the set of TC1-forms in the cohomology class ξ. In view of this definition the choice of the base point $\tilde{x}$ in $\widetilde{X}$ (actually one in each connected component if $\widetilde{X}$ is not connected) provides a unique lift $f^\omega_{\tilde{x}}\colon \widetilde{X} \to \mathbb{R}$ of ω with $f^\omega_{\tilde{x}}(\tilde{x}) = 0$. When X is compact one defines the distance $D(\omega_1, \omega_2)_{\tilde{x}}$ by

$$D(\omega_1, \omega_2)_{\tilde{x}} := \sup_{y \in \widetilde{X}} \left| f^{\omega_1}_{\tilde{x}}(y) - f^{\omega_1}_{\tilde{x}}(y) \right|$$

which, in view of the compacity of X and of the Γ-equivariance of the lifts, is a complete metric. It is not hard to see that different choices of the base point $\tilde{x}$ lead to equivalent distances and therefore to the same induced topology with the same collection of Cauchy sequences.

It is not hard to show that the two definitions of $\mathcal{Z}^1(X)$ viewed as vector spaces equipped with complete metrics are equivalent. To see this one chooses a good cover $\{U_\alpha, \alpha \in A\}$ of X. Indeed, a multi-valued map $\{f_\alpha\colon U_\alpha \to \mathbb{R}\}$ representing ω (cf. the first definition) can be modified to an equivalent multi-valued map $\{f'_\alpha\colon U_\alpha \to \mathbb{R}\}$ by adding appropriate constants on each open set U_α so that $f'_\alpha \cdot \pi_\alpha\colon \pi^{-1}(U_\alpha) \to \mathbb{R}$ defines a Γ-equivariant map on $\widetilde{X}$, hence a representative of a TC1-form (cf. the second definition) in the same cohomology class.

Conversely, a Γ-equivariant map representing ω (in the second definition) and a collection of continuous section $s_\alpha\colon U_\alpha \to \pi^{-1}(U_\alpha)$ (i.e., $\pi \cdot s_\alpha = \mathrm{id}_{U_\alpha}$) give a multi-valued map $\{f \cdot s_\alpha\colon U_\alpha \to \mathbb{R}\}$ representing a TC1-form (first definition) in the same cohomology class.

It is not hard to check that the identifications above make the distances defined by different choices of base points in U_α and $\widetilde{X}$ equivalent and consequently with the same Cauchy sequences.

2.2 Weakly tame and tame real-valued maps and topological closed 1-forms

Fix a field κ. The homology considered is always with coefficients in the fixed field κ (for simplicity in writing omitted from the notation), hence a κ-vector space.

For a continuous map $f\colon X \to \mathbb{R}$ denote

$$X^f_a := f^{-1}((-\infty, a]), \quad X^f_{<a} := f^{-1}((-\infty, a)),$$
$$X^a_f := f^{-1}([a, \infty)), \qquad X^{>a}_f := f^{-1}((a, \infty)).$$

For $a \in \mathbb{R}$ let

$$R^f_a(r) := \dim H_r(X_a, X_{<a}) \in \mathbb{Z}_{\geqslant 0} \sqcup \infty,$$
$$R^a_f(r) := \dim H_r(X^a, X^{>a}) \in \mathbb{Z}_{\geqslant 0} \sqcup \infty.$$

The value $a \in \mathbb{R}$ is called *regular* (w.r.t. to κ) if $R^f_a(r) + R^a_f(r) = 0$ for any r and *critical* if not regular.

Denote by $\mathrm{CR}_r(f) \subset \mathbb{R}$ the set of critical values of f with the property $R^f_a(r) + R^a_f(r) \neq 0$ and $\mathrm{CR}(f) := \bigcup_r \mathrm{CR}_r(f)$.

Definition 2.3 A continuous map $f\colon X \to \mathbb{R}$ is called *weakly tame* if:

1. For any closed interval $I \in \mathbb{R}$ the subspace $f^{-1}(I)$ is an ANR, in particular X is an ANR.
2. For any $a \in \mathbb{R}$ and any r, $R^f_a(r) + R^a_f(r) < \infty$.
3. The set $\mathrm{CR}(f)$ is at most countable.

Let ω be a TC1-form on a connected space X and let $f\colon \widetilde{X} \to \mathbb{R}$ be a lift of ω. The sets $\mathrm{CR}_r(f)$ and $\mathrm{CR}(f)$ are Γ-invariant with respect to the action of Γ on $\mathbb{R}$ by translation (recall $\Gamma \subset \mathbb{R}$) and this action is free. The set of orbits $\mathrm{CR}_r(f)/\Gamma$ or $\mathrm{CR}(f)/\Gamma$ will be denoted by $\mathcal{O}_r(f)$ or $\mathcal{O}(f)$. If f_1 and f_2 are two lifts of ω then $f_2 = f_1 + t$. The translation by t provides a canonical bijective map $T_t\colon \mathcal{O}(f_1) \to \mathcal{O}(f_2)$ which preserves the r-component $\mathcal{O}_r(f)$. Denote then $\mathcal{O}(\omega) := \bigsqcup_{f \in \omega} \mathcal{O}(f)/\!\sim$ with $o_1 \sim o_2$ ($o_i \in \mathcal{O}(f_i)$) iff $T_t(o_1) = o_2$. This definition frees the concept of *orbit of critical values* from the lift f. Suppose X is a compact ANR.

Definition 2.4 • A TC1-form ω is *weakly tame* if one lift f and then any other is weakly tame.

- A TC1-form ω is *tame* if is weakly tame and the set $\mathcal{O}(\omega)$ is finite.

When X is not connected ω is weakly tame, respectively tame, if its restriction to each component is weakly tame, respectively tame.

Examples of tame TC1-forms

1. A locally polynomial[2] closed differential 1-form on a closed smooth manifold with all zeros isolated is tame.
2. A generic simplicial 1-cocycle on a finite simplicial complex defines a TC1-form which is tame. Here generic means that the 1-cocycle takes non-zero values on all 1-simplexes.[3]

Let us check the first example for closed manifolds. The arguments provided remain true when the manifold is compact and the restriction of ω to the boundary has no zeros.

Let $\pi: \tilde{M} \to M$ be the associated Γ-principal covering, $f: \tilde{M} \to \mathbb{R}$ a lift of ω, $\mathcal{X}$ the set of zeros of ω, and $\tilde{\mathcal{X}} = \pi^{-1}(\mathcal{X})$ the set of critical points of f. Let $\tilde{\mathcal{X}}(t) := \tilde{\mathcal{X}} \cap f^{-1}(t)$. Observe that Γ acts freely on the set $\tilde{X}$ and the set of orbits of this action is in bijective correspondence to the set $\mathcal{X}$, hence is finite. Observe also that the restriction of π to $\tilde{\mathcal{X}}(t)$ is injective.

If $t \in \mathbb{R}$ is a regular value then $f^{-1}(t)$ is a codimension one smooth sub-manifold and if t is a critical value then $f^{-1}(t)$ is a codimension one *sub-manifold with finitely many conic singularities*, as many as the cardinality of $\tilde{\mathcal{X}}(t)$. In both cases, t regular or critical value, $f^{-1}(t)$ is a closed subset which is an ANR and then so is $f^{-1}(I)$ for any closed interval I. This verifies requirement 1. in Definition 2.3.

Note that if t is a regular value, then $\tilde{M}_t$ is a manifold with boundary with interior $\tilde{M}_{<t}$, hence $H_r(\tilde{M}_t, \tilde{M}_{<t}) = 0$. If t is a critical value, then $\tilde{M}_t \setminus \tilde{\mathcal{X}}(t)$ is a manifold with boundary with interior $\tilde{M}_{<t}$, hence $H_r(\tilde{M}_t, \tilde{M}_{<t}) = H_r(\tilde{M}_t, \tilde{M}_t \setminus \tilde{\mathcal{X}}(t)) = H_r(D_t, D_t \setminus \tilde{\mathcal{X}}(t))$ with $D_t = \tilde{M}_t \cap D$, where D is a disjoint union of closed small discs embedded in $\tilde{M}$, whose interior is a neighborhood of $\tilde{\mathcal{X}}(t)$. The hypothesis "local polynomial" permits to choose such small discs that make D_t and $S_t = (\partial D) \cap \tilde{M}_t$ compact ANRs and $D_t \setminus \tilde{\mathcal{X}}(t)$ retractible by deformation to S_t. Since (D_t, S_t) is a pair of compact ANRs, $H_r(D_t, D_t \setminus \tilde{\mathcal{X}}(t)) = H_r(D_t, S_t)$ is a vector space of finite dimension. This verifies the finite dimensionality of $H_r(\tilde{M}_t, \tilde{M}_{<t})$. The same arguments verify the finite dimensionality of $H_r(\tilde{M}^t, \tilde{M}^{>t})$, hence requirement 2. in Definition 2.3 is verified for t a critical value. Requirement 3. is obvious in view of the compacity of M.

In the second example the arguments are similar. Note that if t is a simplicial regular value for the lift f then $f^{-1}(t)$ has a collar neighborhood inside the simplicial

[2] Locally polynomial means that locally there exist coordinates such that the coefficients of the form are polynomial functions.

[3] The tameness remains true without the hypothesis *all zeros are isolated* in Case 1 and *generic* in Case 2 but via more elaborated arguments. The restriction of a differential closed 1-form on a manifold M considered in Case 1 to a compact Thom–Mather stratified subset $X \subset M$ is a tame TC1-form on X.

complex $\widetilde{X}$ in which case $(\widetilde{X}_t, \widetilde{X}_{<t})$ can be treated homologically as $(\widetilde{M}_t, \widetilde{M}_{<t})$ above. If t is a simplicial critical value, in view of the genericity, except for a finite set of points $\mathcal{V}_t = \{x_1, \dots, x_k\} \subset f^{-1}(f)$, $f^{-1}(t) \setminus \mathcal{V}_t$ has a collar neighborhood inside $\widetilde{X} \setminus \mathcal{V}_t$. With these observations the homological arguments in the smooth case can be repeated.

3 Topology

3.1 Configurations of points, collision topology, bottleneck topology

Consider a pair (Y, K), Y a locally compact space, and $K \subset Y$ a closed subset. A *configuration* of points in Y is a map $\delta\colon Y \to \mathbb{Z}_{\geqslant 0}$ with finite support. The *total cardinality* of the support is the non-negative integer $\sum_{y \in Y} \delta(y)$. Denote by $\mathrm{Conf}(Y)$ the set of all configurations of points in Y and by $\mathrm{Conf}_N(Y)$ the subset of configurations whose supports have total cardinality N. For a configuration $\delta \in \mathrm{Conf}(Y \setminus K)$ with support $\mathrm{supp}\,\delta := \{y_1, y_2, \dots, y_k\}$ and a collection of disjoint open sets $U_1, U_2, \dots, U_k, V$ with $x_i \in U_i$, $K \subset V$ denote by

$$\mathcal{U}(\delta, U_1, \dots, U_k, V) := \left\{ \delta' \in \mathrm{Conf}(Y \setminus K)\colon \begin{array}{l} \mathrm{supp}\,\delta' \subset \left(\bigcup_{i=1,2,\dots,k} U_i\right) \cup V \\ \sum_{y \in U_i} \delta'(y) = \delta(y_i) \end{array} \right\}$$

and for $\delta \in \mathrm{Conf}(Y)$, and $K = \varnothing$ write

$$\mathcal{U}(\delta, U_1, \dots, U_k) := \mathcal{U}(\delta, U_1, \dots, U_k, \varnothing).$$

On the set $\mathrm{Conf}_N(Y)$ consider the topology generated by the collections of neighborhoods $\{\mathcal{U}(\delta, U_1, \dots, U_k)\}$ of each $\delta \in \mathrm{Conf}_N(Y)$ and refer to it as the *collision topology*. As a topological space $\mathrm{Conf}_N(Y)$ identifies to Y^N/Σ_N, the quotient of the N-fold cartesian product of Y by the group of permutations of N elements.

On the set $\mathrm{Conf}(Y \setminus K)$ consider the topology generated by the collections of neighborhoods $\{\mathcal{U}(\delta, U_1, \dots, U_k, V)\}$ of each $\delta \in \mathrm{Conf}(Y \setminus K)$ and refer to it as the *bottleneck topology*. Note that if $K = \varnothing$ the bottleneck topology and collision topology are the same.

In this paper we will consider only the case $Y = \mathbb{R}$ and $K = (-\infty, 0]$ hence $Y \setminus K = \mathbb{R}_+$.

3.2 Some algebraic topology of a pair (X, ω)

Let κ be a field, X a compact ANR, ω a TC1-form in the cohomology class $\xi = [\omega]$ of degree of irrationality k, and $\Gamma = \Gamma(\xi) \subset \mathbb{R}$ the group defined by ξ. Note that if $k \geqslant 2$ then Γ is dense in $\mathbb{R}$.

Let $\widetilde{X} \to X$ be the associated principal Γ-covering with the free action $\mu\colon \Gamma \times \widetilde{X} \to \widetilde{X}$ and $f\colon \widetilde{X} \to \mathbb{R}$ be a lift of ω. For any $g \in \Gamma$ the homeomorphism $\mu(g, \cdot)\colon \widetilde{X} \to \widetilde{X}$ induces the isomorphism $\langle g \rangle\colon H_r(\widetilde{X}) \to H_r(\widetilde{X})$.

The map f provides two filtrations of $\widetilde{X}$ indexed by $t \in \mathbb{R}$, for $t < t' < t''$,

$$\cdots \subset \widetilde{X}_t \subset \widetilde{X}_{t'} \subset \widetilde{X}_{t''} \subset \cdots, \qquad \cdots \supset \widetilde{X}^t \supset \widetilde{X}^{t'} \supset \widetilde{X}^{t''} \supset \cdots$$

which induce in homology the filtrations

$$\cdots \subseteq \mathbb{I}_t(r) \subseteq \mathbb{I}_{t'}(r) \subseteq \mathbb{I}_{t''}(r) \subseteq \cdots, \qquad \cdots \supseteq \mathbb{I}^t(r) \supseteq \mathbb{I}^{t'}(r) \supseteq \mathbb{I}^{t''}(r) \supseteq \cdots$$

with

$$\mathbb{I}^f_t(r) := \operatorname{img}(H_r(\widetilde{X}_t) \to H_r(\widetilde{X})) \text{ and } \mathbb{I}^t_f(r) := \operatorname{img}(H_r(\widetilde{X}^t) \to H_r(\widetilde{X})).$$

Clearly, $\langle g\rangle(\mathbb{I}_t(r)) = \mathbb{I}_{t+g}(r)$ and $\langle g\rangle(\mathbb{I}^t(r)) = \mathbb{I}^{t+g}(r)$. Note that:

1. The κ-vector space $H_r(\widetilde{X})$ is actually a f.g. $\kappa[\Gamma]$-module (since X is a compact ANR) actually a Noetherian module,
2. $\mathbb{I}_{-\infty}(r) := \bigcap_{t\in\mathbb{R}} \mathbb{I}_t(r)$ and $\mathbb{I}^{\infty}(r) := \bigcap_{t\in\mathbb{R}} \mathbb{I}^t(r)$ are $\kappa[\Gamma]$-submodules,
3. $H_r(\widetilde{X}) = \bigcup_t \mathbb{I}_t(r) = \bigcup_t \mathbb{I}^t(r)$,
4. $H^N_r(X, [\omega]) := H_r(\widetilde{X})/\text{Tor}, H_r(\widetilde{X})$ is a f.g. torsion free $\kappa[G]$-module of rank $\beta^N_{\text{alg},r}(X; [\omega])$ (i.e., the rank of a maximal free submodule), number referred to as the algebraic Novikov–Betti number or simply Novikov–Betti number.
   Note that when $\kappa = \mathbb{R}$ or $\mathbb{C}$ then $\beta^N_r(X; [\xi])$ equals the L_2-Betti number $\beta^{L_2}_r(\widetilde{X})$ of $\widetilde{X}$ (cf. [7, Lemma 1.34]).

Proposition 3.1 (i) $\text{Tor}\, H_r(\widetilde{X}) = \mathbb{I}_{-\infty}(r)$.
(ii) $\text{Tor}\, H_r(\widetilde{X}) = \mathbb{I}^{\infty}(r)$.

Proof The $\kappa[\Gamma]$-module structure of $H_r(\widetilde{X})$ is given by

$$\left(\sum_{i=1}^k a_{g_i} g_i\right) \cdot x := \sum_{i=1}^k a_{g_i} \langle g_i\rangle(x)$$

with $a_{g_i} \in \kappa$, $a_{g_i} \neq 0$ and then if $x \in \mathbb{I}_t(r)$, respectively $\mathbb{I}^t(r)$, one has

$$\left(\sum_{i=1}^k a_{g_i} g_i\right) \cdot x \in \mathbb{I}_{t+\max g_i}(r), \quad \text{respectively} \quad \left(\sum_{i=1}^k a_{g_i} g_i\right) \cdot x \in \mathbb{I}^{t+\min g_i}(r).$$

To check the inclusion $\text{Tor}\, H_r(\widetilde{X}) \subset \mathbb{I}_{-\infty}(r)$ in (i) one starts with $x \in \text{Tor}\, H_r(\widetilde{X})$ which has to belong by 3. above to some $\mathbb{I}_t(r)$. Suppose that

$$(a_0 g_0 + a_1 g_1 + \cdots + a_k g_k) \cdot x = 0$$

with $g_0 < g_1 < \cdots < g_k$, $a_{g_i} \neq 0$. Then $x \in \mathbb{I}_t(r)$ implies

$$a_k x = -\big(a_0(g_0 - g_k) + a_1(g_1 - g_k) + \cdots + a_{k-1}(g_{k-1} - g_k)\big) \cdot x,$$

hence $x \in \mathbb{I}_{t-(g_k-g_{k-1})}(r)$. By repeating the argument, $x \in \mathbb{I}_{t-n(g_k-g_{k-1})}(r)$ for any n, one derives $x \in \mathbb{I}_{-\infty}(r)$. Similarly, to check the inclusion Tor $H_r(\tilde{X}) \subset \mathbb{I}^{\infty}(r)$, one starts with $x \in \mathbb{I}^{t}(r)$, suppose that $g_0 > g_1 > \cdots > g_k$, $a_{g_i} \neq 0$, one derives $x \in \mathbb{I}^{t+(-g_k+g_{k-1})}(r)$, hence $x \in \mathbb{I}^{t+n(-g_k+g_{k-1})}(r)$ for any n, hence $x \in \mathbb{I}^{\infty}(r)$.

To check that $I_{-\infty}(r) \subseteq$ Tor $H_r(\tilde{X})$ and $I^{\infty}(r) \subseteq$ Tor $H_r(\tilde{X})$ one uses the fact that $H_r(\tilde{X})$ is a f.g. $\kappa[\Gamma]$-module. If $x \in I_{-\infty}(r)$ then there exists an infinite collection of negative g's in Γ, say $\cdots < g_r < g_{r-1} < \cdots < g_2 < g_1$, such that $\langle g_r\rangle(x) \in \mathbb{I}_{-\infty}(r)$ and, in view of the fact that $\mathbb{I}_{-\infty}(r)$ is f.g., a finite collection of elements $P_{r_i} \in \kappa[G]$, $i = 1, 2, \ldots, K$, such that

$$0 = \sum_{i=1}^{K} P_{r_i} \cdot (\langle g_{r_i}\rangle(x)) = \left(\sum_{i=1}^{K} P_{r_i} g_{r_i}\right) \cdot x.$$

Hence one obtains $x \in$ Tor $H_r(\tilde{X})$. By a similar argument one concludes that $x \in \mathbb{I}^{\infty}(r)$ implies $x \in$ Tor $H_r(\tilde{X})$. □

As an immediate consequence one has

$$\mathbb{I}_a(r) \cap \mathbb{I}^{\infty}(r) = \mathbb{I}^{\infty}(r) = \text{Tor } H_r(\tilde{X}),$$
$$\mathbb{I}_{-\infty}(r) \cap \mathbb{I}^{b}(r) = \mathbb{I}_{-\infty}(r) = \text{Tor } H_r(\tilde{X}).$$

Denote by $i_{\alpha}^{\beta}(r)\colon\ H_r(\tilde{X}_{\alpha}) \to H_r(\tilde{X}_{\beta})$, $\alpha < \beta$, the inclusion induced linear map and for $a \in \mathbb{R}$ consider the diagram

$$\begin{array}{ccccc}
\varprojlim_{a>t\to-\infty} H_r(\tilde{X}_t) & \longrightarrow & H_r(\tilde{X}_a) & \longrightarrow & \varinjlim_{a<t\to\infty} H_r(\tilde{X}_t) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{I}_{-\infty} = \bigcap_{t\in\mathbb{R}} \mathbb{I}_t(r) & \longrightarrow & \mathbb{I}_a(r) & \longrightarrow & \bigcup_{t\in\mathbb{R}} \mathbb{I}_t(r) = \mathbb{I}_{\infty}(r)
\end{array} \qquad (3.1)$$

where the direct and inverse limits are taken w.r. to $i_{\alpha}^{\beta}(r)$. Clearly,

$$\varprojlim_{t\to-\infty} H_r(\tilde{X}_t) = \varprojlim_{a>t\to-\infty} H_r(\tilde{X}_t),$$
$$\varinjlim_{t\to\infty} H_r(X_t) = \varinjlim_{a<t\to\infty} H_r(\tilde{X}_t),$$
$$\varinjlim_{t\to\infty} H_r(X_t) = H_r(\tilde{X}).$$

Theorem 3.2 *The left and right vertical arrows in the diagram (3.1) are isomorphisms. More precisely, for each $l \in \Gamma \subset \mathbb{R}$ there exist the subspaces $V, W \subset H_r(X_a)$ depending on l and the injective linear map $\alpha\colon\ V \to V$ such that $H_r(\tilde{X}_a) = V \oplus W$ and the diagram (3.1) is isomorphic to the diagram*

$$\begin{array}{ccccc} \bigcap_{k\in\mathbb{Z}}\alpha^k(V) & \longrightarrow & V\oplus W & \longrightarrow & \overline{V} \\ \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle p_V} & & \Big\downarrow{\scriptstyle \mathrm{id}} \\ \bigcap_{k\in\mathbb{Z}}\alpha^k(V) & \longrightarrow & V & \longrightarrow & \overline{V} \end{array}$$

where $\overline{V} := \varinjlim \{\cdots V \xrightarrow{\alpha} V \cdots\}$, *the bottom horizontal arrows are inclusions and the middle vertical arrow is the first component projection.*

Proof Consider the isomorphism $\mu_l(r)\colon\ H_r(X_{a-l}) \to H_r(X_a)$ induced by the homeomorphism $\mu_l\colon\ X_{a-l} \to X_a$ and the inclusion induced linear map $i^a_{a-l}(r)\colon\ H_r(X_{a-l}) \to H_r(X_a)$. Take $V = \mathrm{img}\,(i^a_{a-l}(r))$ and W a complement of $V \subset H_r(\widetilde{X}_a)$, hence $H_r(X_a) = V\oplus W$. One can decompose $H_r(X_{a-l})$ as $H_r(X_{a-l}) = V'\oplus W'$, in order to have $i^a_{a-l}(r) = \left(\begin{smallmatrix} u & 0\\ 0 & 0\end{smallmatrix}\right)$ with $u\colon\ V' \to V$ an isomorphism. With respect to this decompositions the isomorphism

$$\mu_{-l}(r) = \mu_l(r)^{-1}\colon\ V\oplus W \to V'\oplus W'$$

is given by the matrix $\left(\begin{smallmatrix} \alpha_{1,1} & \alpha_{1,2}\\ \alpha_{2,1} & \alpha_{2,2}\end{smallmatrix}\right)$ and the composition $v = i^a_{a-l}(r)\cdot\mu_{-l}(r)$ is given by $\left(\begin{smallmatrix} \alpha & \beta\\ 0 & 0\end{smallmatrix}\right)$ with $\alpha\colon\ V \to V$ injective. This is our claimed injective linear map α. In view of the commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_r(X_{a-2l}) & \xrightarrow{i^{a-l}_{a-2l}(r)} & H_r(X_{a-l}) & \xrightarrow{i^{a}_{a-l}(r)} & H_r(X_a) & \xrightarrow{i^{a+l}_{a}(r)} & H_r(X_{a+l}) & \longrightarrow & \cdots \\ & & \Big\uparrow{\scriptstyle \mu_{-2l}(r)} & & \Big\uparrow{\scriptstyle \mu_{-l}(r)} & & \Big\uparrow{\scriptstyle =} & & \Big\uparrow{\scriptstyle \mu_{l}(r)} & & \\ \cdots & \longrightarrow & H_r(X_a) & \longrightarrow & H_r(X_a) & \longrightarrow & H_r(X_a) & \longrightarrow & H_r(X_a) & \longrightarrow & \cdots \\ & & \Big\uparrow{\scriptstyle =} & & \Big\uparrow{\scriptstyle =} & & \Big\uparrow{\scriptstyle =} & & \Big\uparrow{\scriptstyle =} & & \\ \cdots & \xrightarrow{v} & V\oplus W & \xrightarrow{v} & V\oplus W & \xrightarrow{v} & V\oplus W & \xrightarrow{v} & V\oplus W & \xrightarrow{v} & \cdots \end{array}$$

whose vertical arrows are isomorphisms and in view of the description of v one obtains:

$$\varprojlim_{k\to\infty} H_r(\widetilde{X}_{a-kl}) = \bigcap_k \alpha^k(V) \subset V,$$
$$\varinjlim_{k'\to\infty} H_r(\widetilde{X}_{a+k'l}) = \varinjlim\{\cdots \xrightarrow{\alpha} V \xrightarrow{\alpha} V \xrightarrow{\alpha} V \xrightarrow{\alpha} \cdots\}.$$

Since

$$\varprojlim_{k\to\infty} H_r(\widetilde{X}_{a-kl}) = \varprojlim_{t\to-\infty} H_r(\widetilde{X}_t),$$
$$\varinjlim_{k\to\infty} H_r(X_{a+kl}) = \varinjlim_{a<t\to\infty} H_r(X_t),$$

the statement follows. □

3.3 Novikov–Betti numbers (a topological definition)

Recall that for $f\colon \widetilde{X} \to \mathbb{R}$, a lift of a tame TC1-form ω, the vector space $\mathbb{I}^f_t(r)/\mathbb{I}^f_{<t}(r)$ is zero when t is a regular value and of finite dimension when t is a critical value, and the isomorphism $\langle g\rangle\colon\ H_r(\widetilde{X}) \to H_r(\widetilde{X})$ induces an isomorphism

$$\langle g\rangle_t\colon\ \mathbb{I}^f_t(r)/\mathbb{I}^f_{<t}(r) \to \mathbb{I}^f_{t+g}(r)/\mathbb{I}^f_{<(t+g)}(r).$$

Consider the κ-vector space

$$\mathbb{NH}_r(f) := \bigoplus_{t\in\mathbb{R}} \mathbb{I}^f_t(r)/\mathbb{I}^f_{<t}(r)$$

and note that this sum

- involves only components corresponding to t critical values, hence at most countably many,
- is a $\kappa[\Gamma]$-module whose multiplication by g is provided by the component-wise isomorphism $\langle g\rangle_t$,
- is independent of the lift f up to an isomorphism since

$$\mathbb{I}^f_t(r)/\mathbb{I}^f_{<t}(r) = \mathbb{I}^{f+c}_{t+c}(r)/\mathbb{I}^{f+c}_{<(t+c)}(r),$$

- is a free $\kappa[\Gamma]$-module canonically isomorphic to $V \times_\kappa \kappa[\Gamma]$ where V is the finite dimensional vector space

$$V = \bigoplus_{o\in\mathcal{O}(f)/\Gamma} \mathbb{I}^f_{a^o}(r)/\mathbb{I}^f_{<a^o}(r)$$

  for any choice $a^o \in o \in \mathcal{O}_r(f)$. Different choices of a^o lead to isomorphic vector spaces V.

One defines

$$\beta^N_{\mathrm{top},r}(X;\omega) := \dim_\kappa V = \operatorname{rank}(\mathbb{NH}_r(f))$$

which will be shown in Sect. 5.2 to be the same as

$$\beta^N_{\mathrm{alg},r}(X;\xi(\omega)) := \sup\{\operatorname{rank} L\colon L \text{ a free submodule of } H_r(\widetilde{X})\}.$$

One can provide a similar definition using $\mathbb{I}^t_f$ instead of $\mathbb{I}^f_t$. Clearly $\mathbb{I}^f_t = \mathbb{I}^t_{-f}$ and $\mathbb{I}^f_{<t} = \mathbb{I}^{>t}_{-f}$ with $-f$ being a lift of the TC1-form $-\omega$. From the algebraic perspective the first is based on the group Γ, the second on the group Γ' canonically isomorphic to Γ by the isomorphism $g \to g' = -g$. This leads to the same numbers $\beta^N_{\mathrm{alg},r}$ and $\beta^N_{\mathrm{top},r}$.

Both numbers $\beta^N_{\mathrm{top},r}(X;\omega)$ and $\beta^N_{\mathrm{alg},r}(X;\xi(\omega))$ whose equality is verified in Sect. 5.2 are referred to as the Novikov–Betti numbers.

4 The maps δ^f_r and γ^f_r

In this section $f: X \to \mathbb{R}$ will be a weakly tame map, i.e., the requirements 1.–3. in Definition 2.3 are satisfied.

Recall from the previous section the notations:

$$\begin{aligned}
&X^f_a := f^{-1}((-\infty, a]), \qquad X^f_{<a} := f^{-1}((-\infty, a)),\\
&X^f_\infty = X, \qquad X^a_f := f^{-1}([a, \infty)),\\
&X^{>a}_f := f^{-1}((a, \infty)), \qquad X^{-\infty}_f = X,\\
&\mathbb{I}^f_a(r) := \operatorname{img}(H_r(X^f_a) \to H_r(X)),\\
&\mathbb{I}^f_{<a}(r) = \operatorname{img}(H_r(X^f_{<a}) \to H_r(X)) = \bigcup_{\alpha<a} \mathbb{I}^f_\alpha(r),\\
&\mathbb{I}^a_f(r) := \operatorname{img}(H_r(X^a_f) \to H_r(X)),\\
&\mathbb{I}^{>a}_f(r) = \operatorname{img}(H_r(X^{>a}_f) \to H_r(X)) = \bigcup_{\beta>a} \mathbb{I}^\beta_f(r),\\
&\mathbb{I}^f_{-\infty}(r) = \bigcap_{a\in\mathbb{R}} \mathbb{I}^f_a(r), \qquad \mathbb{I}^\infty_f(r) = \bigcap_{a\in\mathbb{R}} \mathbb{I}^a_f(r).
\end{aligned}$$

For any $a, b \in \mathbb{R}$ define $\mathbb{F}^f_r(a, b) := \mathbb{I}^f_a(r) \cap \mathbb{I}^b_f(r)$, and when $a < b$ define

$$\mathbb{T}^f_r(a, b) := \ker(H_r(X^f_a) \to H_r(X^f_b)).$$

Extend the definitions $\mathbb{T}^f_r$ to $\mathbb{T}^f_r(< a, b), \mathbb{T}^f_r(a, < b), \mathbb{T}^f_r(-\infty, b), \mathbb{T}^f_r(a, \infty)$ as follows:

$$\begin{aligned}
\mathbb{T}_r(a, \infty) &= \varinjlim_{a<b\to\infty} \mathbb{T}_r(a, b),\\
\mathbb{T}_r(< a, b) &= \varinjlim_{a>a'\to a} \mathbb{T}_r(a', b) \quad \text{for } a \leqslant b \leqslant \infty,\\
\mathbb{T}_r(a, < b) &= \varinjlim_{a<b'\to b} \mathbb{T}_r(a, b') \quad \text{for } a < b' < b,\\
\mathbb{T}_r(-\infty, b) &= \varprojlim_{b>a\to-\infty} \mathbb{T}_r(a, b).
\end{aligned}$$

Note that:

- for $a < b < c \leqslant \infty$ the obvious linear map

$$\ker(\mathbb{T}^f_r(a, c) \to \mathbb{T}^f_r(b, c)) \to \mathbb{T}^f_r(a, b)$$

is an isomorphism,

- for a, b the homology exact sequence of the pair (X_b^f, X_a^f) implies the short exact sequence

$$0 \to \operatorname{coker}(H_r(X_a^f) \to H_r(X_b^f)) \to \mathbb{H}_r(X_b^f, X_a^f) \to \mathbb{T}_{r-1}^f(a, b) \to 0, \qquad (4.1)$$

- the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{T}_r(a,b) & \xrightarrow{=} & \mathbb{T}_r(a,b) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{T}_r(a,\infty) & \longrightarrow & H_r(X_a) & \twoheadrightarrow & \mathbb{I}_a(r) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{T}_r(b,\infty) & \longrightarrow & H_r(X_b) & \twoheadrightarrow & \mathbb{I}_b(r) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \operatorname{coker}(\mathbb{T}_r^f(a,\infty) \to \mathbb{T}_r^f(b,\infty)) & \longrightarrow & \operatorname{coker}(H_r(X_a^f) \to H_r(X_b^f)) & \twoheadrightarrow & \mathbb{I}_b(r)/\mathbb{I}_a(r) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

implies the exactness of the sequence

$$\begin{array}{ccccc}
\operatorname{coker}(\mathbb{T}_r^f(a,\infty) \to \mathbb{T}_r^f(b,\infty)) & \longrightarrow & \operatorname{coker}(H_r(X_a^f) \to H_r(X_b^f)) & \longrightarrow & \mathbb{I}_b^f(r)/\mathbb{I}_a^f(r) \\
\uparrow & & & & \downarrow \\
0 & & & & 0
\end{array} \qquad (4.2)$$

- the short exact sequences (4.1) and (4.2) imply the isomorphism

$$\begin{aligned} &H_r(X_a^f, X_{<a}^f) \\ &\quad \simeq \mathbb{I}_a^f(r)/\mathbb{I}_{<a}^f(r) \oplus \operatorname{coker}(\mathbb{T}_r^f(< a, \infty) \to \mathbb{T}_r^f(a, \infty)) \oplus \mathbb{T}_{r-1}^f(< a, a). \end{aligned}$$

4.1 The assignments $\hat{\delta}_r^f$ and δ_r^f

Call a *box* a subset $B \subset \mathbb{R}^2$ of the form $B = (a', a] \times [b, b')$ for $-\infty \leqslant a' < a$, $b < b' \leqslant \infty$, and define

$$\mathbb{F}_r(B) := \frac{\mathbb{I}_a(r) \cap \mathbb{I}^b(r)}{\mathbb{I}_{a'}(r) \cap \mathbb{I}^b(r) + \mathbb{I}_a(r) \cap \mathbb{I}^{b'}(r)}.$$

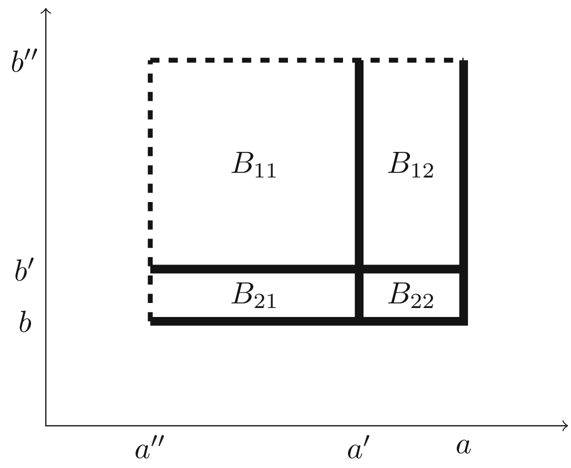


Fig. 1 Box B divided in four disjoint boxes

Let

$$\pi^B_{(a,b)} = \pi^B_{(a,b)}(r) \colon \mathbb{F}_r(a,b) \to \mathbb{F}_r(B)$$

be the canonical projection.[4]

For $B = B' \sqcup B''$ with $B' = (a', a''] \times [b, b')$, $B'' = (a'', a] \times [b, b')$, $-\infty \leqslant a' < a'' < a$, $b < b' < b'' \leqslant \infty$, or with $B' = (a', a] \times [b'', b')$, $B'' = (a', a] \times [b, b'')$, $-\infty \leqslant a' < a$, $b < b'' < b' \leqslant \infty$, the inclusion $B' \subseteq B$ induces the *injective* linear map

$$i^B_{B'} = i^B_{B'}(r) \colon \mathbb{F}_r(B') \to \mathbb{F}_r(B)$$

and the inclusion $B'' \subseteq B$ induces the *surjective* linear map

$$\pi^{B''}_B = \pi^{B''}_B(r) \colon \mathbb{F}_r(B) \to \mathbb{F}_r(B'').$$

For $-\infty \leqslant a'' < a' < a$, $b < b' < b'' \leqslant \infty$ one denotes by

$$\begin{aligned} B_{11} &:= (a'', a'] \times [b', b''), \quad & B_{12} &:= (a', a] \times [b', b''), \\ B_{21} &:= (a'', a'] \times [b, b'), \quad & B_{22} &:= (a', a] \times [b, b'), \end{aligned}$$

and by

$$\begin{aligned} B_{1\cdot} &:= B_{11} \sqcup B_{12}, \quad & B_{\cdot 1} &:= B_{11} \sqcup B_{21}, \\ B_{\cdot 2} &:= B_{12} \sqcup B_{22}, \quad & B_{2\cdot} &:= B_{21} \sqcup B_{22}, \\ B &:= B_{1\cdot} \sqcup B_{2\cdot}, \quad & B &:= B_{\cdot 1} \sqcup B_{\cdot 2}. \end{aligned}$$

In view of the definitions and the notations above one has

[4] When implicit in the context, r will be dropped off the notation.

Proposition 4.1 (i) *The sequence*

$$0 \to \mathbb{F}_r(B') \xrightarrow{i^B_{B'}} \mathbb{F}_r(B) \xrightarrow{\pi^{B''}_B} \mathbb{F}_r(B'') \to 0$$

is exact.

(ii) *For B_{ij} as in Fig. 1, the diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{F}_r(B_{11}) & \xrightarrow{i^{B_{1\cdot}}_{B_{11}}} & \mathbb{F}_r(B_{1\cdot}) & \xrightarrow{\pi^{B_{12}}_{B_{1\cdot}}} & \mathbb{F}_r(B_{12}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i^{B_{\cdot 1}}_{B_{11}}} & & \downarrow{\scriptstyle i^{B}_{B_{1\cdot}}} & & \downarrow & & \\
0 & \longrightarrow & \mathbb{F}_r(B_{\cdot 1}) & \xrightarrow{i^{B}_{B_{\cdot 1}}} & \mathbb{F}_r(B) & \xrightarrow{\pi^{B_{\cdot 2}}_{B}} & \mathbb{F}_r(B_{\cdot 2}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \pi^{B_{2\cdot}}_{B}} & & \downarrow{\scriptstyle \pi^{B_{\cdot 2}}_{B_{22}}} & & \\
0 & \longrightarrow & \mathbb{F}_r(B_{21}) & \xrightarrow{i^{B_{2\cdot}}_{B_{21}}} & \mathbb{F}_r(B_{2\cdot}) & \xrightarrow{\pi^{B_{22}}_{B_{2\cdot}}} & \mathbb{F}_r(B_{22}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

is commutative with all rows and columns exact sequences.

One denotes by $\pi^{B_{22}}_B$ the composition $\pi^{B_{22}}_B := \pi^{B_{22}}_{B_{\cdot 2}} \cdot \pi^{B_{\cdot 2}}_B = \pi^{B_{22}}_{B_{2\cdot}} \cdot \pi^{B_{2\cdot}}_B$, and by $i^B_{B_{11}}$ the composition $i^B_{B_{11}} := i^B_{B_{1\cdot}} \cdot i^{B_{1\cdot}}_{B_{11}} = i^B_{B_{\cdot 1}} \cdot i^{B_{\cdot 1}}_{B_{11}}$, and in general

$$\pi^{B''}_B = \pi^{B''}_B(r)\colon\ \mathbb{F}_r(B) \to \mathbb{F}_r(B'')$$

when the box B'' is located in the lower-right corner of the box B, for example $B'' = B_{22}$, and

$$i^B_{B'} = i^B_{B'}(r)\colon\ \mathbb{F}_r(B') \to \mathbb{F}_r(B)$$

when the box B' is located in the upper-left corner of the box B, for example $B' = B_{11}$. The map $\pi^{B'}_B$ is always surjective and $i^B_{B'}$ is always injective.

For $\epsilon > 0$ one denotes $B(a, b; \epsilon) := (a - \epsilon] \times [b, b + \epsilon)$. Suppose $-\infty \leqslant a' < a$, $b < b' \leqslant \infty$.

- Define

$$\hat{\delta}_r(a, b) := \varinjlim_{\epsilon \to 0} \mathbb{F}_r(B(a, b; \epsilon))$$

w.r. to the surjective linear maps $\mathbb{F}_r(B(a,b;\epsilon)) \xrightarrow{\pi^{B(a,b;\epsilon')}_{B(a,b;\epsilon)}} \mathbb{F}_r(B(a,b;\epsilon')), \epsilon > \epsilon'$, and denote by

$$\pi^{(a,b)}_B = \pi^{(a,b)}_B(r) \colon \mathbb{F}_r(B) \to \hat{\delta}_r(a,b)$$

the composition $\mathbb{F}_r(B) \xrightarrow{\pi^{B(a,b;\epsilon)}_B} \mathbb{F}_r(B(a,b;\epsilon)) \xrightarrow{\pi^{(a,b)}_{B(a,b;\epsilon)}} \hat{\delta}_r(a,b)$ for $\epsilon < \inf\{(a'-a),(b'-b)\}$ and by

$$\pi^{(a,b)}_{(a,b)} = \pi^{(a,b)}_{(a,b)}(r) \colon \mathbb{F}_r(a,b) \to \hat{\delta}_r(a,b)$$

the composition $\mathbb{F}_r(a,b) \xrightarrow{\pi^{B(a,b;\epsilon)}_{(a,b)}} \mathbb{F}_r(B(a,b;\epsilon)) \xrightarrow{\pi^{(a,b)}_{B(a,b;\epsilon)}} \hat{\delta}_r(a,b)$. Both compositions are surjective maps independent of ϵ. One has

$$\hat{\delta}_r(a,b) = \frac{\mathbb{I}_a(r) \cap \mathbb{I}^b(r)}{\mathbb{I}_{<a}(r) \cap \mathbb{I}^b(r) + \mathbb{I}_a(r) \cap \mathbb{I}^{>b}(r)}$$

with $\pi^{a,b}_{a,b}$ the canonical projection.

- Define

$$\mathbb{F}_r((a',a]\times b) := \varinjlim_{\epsilon\to 0} \mathbb{F}_r((a',a]\times[b,b+\epsilon)),$$

w.r. to the surjective maps $\pi^{(a',a]\times[b,b+\epsilon')}_{(a',a]\times[b,b+\epsilon)}(r) \colon \mathbb{F}_r((a',a]\times[b,b+\epsilon)) \to \mathbb{F}_r((a',a]\times[b,b+\epsilon')), \epsilon > \epsilon'$, and denote by

$$\pi^{(a',a]\times b}_B = \pi^{(a',a]\times b}_B(r) \colon \mathbb{F}_r(B) \to \mathbb{F}_r((a',a]\times b)$$

and

$$\pi^{(a,b)}_{(a',a]\times b} = \pi^{(a,b)}_{(a',a]\times b}(r) = \mathbb{F}_r((a',a]\times b) \to \hat{\delta}_r(a,b)$$

the canonical surjective maps induced by passing to limit when $\epsilon \to 0$. One has

$$\mathbb{F}_r((a',a]\times b) = \frac{\mathbb{I}_a(r) \cap \mathbb{I}^b(r)}{\mathbb{I}_{a'}(r) \cap \mathbb{I}^b(r) + \mathbb{I}_a(r) \cap \mathbb{I}^{>b}(r)}.$$

- Define

$$\mathbb{F}_r(a\times[b,b')) := \varinjlim_{\epsilon\to 0} \mathbb{F}_r((a-\epsilon,a]\times[b,b'))$$

w.r. to the surjective maps $\pi^{(a-\epsilon',a]\times[b,b')}_{(a-\epsilon,a]\times[b,b')}(r)\colon \mathbb{F}_r((a-\epsilon,a]\times[b,b'))\to\mathbb{F}_r((a-\epsilon',a]\times[b,b'))$, $\epsilon>\epsilon'$, and denote by

$$\pi^{a\times[b,b')}_B=\pi^{a\times[b,b')}_B(r)\colon \mathbb{F}_r(B)\to\mathbb{F}_r(a\times[b,b'))$$

and

$$\pi^{a,b}_{a\times[b,b')}=\pi^{a,b}_{a\times[b,b')}(r)\colon \mathbb{F}_r(a\times[b,b'))\to\hat{\delta}_r(a,b)$$

the canonical surjective maps induced by passing to limit when $\epsilon\to 0$. One has

$$\mathbb{F}_r(a\times[b,b'))=\frac{\mathbb{I}_a(r)\cap\mathbb{I}^b(r)}{\mathbb{I}_{<a}(r)\cap\mathbb{I}^b(r)+\mathbb{I}_a(r)\cap\mathbb{I}^{b'}(r)}.$$

- Define
$$\mathbb{F}_r(\mathbb{R}\times b):=\mathbb{I}^b(r)/\mathbb{I}^{>b}(r)=\bigcup_{a\in\mathbb{R}}\mathbb{F}_r((-\infty,a]\times b) \tag{4.3}$$
and
$$\mathbb{F}_r(a\times\mathbb{R}):=\mathbb{I}_a(r)/\mathbb{I}_{<a}(r)=\bigcup_{b\in\mathbb{R}}\mathbb{F}_r(a\times[b,\infty)) \tag{4.4}$$
with $\mathbb{F}_r((-\infty,a']\times b)\xrightarrow{\subseteq}\mathbb{F}_r((-\infty,a]\times b)$, $a'<a$, and $\mathbb{F}_r(a\times[b',\infty))\xrightarrow{\subseteq}\mathbb{F}_r(a\times[b,\infty))$, $b'>b$, induced by the linear injective maps i'^B_B in Proposition 4.1 (i).

Note that:

- $\hat{\delta}_r(a,b)\neq 0$ implies $a,b\in \mathrm{CR}_r(f)$,
- $\mathbb{F}_r(a\times\cdot)\neq 0$ implies $a\in\mathrm{CR}_r(f)$,
- $\mathbb{F}_r(\cdot\times b)\neq 0$ implies $b\in\mathrm{CR}_r(f)$,
- $\varinjlim_{\epsilon\to 0}\mathbb{F}_r((a-\epsilon,a]\times b)=\hat{\delta}_r(a,b)$,
- $\varinjlim_{\epsilon\to 0}\mathbb{F}_r(a\times[b,b+\epsilon))=\hat{\delta}_r(a,b)$,
- $\varinjlim_{\epsilon,\epsilon'\to 0}\mathbb{F}_r((a-\epsilon,a]\times[b,b+\epsilon'))=\hat{\delta}_r(a,b)$.

The above definitions combined with Proposition 4.1 lead to the following proposition.

Proposition 4.2 (i) *For $-\infty\leqslant a'<a''<a$, $b\in\mathbb{R}$ the sequence*

$$0\to\mathbb{F}_r((a',a'']\times b)\xrightarrow{i}\mathbb{F}_r((a',a]\times b)\xrightarrow{\pi}\mathbb{F}_r((a'',a]\times b)\to 0$$

is exact, and for $a\in\mathbb{R}$, $b<b''<b'\leqslant\infty$ the sequence

$$0\to\mathbb{F}_r(a\times[b'',b'))\xrightarrow{i}\mathbb{F}_r(a\times[b,b'))\xrightarrow{\pi}\mathbb{F}_r(a\times[b,b''))\to 0$$

is exact. In both sequences i and π are the linear maps induced by the injective linear maps $i^B_{B'}$ and the surjective linear maps $\pi^{B''}_B$.

(ii) (a) *For any* $a \in \mathbb{R}$ *and* $b < b' \leqslant \infty$*,*

$$\dim \mathbb{F}_r(a \times [b, b')) \leqslant \dim \left(\mathbb{I}_a(r) \cap \mathbb{I}^b(r) / \mathbb{I}_{<a}(r) \cap \mathbb{I}^b(r)\right) \leqslant \dim (\mathbb{I}_a(r)/\mathbb{I}_{<a}(r)) \leqslant \dim H_r(X_a, X_{<a}),$$

and when a *is a regular value* $\dim \mathbb{F}_r(a \times [b, b')) = \dim \mathbb{F}_r(a \times \mathbb{R}) = 0$.

(b) *For any* $b \in \mathbb{R}$ *and* $-\infty \leqslant a' < a$*,*

$$\mathbb{F}_r((a', a] \times b) \leqslant \dim \left(\mathbb{I}_a(r) \cap \mathbb{I}^b(r) / \mathbb{I}_a(r) \cap \mathbb{I}^{>b}(r)\right) \leqslant \dim (\mathbb{I}^b(r)/\mathbb{I}^{>b}(r)) \leqslant \dim H_r(X^b, X^{>b}),$$

and when b *is a regular value* $\dim \mathbb{F}_r((a, a] \times b) = \dim \mathbb{F}_r(\mathbb{R} \times b) = 0$.

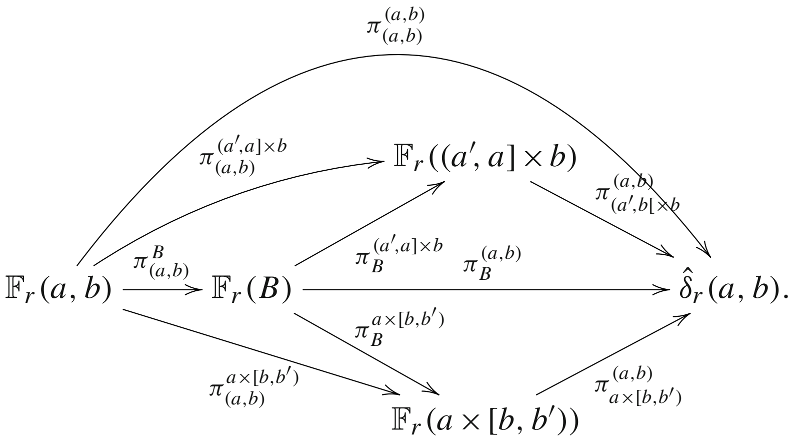
The inequalities above hold (in view of (4.3) and (4.4)) for $[b, b')$*, respectively* $(a', a]$*, replaced by* $[b, \infty)$ *or* $\mathbb{R}$*, respectively* $(-\infty, a]$ *or* $\mathbb{R}$*.*

(iii) (a) *If either* a *or* b *are regular values then* $\hat{\delta}_r(a, b) = 0$.

(b) *For any* $a \in \mathbb{R}$ *the set* $\operatorname{supp} \delta^f_r \cap (a \times \mathbb{R})$ *is finite and if* a *regular value is empty.*

(c) *For any* $b \in \mathbb{R}$ *the set* $\operatorname{supp} \delta^f_r \cap (\mathbb{R} \times b)$ *is finite and if* b *regular value is empty.*

The relation between the surjective linear maps $\pi^{\cdots}_{\cdots}$ is summarized by the following commutative diagram with $B = (a', a] \times [b, b')$:



Definition 4.3 For $(a, b) \in \mathrm{CR}_r(f) \times \mathrm{CR}_r(f)$ a splitting

$$i_{(a,b)}(r) \colon \hat{\delta}_r(a, b) \to \mathbb{F}_r(a, b) \subset H_r(X)$$

is a right inverse of the canonical projection $\pi^{(a,b)}_{(a,b)}(r) \colon \mathbb{F}_r(a, b) \to \hat{\delta}^f_r(a, b)$, i.e., $\pi^{(a,b)}_{(a,b)}(r) \cdot i_{(a,b)}(r) = \mathrm{id}$.

For $-\infty \leqslant a' < a, \ b < b' \leqslant \infty$, let K be either one of the following sets:

1. a bounded or unbounded box $B = (a', a] \times [b, b')$,

2. a bounded or unbounded horizontal open-closed interval $I = (a', a] \times b$,
3. a bounded or unbounded closed-open vertical interval $J = a \times [b, b')$,
4. $a \times \mathbb{R}$,
5. $\mathbb{R} \times b$,
6. $\mathbb{R}^2$.

Call $(a, b) \in \mathbb{R}^2$ the *relevant corner* in Case 1 and the *relevant end* in Case 2 or 3. The interval $(a', a''] \times b$, when viewed as a subinterval of $(a', a] \times b$, is called the *left* (*open-closed*) *subinterval* and $a \times [b'', b')$, when viewed as a subinterval of $a \times [b, b')$, is called the *upper* (*closed-open*) *subinterval*.

For $(a, b) \in K$ a splitting $i_{(a,b)}(r)$ provides the injective linear map

$$i^K_{(a,b)}(r)\colon\ \hat{\delta}^f_r(a, b) \to \mathbb{F}_r(K)$$

defined as follows:

- If (a, b) is the relevant corner or the relevant end then $i^K_{(a,b)}(r)$ is the composition

$$i^K_{(a,b)} := \pi^K_{(a,b)}(r) \cdot i_{(a,b)}(r)$$

  with $\pi^K_{(a,b)}(r)\colon\ \mathbb{F}_r(a, b) \to \mathbb{F}_r(K)$ the canonical projection.
- If (a, b) is not the relevant corner or end then:

  (a) In Cases 1–3, one defines

$$i^K_{(a,b)}(r) := i^K_{K'}(r) \cdot i^{K'}_{a,b}(r)$$

  with $K' \subset K$, the only upper-left box, respectively left subinterval, respectively upper subinterval, having (a, b) as the relevant corner, respectively end.

  (b) In Cases 4–6, one defines $i^K_{(a,b)}(r)$ as the direct limit of $i^{K'}_{(a,b)}(r)$ where K' runs among the subsets of K of the same type and located as upper left box, respectively left interval, respectively upper interval, which make $i^K_{K'}(r)$ injective.

Choose a collection of splittings $\mathcal{S} := \{i_{(a,b)}(r)\colon (a, b) \in \mathrm{CR}(f) \times \mathrm{CR}(f),\ r \in \mathbb{Z}_{\geqslant 0}\}$. Let K be a set as in Cases 1–4, 6 above and $A \subseteq \mathrm{CR}(f) \times \mathrm{CR}(f)$. Denote

$$^{\mathcal{S}}I_A(r) := \bigoplus_{\alpha,\beta \in A} i_{(\alpha,\beta)}(r)\colon \quad \bigoplus_{(\alpha,\beta)\in A} \hat{\delta}_r(\alpha, \beta) \to \mathbb{H}_r(X),$$

$$^{\mathcal{S}}I^K_{A\cap K}(r) := \bigoplus_{(\alpha,\beta)\in A\cap K} i^K_{(\alpha,\beta)}(r)\colon \quad \bigoplus_{(\alpha,\beta)\in A} \hat{\delta}_r(\alpha, \beta) \to \mathbb{F}_r(K).$$

Proposition 4.4 *For any choice of* $\mathcal{S}$ *the following hold:*

(i) *The maps* $^{\mathcal{S}}I(r)$ *and* $^{\mathcal{S}}I^K_{A\cap K}(r)$ *are injective.*

(ii) *If* $A = \mathrm{CR}(f) \times \mathrm{CR}(f)$ *and* K *is in either Case* 2, 3, 4, *or* 5, *then* $^{\mathcal{S}}I^K_K(r) := {}^{\mathcal{S}}I^K_{A\cap K}(r)$ *is an isomorphism.*

Proof (i) It suffices to verify the statement for A a finite set. This is done by induction on cardinality of A as follows. When $\# A = 1$ this follows from the fact that $i_r(\alpha, \beta)$ is a splitting. When $\#(K \cap A) \geqslant 2$ one can write $K = K_1 \sqcup K_2$ with $\#(K_i \cap A) < \#(K \cap A)$. In view of the definition of ${}^{\mathbb{S}}I^{\cdots}_{\cdots}$ the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{F}_r(K_1) & \xrightarrow{i^K_{K_1}} & \mathbb{F}_r(K) & \xrightarrow{\pi^{K_2}_K} & \mathbb{F}_r(K_2) & \longrightarrow & 0 \\
 & & \uparrow {\scriptstyle I_{A\cap K_1}(r)} & & \uparrow {\scriptstyle I_{A\cap K}(r)} & & \uparrow {\scriptstyle I_{A\cap K_2}(r)} & & \\
0 & \longrightarrow & \bigoplus_{\alpha.\beta \in A\cap K_1} \hat{\delta}(\alpha, \beta) & \longrightarrow & \bigoplus_{\alpha.\beta \in A\cap K} \hat{\delta}(\alpha, \beta) & \longrightarrow & \bigoplus_{\alpha.\beta \in A\cap K_2} \hat{\delta}(\alpha, \beta) & \longrightarrow & 0.
\end{array}$$

Since both rows are short exact sequences with the left and right vertical arrows injective, the mid vertical arrow is injective too.

(ii) In view of the weakly tame property of f, if $a \notin \mathrm{CR}(f)$ then $\mathbb{F}_r(a \times \mathbb{R}) = 0$. If $a \in \mathrm{CR}(f)$ then $\mathbb{F}_r(a \times \mathbb{R}) = \mathbb{I}_a(r)/\mathbb{I}_{<a}(r)$ in view of (4.4) and by Proposition 4.2 (ii) is of finite dimension. Denote by $\mathbb{S}(a, b) = \mathbb{I}_a(r) \cap \mathbb{I}^b(r)/\mathbb{I}_{<a}(r) \cap \mathbb{I}^b(r)$ and observe that

$$\mathbb{F}_r(a \times \mathbb{R}) = \mathbb{S}(a, -\infty) \supseteq \mathbb{S}(a, b) \supseteq \mathbb{S}(a, b') \supseteq \mathbb{S}(a, \infty) = 0$$

for $b < b'$. The finite dimensionality of $\mathbb{F}(a \times \mathbb{R})$ implies the existence of a finite collection of critical values $b_1 < b_2 < \cdots < b_N$ such that $\mathbb{S}(a, -\infty) = \mathbb{S}(a, b_1) \supset \mathbb{S}(a, b_1) \supset \cdots \supset \mathbb{S}(a, b_{N-1}) \supset \mathbb{S}(a, b_N) \supset 0$ and in view of the definition of $\mathbb{S}(a, b)$, $\mathbb{S}(a, b') = \mathbb{S}(a, b_i)$ if $b' \in (b_{i-1}, b_i]$. This implies that $\hat{\delta}_r(a, b_i) = \mathbb{S}(a, b_i)/\mathbb{S}(a, b_{i+1})$ and therefore

$$\dim \mathbb{F}_r(a \times \mathbb{R}) = \sum_{i=1}^{N} \dim \hat{\delta}_r(a, b_i) \leqslant \sum_{t \in \mathbb{R}} \dim \hat{\delta}_r(a, t).$$

By (i) one has $\sum_{t\in\mathbb{R}} \dim \hat{\delta}_r(a, t) \leqslant \dim \mathbb{F}_r(a \times \mathbb{R})$. These two inequalities and (i) imply ${}^{\mathbb{S}}I^K_{A\cap K}(r)$ is an isomorphism and

$$\mathrm{supp}\ \hat{\delta}_r \cap a \times \mathbb{R} = \{(a, b_1), (a, b_2), \ldots, (a, b_N)\}.$$

If $b \notin \mathrm{CR}(f)$, then $\mathbb{F}(\mathbb{R} \times b) = 0$. If $b \in \mathrm{CR}(f)$ then by (4.3), $\mathbb{F}_r(\mathbb{R} \times b) = \mathbb{I}^b(r)/\mathbb{I}^{>b}(r)$ and by Proposition 4.2 (ii) is of finite dimension. Denote by $\mathbb{U}(a, b) = \mathbb{I}_a(r) \cap \mathbb{I}^b(r)/\mathbb{I}_a(r) \cap \mathbb{I}^{>b}(r)$ and observe that

$$\mathbb{F}_r(\mathbb{R} \times b) = \mathbb{U}(\infty, b) \supseteq \mathbb{U}(a', b) \supseteq \mathbb{U}(a, b) \supseteq \mathbb{S}(-\infty, b) = 0$$

for $a' > a$. The finite dimensionality of $\mathbb{F}_r(\mathbb{R} \times b)$ implies the existence of a finite collection of critical values $a_1 > a_2 > \cdots > a_N$ such that $\mathbb{U}(\infty, b) = \mathbb{U}(a_1, b) \supset$

$\mathbb{U}(a_2, b) \supset \cdots \supset \mathbb{U}(a_{N-1}, b) \supset \mathbb{U}(a_N, b) \supset 0$ and in view of the definition of $\mathbb{U}(a, b)$, $\mathbb{U}(a', b) = \mathbb{U}(a_i, b)$ if $a' \in [a_{i+1}, a_i)$. Therefore

$$\dim \mathbb{F}_r(\mathbb{R} \times b) = \sum_{i=1}^{N} \dim \hat{\delta}_r(a_i, b) \leqslant \sum_{t \in \mathbb{R}} \dim \hat{\delta}_r(t, b).$$

By (i) one has $\sum_{t \in \mathbb{R}} \dim \hat{\delta}_r(t, b) \leqslant \dim \mathbb{F}_r(\mathbb{R} \times b)$. These two inequalities and (i) imply ${}^{\mathcal{S}}I^{K}_{A \cap K}(r)$ is an isomorphism and

$$\operatorname{supp} \hat{\delta}_r \cap a \times \mathbb{R} = \{(a_1, b), (a_2, b), \ldots, (a_N, b)\}. \qquad \square$$

Define

$$\delta^f_r(a, b) := \dim \hat{\delta}^f_r(a, b).$$

As a consequence of Proposition 4.4 for any $a \in \mathbb{R}$ the set $\operatorname{supp} \delta^f_r \cap (a \times \mathbb{R})$ is finite and of total cardinality

$$\dim \mathbb{I}_a(r)/\mathbb{I}_{<a}(r) = \sum_{(a,x) \in \operatorname{supp} \delta^f_r \cap (a \times \mathbb{R})} \delta^f_r(a, x) \tag{4.5}$$

hence equal to zero when a is a regular value. Similarly, for any $b \in \mathbb{R}$ the set $\operatorname{supp} \delta^f_r \cap (\mathbb{R} \times b)$ is finite of total cardinality

$$\dim \mathbb{I}^b(r)/\mathbb{I}^{>b}(r) = \sum_{(x,b) \in \operatorname{supp} \delta^f_r \cap (\mathbb{R} \times b)} \delta^f_r(x, b)$$

hence equal to zero when b is a regular value.

4.2 The assignments $\hat{\gamma}^f_r$ and γ^f_r

Call a *box above diagonal*, abbreviated *ad-box*, a subset $B \subset \mathbb{R}^2_+ = \{(x, y) : x < y\}$ of the form $B = (a', a] \times (b', b]$, with $a' < a \leqslant b' < b$. For $a' < a \leqslant b' < b$ the inclusions $X_{a'} \subset X_a \subseteq X_{b'} \subset X_b$ induce in homology the following commutative cartesian diagram:

$$\begin{array}{ccc}
\mathbb{T}_r(a', b) & \xrightarrow{\;i^a_{a'}(r)\;} & \mathbb{T}_r(a, b) \\
\uparrow \subseteq & \overset{u(r)}{\nearrow} & \uparrow \subseteq \\
\mathbb{T}_r(a', b') & \longrightarrow & \mathbb{T}_r(a, b')
\end{array}$$

with the property that $\operatorname{img}(u(r)) = \operatorname{img}(i^a_{a'}(r)) \cap \mathbb{T}_r(a, b')$. Here $i^a_{a'}(r) \colon \mathbb{T}_r(a', b) \to \mathbb{T}_r(a, b)$ is the restriction of $i^a_{a'}(r) \colon H_r(X_{a'}) \to H_r(X_b)$. In order to avoid heavy notation, when implicit from the context we will simply write $i(r)$ instead of $i^a_{a'}(r)$.

Define

$$\mathbb{T}_r(B) := \mathbb{T}_r(a,b)/i^a_{a'}(r)(\mathbb{T}_r(a',b)) + \mathbb{T}_r(a,b').$$

For $-\infty \leqslant a'' \leqslant a' < a$, $b'' \leqslant b' < b$, one considers the ad-boxes

$$\begin{aligned} &B_{11} := (a'',a'] \times (b'',b'], && B_{12} := (a'',a'] \times (b',b],\\ &B_{21} := (a',a] \times (b'',b'], && B_{22} := (a',a] \times (b',b],\\ &B_{1\cdot} := B_{11} \sqcup B_{12}, && B_{\cdot 1} := B_{11} \sqcup B_{21},\\ &B_{2\cdot} := B_{12} \sqcup B_{22}, && B_{\cdot 2} := B_{11} \sqcup B_{12},\\ &B_{2\cdot} := B_{21} \sqcup B_{22}, && B := B_{1\cdot} \sqcup B_{2\cdot} = B_{\cdot 1} \sqcup B_{\cdot 2}. \end{aligned}$$

Suppose that B_1, B_2, B are three ad-boxes with $B_1 \sqcup B_2 = B$ in either one of the two relative positions:

- B_1 the left side ad-box and B_2 the right side ad-box, for example $B = B_{\cdot 2} = B_{12} \sqcup B_{22}$, $B_1 = B_{12}$, $B_2 = B_{22}$ with B_{ij} as in Fig. 2,
- B_1 the down side ad-box and B_2 the upper side ad-box, for example $B = B_{1\cdot} = B_{11} \sqcup B_{12}$, $B_1 = B_{11}$, $B_2 = B_{12}$ with B_{ij} as in Fig. 2.

The inclusion $B_1 \subseteq B$ induces the *injective* linear map $i^B_{B_1}(r)\colon \mathbb{T}_r(B_1) \to \mathbb{T}_r(B)$ and the inclusion $B_2 \subset B$ the *surjective* linear map $\pi^{B_2}_B(r)\colon \mathbb{T}_r(B) \to \mathbb{T}_r(B_2)$. One still calls an *ad-box* the set $(-\infty, a] \times (b', b]$, $a \leqslant b'$, and defines

$$\mathbb{T}_r((-\infty,a] \times (b',b]) := \mathbb{T}_r(a,b)/\big(\bigcap_{\{a' : a' < a\}} i^a_{a'}(r)(\mathbb{T}_r(a',b)) + \mathbb{T}_r(a,b')\big).$$

By elementary but tedious arguments, for details see Appendix, one can show.

Proposition 4.5 (i) *If B_1, B_2, B are ad-boxes such that $B = B_1 \sqcup B_2$ then the sequence*

$$0 \to \mathbb{T}_r(B_1) \xrightarrow{i^B_{B_1}(r)} \mathbb{T}_r(B) \xrightarrow{\pi^{B_2}_B(r)} \mathbb{T}_r(B_2) \to 0$$

is exact.

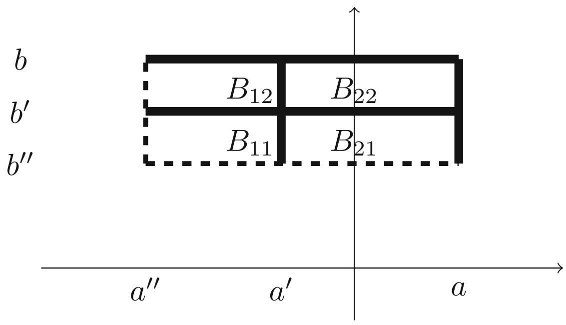


Fig. 2 Ad-box B divided in four disjoint ad-boxes

(ii) *If $B_{11}, B_{12}, B_{21}, B_{22}$ are ad-boxes as in Fig. 2 then the diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{T}_r(B_{11}) & \xrightarrow{i_{B_{11}}^{B_{1\cdot}}} & \mathbb{T}_r(B_{\cdot 1}) & \xrightarrow{\pi_{B_{1\cdot}}^{B_{12}}} & \mathbb{T}_r(B_{21}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{T}_r(B_{1\cdot}) & \xrightarrow{i_{B_{1\cdot}}^{B}} & \mathbb{T}_r(B) & \xrightarrow{\pi_{B}^{B_{\cdot 2}}} & \mathbb{T}_r(B_{2\cdot}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{T}_r(B_{12}) & \xrightarrow{i_{B_{21}}^{B_{2\cdot}}} & \mathbb{T}_r(B_{\cdot 2}) & \xrightarrow{\pi_{B_{2\cdot}}^{B_{22}}} & \mathbb{T}_r(B_{22}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

is commutative with all rows and columns exact sequences.

As a consequence, for any $-\infty \leqslant a' < a \leqslant b' < b$ and $\epsilon > \epsilon'$ the inclusion induced linear maps

$$\pi_{(a',a]\times(b-\epsilon,\ b]}^{(a',a]\times(b-\epsilon',b]}(r)\colon\ \mathbb{T}_r((a',a]\times(b-\epsilon\ ,b]) \to \mathbb{T}_r((a',a]\times(b-\epsilon',b]),$$
$$\pi_{(a-\epsilon,a]\times(b',b]}^{(a-\epsilon',a]\times(b',b]}(r)\colon\ \mathbb{T}_r((a-\epsilon,a]\times(b',b]) \to \mathbb{T}_r((a-\epsilon',a]\times(b',b])$$

and

$$\pi_{B(a,b;\epsilon)}^{B(a,b;\epsilon')}(r)\colon\ \mathbb{T}_r(B(a,b;\epsilon)) \to \mathbb{T}_r(B(a,b;\epsilon')),$$

where $B(a,b;\epsilon) := (a-\epsilon]\times(b-\epsilon,b]$, are surjective.

Define

- for $a < b$,

$$\hat{\gamma}_r^f(a,b) := \varinjlim_{\epsilon\to 0} T_r(B(a,b;\epsilon)),$$

  with respect to the maps $\pi_{B(a,b;\epsilon)}^{B(a,b;\epsilon')}(r)$,
- for $-\infty \leqslant a' < a < b$,

$$\mathbb{T}_r((a',a]\times b) := \varinjlim_{\epsilon\to 0} \mathbb{T}_r((a',a]\times(b-\epsilon,b])$$

  with respect to the maps $\pi_{(a',a]\times(b-\epsilon,b]}^{(a',a]\times(b-\epsilon',b]}(r)$, and
- for $a \leqslant b' < b$,

$$\mathbb{T}_r(a\times(b',b]) := \varinjlim_{\epsilon\to 0} \mathbb{T}_r((a-\epsilon,a]\times(b',b])$$

with respect to the maps $\pi^{(a-\epsilon',a]\times(b',b]}_{(a-\epsilon,a]\times(b',b]}(r)$.

These maps induce for $-\infty \leqslant a' < a'' < a \leqslant b' < b'' < b \leqslant \infty$ the following exact sequences:

$$0 \to \mathbb{T}_r((a', a'']\times b) \xrightarrow{\iota^I_{I_1}} \mathbb{T}_r((a', a]\times b) \xrightarrow{\pi^{I_1}_I} \mathbb{T}_r((a'', a]\times b) \to 0,$$
$$0 \to \mathbb{T}_r(a\times(b', b'']) \xrightarrow{\iota^I_{I_1}} \mathbb{T}_r(a\times(b', b]) \xrightarrow{\pi^{I_2}_I} \mathbb{T}_r(a\times(b'', b]) \to 0.$$

For $-\infty \leqslant a < b < \infty$ define

- $$\mathbb{T}_r((a,b)\times b) := \varinjlim_{b-a>\epsilon\to 0} \mathbb{T}_r((a, b-\epsilon]\times b)$$

  with respect to the injective maps induced by inclusions $(a, b-\epsilon]\times b$ into $(a, b-\epsilon']\times b$ for $\epsilon > \epsilon'$,
- $$\mathbb{T}_r(a\times(a,b]) := \varinjlim_{b-a>\epsilon\to 0} T_r(a\times(a+\epsilon, b])$$

  with respect to the surjective maps induced by inclusions $a\times(a+\epsilon, b]$ into $a\times(a+\epsilon', b]$ for $\epsilon > \epsilon'$ and $a+\epsilon < b$,
- $$\mathbb{T}_r(a\times(a,\infty)) := \varinjlim_{a<b\to\infty} \mathbb{T}_r(a\times(a,b])$$

  with respect to the maps induced by inclusions $a\times(a,b_1]$ into $a\times(a,b_2]$ for $a < b_1 < b_2 < b$.

The reader can also check that

$$\mathbb{T}_r((a,b)\times b) = \varinjlim_{\epsilon\to 0} \mathbb{T}_r((a, b-\epsilon]\times(b-\epsilon, b])$$

with respect to the linear maps $t^{B_{\epsilon'}}_{B_\epsilon} = \iota^{B_{\epsilon'}}_{(a,b-\epsilon]\times(b-\epsilon',b]} \cdot \pi^{(a,b-\epsilon]\times(b-\epsilon',b]}_{B_\epsilon}$ where $B_\epsilon = (a', b-\epsilon]\times(b-\epsilon, b]$ and $\epsilon' < \epsilon$.

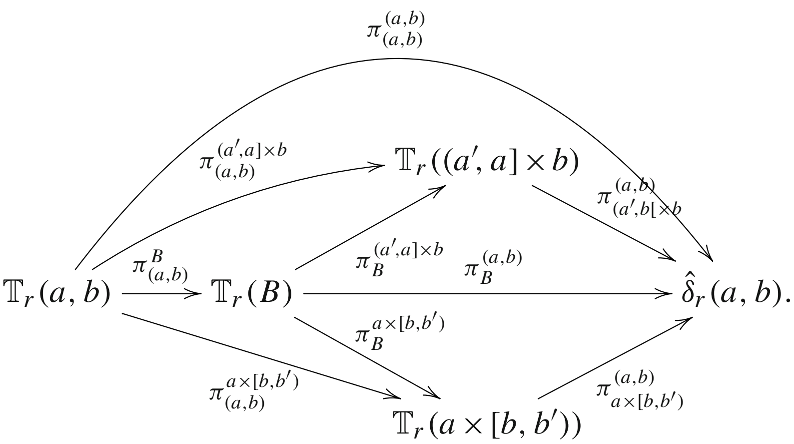
One can verify that

$$\begin{aligned}
\hat{\gamma}_r(a,b) &= \mathbb{T}_r(a,b)/i(r)(\mathbb{T}_r(<a,b)) + \mathbb{T}_r(a,<b),\\
\mathbb{T}_r(a\times(b',b]) &= \mathbb{T}_r(a,b)/i(r)(\mathbb{T}_r(<a,b)) + \mathbb{T}_r(a,b'),\\
\mathbb{T}_r((a',a]\times b) &= \mathbb{T}_r(a,b)/i(r)(\mathbb{T}_r(a',b)) + \mathbb{T}_r(a,<b),\\
\mathbb{T}_r(a\times(a,b]) &= \mathbb{T}_r(a,b)/i(r)(\mathbb{T}_r(<a,b)),\\
\mathbb{T}_r(a\times(a,\infty)) &= \mathbb{T}_r(a,\infty))/i(r)(\mathbb{T}_r(<a,\infty)),\\
\mathbb{T}_r(a',b)\times b) &= \mathbb{T}_r(<b,b)/i(r)(\mathbb{T}_r(a',b)),\\
\mathbb{T}_r(-\infty,b)\times b) &= \mathbb{T}_r(<b,b)/\textstyle\bigcap_{\{a':a'<b\}} i(r)(\mathbb{T}_r(a',b)),
\end{aligned} \tag{4.6}$$

and that

$$\begin{aligned}
\varinjlim_{\epsilon\to 0} \mathbb{T}_r((a-\epsilon,a]\times b) &= \hat{\gamma}_r(a,b),\\
\varinjlim_{\epsilon\to 0} \mathbb{T}_r(a\times(b-\epsilon,b]) &= \hat{\gamma}_r(a,b),\\
\varinjlim_{\epsilon,\epsilon'\to 0} \mathbb{T}_r((a-\epsilon]\times(b-\epsilon',b]) &= \hat{\gamma}_r(a,b).
\end{aligned}$$

The relations between the canonical projections $\pi^{\cdots}_{\cdots}$, implicit in (4.6), can be summarized by the following diagram:



In view of the above definitions Proposition 4.5 leads to the following

Proposition 4.6 (i) (a) *For $a<b'<b$ one has*

$$\dim \mathbb{T}_r(a\times(b',b]) \leqslant \dim(\operatorname{coker}(H_r(X_{<a})\to H_r(X_a))) \leqslant \dim H_r(X_a,X_{<a}).$$

(b) *For $a'<a<b$ one has*

$$\begin{aligned}
\dim \mathbb{T}_r((a',a]\times b) &\leqslant \dim(\operatorname{coker}(H_{r+1}(X_{<b},X_a)\to H_{r+1}(X_b,X_a)))\\
&\leqslant \dim H_{r+1}(X_b,X_{<b}).
\end{aligned}$$

The same inequalities hold for $(b',b]$ replaced by (a,∞) and $(a',a]$ replaced by $(-\infty,b)$.

(ii) (a) *If either a or b are regular values then* $\hat{\gamma}_r(a,b) = 0$.
(b) *For any a,* $\operatorname{supp} \gamma_r^f \cap (a \times (a, \infty))$ *is a finite set and when a regular value is empty.*
(c) *For any b,* $\operatorname{supp} \gamma_r^f \cap ((-\infty, b) \times b)$ *is a finite set and when b regular value is empty.*

Proof (i) To check part (a) observe that from the commutative diagram (4.7) with all rows and columns exact

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 & & \mathbb{T}_r(a,b)/i(\mathbb{T}_r(<a,b)) & \longrightarrow & H_r(X_a)/i(H_r(X_{<a})) & & \\
 & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathbb{T}_r(a,b) & \longrightarrow & H_r(X_a) & \longrightarrow & H_r(X_b) \\
 & & \uparrow & & \uparrow i & & \uparrow = \\
0 & \longrightarrow & \mathbb{T}_r(<a,b) & \longrightarrow & H_r(X_{<a}) & \longrightarrow & H_r(X_b)
\end{array} \tag{4.7}$$

one derives the injectivity of $\mathbb{T}_r(a,b)/i(\mathbb{T}_r(<a,b)) \to H_r(X_a)/iH_r(X_{<}a)$. In the sequence

$$\begin{array}{ccc}
\mathbb{T}_r(a,b)/(i(r)(\mathbb{T}_r(<a,b)) + \mathbb{T}_r(a,b')) & \longleftarrow & \mathbb{T}_r(a,b)/i(r)(\mathbb{T}_r(<a,b)) \\
 & \swarrow & \\
H_r(X_a)/iH_r(X_{<}a) & \longrightarrow & H_r(X_a, X_{<a})
\end{array}$$

the horizontal right to left arrow is surjective, and both the horizontal left to right arrow and the oblique right to left arrow are injective. Then, in view of the finite dimensionality of $H_r(X_a, X_{<a})$, the statement follows.

To check part (b) one considers the commutative diagram (4.8) with all rows and columns exact

$$\begin{array}{ccccc}
0 & & 0 & & \\
\uparrow & & \uparrow & & \\
H_{r+1}(X_b, X_a)/iH_{r+1}(X_{<b}, X_a) & \longrightarrow & \mathbb{T}_r(a,b)/i(r)(\mathbb{T}_r(a,<b)) & & \\
\uparrow & & \uparrow & & \\
H_{r+1}(X_b, X_a) & \longrightarrow & T_r(a,b) & \longrightarrow & 0 \\
\uparrow i & & \uparrow i(r) & & \\
H_{r+1}(X_{<b}, X_a) & \longrightarrow & \mathbb{T}_r(a,<b) & \longrightarrow & 0
\end{array} \tag{4.8}$$

and one derives the surjectivity of

$$H_{r+1}(X_b, X_a)/iH_{r+1}(X_{<b}, X_a) \to \mathbb{T}_r(a, b)/i(r)(\mathbb{T}_r(a, < b)).$$

From the long exact sequence of the triple $(X_a \subset X_{<b} \subset X_b)$ one derives the injectivity of $H_{r+1}(X_b, X_a)/iH_{r+1}(X_{<b}, X_a) \to H_{r+1}(X_b, X_{<b})$. Then, in view of the finite dimensionality of $H_{r+1}(X_b, X_{<b})$, the diagram

$$\begin{array}{ccc} \mathbb{T}_r(a,b)/(i(r)(T_r(a',b)) + \mathbb{T}_r(a,<b)) & \longleftarrow & \mathbb{T}_r(a,b)/i(r)(T_r(a,<b)) \\ & \nearrow & \\ H_{r+1}(X_b, X_a)/iH_{r+1}(X_{<b}, X_a) & \longrightarrow & H_{r+1}(X_b, X_{<b}) \end{array}$$

implies the statement (b). The extension to intervals (a, ∞), $(-\infty, b)$ follows in view of the definitions of $\mathbb{T}_r(a \times (a, \infty))$ and $\mathbb{T}_r((-\infty, b) \times b)$. (ii) follows from (i). □

Suppose that K is one of the following type of sets:

1. a bounded or unbounded ad-box $B = (a', a] \times (b', b]$, $-\infty \leqslant a' < a < b$, $b' < b \leqslant \infty$,
2. a horizontal open-closed interval $I = (a', a] \times b$, $-\infty \leqslant a' < a$,
3. a vertical open-closed interval $J = a \times (b', b]$, $a \leqslant b' < b$,
4. $(-\infty, b) \times b$,
5. $a \times (a, \infty)$,

and observe that in view of Proposition 4.6, when K is of Type 2–4 or 5, the vector space $\mathbb{T}_r(K)$ has finite dimension.

Denote by $(\mathrm{CR}(f) \times \mathrm{CR}(f))_+ := \{(a, b) : a, b \in \mathrm{CR}(f), a < b\}$.

Splittings

1. For any $(a, b) \in (\mathrm{CR}(f) \times \mathrm{CR}(f))_+$ a *splitting*

$$i_{(a,b)}(r) \colon \hat{\gamma}_r(a, b) \to \mathbb{T}_r(a, b)$$

   is a right inverse of the canonical projection $\pi^{(a,b)}_{(a,b)}(r) \colon \mathbb{T}_r(a, b) \to \hat{\gamma}^f_r(a, b)$.
2. For any K in either one of the situations above and $(a, b) \in K$ one defines

$$i^K_{(a,b)}(r) \colon \hat{\gamma}_r(a, b) \to \mathbb{T}_r(K)$$

   first for the case the point (a, b) is the relevant corner or vertex of K then for an arbitrary point of K as in the previous subsection.
3. For a collection of splittings $\mathcal{S} = \{i_{(a,b)}(r) : (a, b) \in (\mathrm{CR}(f) \times \mathrm{CR}(f))_+\}$, set $A \subset (\mathrm{CR}(f) \times \mathrm{CR}(f))_+$, and K one of Types 1–5 above, one defines

$${}^{\mathcal{S}}I^K_{A \cap K}(r) = \bigoplus_{(\alpha,\beta) \in A \cap K} i^K_{(\alpha,\beta)}(r) \colon \bigoplus_{(\alpha,\beta) \in A} \hat{\delta}_r(\alpha, \beta) \to \mathbb{T}_r(K).$$

Proposition 4.6 can be refined to Proposition 4.7.

Proposition 4.7 *For any choice of* $\mathcal{S}$:

(i) *The maps* ${}^{\mathcal{S}}I_A^K(r)$ *are injective.*
(ii) *If* $A = \mathrm{CR}(f) \times \mathrm{CR}(f)_+$ *and* K *is either of Type 2, 3, 4 or 5, then* ${}^{\mathcal{S}}I_{A\cap K}^K$ *is an isomorphism. In particular for* $a' < a \leqslant b' < b$,

$$\mathbb{T}_r(a \times (b', b]) \simeq \bigoplus_{b'<t\leqslant b} \hat{\gamma}_r(a, t), \tag{4.9}$$

$$\mathbb{T}_r((a', a] \times b) \simeq \bigoplus_{a'<t\leqslant a} \hat{\gamma}_r(t, b). \tag{4.10}$$

The isomorphism (4.9) *continues to hold for* $b' = a$ *and/or* $b = \infty$ *and the isomorphism* (4.10) *continues to hold for* $a = b$ *and/or* $a' = -\infty$.

Proof (i) follows by similar arguments as in Proposition 4.4 (i).
(ii) The finite dimensionality of $\mathbb{T}_r(a \times (a, \infty))$, cf. Proposition 4.6 (ii), implies the existence of a finite collection of critical values $a = b_0 < b_1 < b_2 < \cdots < b_N < \infty$ such that the only jumps of the integer-valued function in t, $\dim \mathbb{T}_r(a \times (a, t])$, are $\{b_1, \ldots, b_N\} \subset (a, \infty)$. Then, in view of the definition of this function one has

- $\mathbb{T}_r(a \times (a, b_i]) \subset \mathbb{T}_r(a \times (a, b_{i+1}])$ (strict inclusion) for all $i = 0, 1, \ldots, N$,
- $\mathbb{T}_r(a \times (a, b_i]) = \mathbb{T}_r(a \times (a, t])$ for $b_i \leqslant t < b_{i+1}$.

The second equality implies $\mathbb{T}_r(a \times (a, b_{i+1}])/\mathbb{T}_r(a \times (a, b_i]) = \hat{\gamma}_r(a, b_{i+1})$. (i) shows that there are no other (a, t) in the set $\operatorname{supp} \hat{\gamma}_r \cap (a \times (a, \infty))$ but (a, b_i), $i = 1, 2, \ldots, N$. This establishes (4.9).

Similarly, the finite dimensionality of $\mathbb{T}_r((-\infty, b) \times b)$, which follows from Proposition 4.6 (ii), implies the existence of a finite collection of numbers $-\infty < a_N < \cdots < a_2 < a_1 < a_0 = b$, the only jumps of the integer-valued function $\dim \mathbb{T}_r((t, b) \times b)$. Then in view of this function one has

- $\mathbb{T}_r((a', a_{i+1}] \times b) \subset \mathbb{T}_r((a', a_i] \times b)$ for all i with $a' < a_{i+1}$,
- $\mathbb{T}_r((a', t] \times b) = \mathbb{T}_r((a', a_{i+1}] \times b)$ for $a_{i+1} \leqslant t < a_i$.

The second equality implies $\mathbb{T}_r((a_i, a] \times b)/\mathbb{T}_r((a_{i+1}, a] \times b) = \hat{\gamma}_r(a_i, b)$. This implies (4.10). (i) of this theorem implies that there are no other (t, b) but (a_i, b) with $i = 1, 2, \ldots, N$, in the set $\operatorname{supp} \hat{\gamma}_r \cap ((-\infty, b) \times b)$. □

Corollary 4.8 (i) $\operatorname{coker}(H_r(X_{<a}) \to H_r(X_a)) \simeq \left(\bigoplus_{t\in\mathbb{R}} \hat{\delta}_r(a, t)\right) \oplus \left(\bigoplus_{t\in\mathbb{R}} \hat{\gamma}_r(a, t)\right)$.
(ii) $\ker(H_r(X_{<b}) \to H_r(X_b)) \simeq \mathbb{T}_r((-\infty, b) \times b) \simeq \bigoplus_{t\in(-\infty,a)} \hat{\gamma}_r^f(t, a)$.

Proof (i) In view of (4.2) one has

$$\operatorname{coker}(H_r(X_{<a}) \to H_r(X_a)) \simeq \mathbb{I}_a(r)/\mathbb{I}_{<a}(r) \oplus \operatorname{coker}(\mathbb{T}_r(< a, \infty) \to \mathbb{T}_r(a, \infty)).$$

In view of the fifth equality (4.6) and of the equality (4.9) one has

$$\operatorname{coker}(\mathbb{T}_r(<a,\infty)\to\mathbb{T}_r(a,\infty))\simeq\mathbb{T}_r(a\times(a,\infty))\simeq\bigoplus_{t\in(a,\infty)}\hat{\gamma}_r(a,t).$$

In view of the equality (4.5),

$$\mathbb{I}_a(r)/\mathbb{I}_{<a}(r)\simeq\bigoplus_{t\in\mathbb{R}}\hat{\delta}^f_r(a,t).$$

In view of the exact sequence (4.2),

$$\operatorname{coker}(H_r(X^f_a)\to H_r(X^f_b))\simeq\operatorname{coker}(\mathbb{T}^f_r(a,\infty)\to\mathbb{T}^f_r(b,\infty))\oplus\mathbb{I}^f_b(r)/\mathbb{I}^f_a(r).$$

Combining these three isomorphisms (i) follows.

(ii) In view of the seventh equality (4.6) one has

$$\mathbb{T}_r((-\infty,b)\times b)=\mathbb{T}(<b,b)/i(r)\mathbb{T}_r(-\infty,b).$$

In view of Proposition 4.7,

$$\mathbb{T}_r((-\infty,b)\times b)\simeq\bigoplus_{t\in(-\infty,b)}\hat{\gamma}_r(t,b).$$

Putting together these isomorphisms one obtains (ii). □

Define $\gamma^f_r(a,b):=\dim\hat{\gamma}^f_r(a,b)$. To summarize, the map $\gamma^f_r:\mathbb{R}^2\to\mathbb{Z}_{\geqslant 0}$ satisfies:

- For any $a\in\mathbb{R}$, $\operatorname{supp}\gamma^f_r\cap(a\times\mathbb{R})$ is a finite set of total cardinality

$$\dim\mathbb{T}_r(a\times(a,\infty))=\sum_{t\in(-\infty,a)}\gamma^f_r(a,t),$$

  in particular for a regular value is zero.
- For any $b\in\mathbb{R}$ the set $\operatorname{supp}\gamma^f_r\cap(\mathbb{R}\times b)$ is finite and of total cardinality

$$\dim\mathbb{T}_r((-\infty,b)\times b)=\sum_{x\in(-\infty,b)}\gamma^f_r(x,b),$$

  in particular for b regular value is zero.

As a consequence of Corollary 4.8 one obtains:

$$\dim H_r(X_a,X_{<a})=\sum_{t\in\mathbb{R}}\delta_r(a,t)+\sum_{t\in(a,\infty)}\gamma_r(a,t)+\sum_{t\in(-\infty,a)}\gamma_{r-1}(t,a). \tag{4.11}$$

4.3 The case of TC1-form

Suppose that $f\colon \widetilde{X} \to \mathbb{R}$ is the lift of a tame TC1-form ω. Then f is a Γ-equivariant map, the sets $\mathrm{CR}_r(f)$ are Γ-invariant w.r. to the translations by the elements of Γ, the set $\mathcal{O}_r(f) = \mathrm{CR}_r(f)/\Gamma$ is finite and the maps δ^f_r and γ^f_r are Γ-invariant, i.e.,

$$\delta^f_r(a,b) = \delta^f_r(a+g, b+g), \quad \gamma^f_r(a,b) = \gamma^f_r(a+g, b+g).$$

Theorem 3.2 implies $\underset{a\to -\infty}{\underleftarrow{\lim}}\ \mathbb{T}_r(a,b) = \mathbb{T}_r(-\infty, b) = 0$. Consequently, in view of (4.5) and (4.11), one has

Corollary 4.9 *For f a lift of a tame TC1-form ω and for any choice $a^o \in o \in \mathcal{O}_r(f)$:*

(i)

$$\beta^N_{\mathrm{top},r}(X;\omega) = \sum_{o\in\mathcal{O}(f)} \sum_{t\in\mathbb{R}} \delta^f_r(a^o, t).$$

(ii)

$$\sum_{o\in\mathcal{O}(f)} \dim H_r(\widetilde{X}_{a^o}, \widetilde{X}_{<a^o}) = \sum_{o\in\mathcal{O}(f)} \Bigg(\sum_{t\in\mathbb{R}} \delta_r(a^o, t) + \sum_{t\in\mathbb{R}_+} \gamma_r(a^o, a^o + t) + \sum_{t\in\mathbb{R}_+} \gamma_{r-1}(a^o - t, a^o) \Bigg).$$

In case ω is a Morse closed differential 1-form on a closed manifold M and $f\colon \widetilde{M} \to \mathbb{R}$ is a lift of ω then $\sum_{o\in\mathcal{O}(f)} \dim H_r(\widetilde{X}_{a^o}, \widetilde{X}_{<a^o})$ is exactly the number of zeros of ω of Morse index r. Indeed, let $\widetilde{\mathcal{X}}_r$ be the set of critical points of index r and $\mathcal{X}_r$ be the finite set of zeros of ω of Morse index r. The group Γ acts freely on $\widetilde{\mathcal{X}}_r$ and the orbits of this action identify to $\mathcal{X}_r$. Let $\pi\colon \widetilde{\mathcal{X}}_r \to \mathcal{X}_r$ be the quotient map and let $\widetilde{\mathcal{X}}_{r,a} := \widetilde{\mathcal{X}}_r \cap f^{-1}(a)$. Note that the restriction of π to $\widetilde{\mathcal{X}}_{r,a}$ is injective.

Choose for any $o \in \mathcal{O}(f)$ a critical value a^o and observe that $\bigcup_{o\in\mathcal{O}(f)} \widetilde{\mathcal{X}}_{r,a^o}$ identifies by π to $\mathcal{X}_r$.

Classical Morse theory identifies the κ-vector space generated by $\widetilde{\mathcal{X}}_{r,a}$ with the vector space $H_r(\widetilde{M}_a, \widetilde{M}_{<a})$ and therefore the cardinality of set $\mathcal{X}_r$ is the dimension of the vector space $\bigoplus_{o\in\mathcal{O}(f)} H_r(\widetilde{M}_{a^o}, \widetilde{M}_{<a^o})$ calculated by Corollary 4.9 (ii), equivalently the rank of the free $\kappa[\Gamma]$-module $\bigoplus_{a\in\mathrm{CR}(f)} H_r(\widetilde{M}_a, \widetilde{M}_{<a})$.

5 The configurations δ^ω_r and γ^ω_r

5.1 The supports of δ^f_r and γ^f_r

In view of Propositions 4.2 and 4.6, the supports of δ^f_r and γ^f_r are located on finitely many diagonals Δ^δ_s and Δ^γ_s, for a finite collection of values of s ($s > 0$ in the case of

γ_r^f). Here we denote by $\Delta_s := \{(x, y) \in \mathbb{R}^2 : y - x = s\}$. One way to produce these diagonals goes as follows. Since the procedure is the same for δ_r^f and γ_r^f we describe in details only the case of δ_r^f. Choose one $a^o \in o \in \mathcal{O}_r(f)$ for each orbit o. Each point in the set

$$\text{supp } \delta_r^f \cap a_o \times \mathbb{R} = \left\{(a^o, b^1_{a^o}), (a^o, b^2_{a^o}), \ldots, (a^o, b^{n(o)})\right\}$$

defines a diagonal corresponding to $s^i = b^i_{a^o} - a^o$, so ultimately one obtains a collection of at most $\sum_{o \in \mathcal{O}_r(f)} n(o)$ such diagonals. The index s which appears is always a difference of critical values of the lift f. In view of the equality $\delta_r^f(a, b) = \delta_r^f(a + g, b + g)$ the integer $n(o)$ is independent of the choice of a^o. Note that different orbits o' and o'' lead to the same diagonal once the equality $b^i_{a^{o'}} - a^{o'} = b^j_{a^{o''}} - a^{o''}$ holds, so the same diagonal can appear multiple times, different choices of a^o lead to the same diagonals and different lifts of ω also lead to the same diagonals with the same number of occurrences. Similarly one can produce diagonals by choosing $b_o \in o$ and considering the diagonals corresponding to the points $(b_i^{b_o}, b_o)$, $1 \leqslant i \leqslant m(o)$, in the set supp $\delta_r^f \cap \mathbb{R} \times b_o$, hence with $s = b_o - a_i^{b_o}$. Again the integer $m(o)$ depends only on the orbit o. The outcome of both procedures is expected to be the same as argued below.

To calculate the number of diagonals and the "correct multiplicity" associated with each diagonal, the following definitions are of help. Note that both cases δ_r^f and γ_r^f are similar so one treats for convenience only the case of δ_r^f and one points out the minor notational differences when the case.

Two points $(x, y) \in \mathbb{R}^2$ and $(x', y') \in \mathbb{R}^2$ are Γ-equivalent, written $(x, y) \sim (x', y')$, iff there exists $g \in \Gamma$ such that $x = g + x'$, $y = g + y'$.

Definition 5.1 A subset $\mathcal{B}^f \subset \text{supp } \delta_r^f \subset \text{CR}(f) \times \text{CR}(f)$ is called a *base* for supp δ_r^f[5] if the following hold:

1. For any $(\alpha, \beta) \in \text{supp } \delta_r^f$ there exist $g \in \Gamma$ and $(a, b) \in B^f$ such that $(\alpha, \beta) = (g + a, g + b)$.
2. If (a, b) and (a', b') are in $\mathcal{B}^f$ then $(a, b) \sim (a', b')$ implies $(a, b) = (a', b')$.

In view of 2. the pair (a, b) and g claimed by 1. are unique.

Observe that the following hold:

1. If $\mathcal{B}_1^{f_1}$ and $\mathcal{B}_2^{f_2}$ are two bases for the supports of $\delta_r^{f_1}$ and $\delta_r^{f_2}$, f_1 and f_2 are lifts of ω, then there exists a canonical bijective correspondence $\theta : \mathcal{B}_1 \to \mathcal{B}_2$ with the property that $\delta^{f_2}(\theta(a, b)) = \delta^{f_1}(a, b)$.
2. If for any $o \in \mathcal{O}_r(f)$ one chooses $a^o \in o$ and let $b^1_{a^o}, b^2_{a^o}, \ldots, b^{n(o)}_{a^o}$ be all critical values[6] such that supp $\delta_r^f \cap a^o \times R = \{(a^o, b^1_{a^o}), (a^o, b^2_{a^o}), \ldots, (a^o, b^{n(o)}_{a^o})\}$, then

[5] For γ^f, supp $\delta_r^f \subset (\text{CR}(f) \times \text{CR}(f))_+$.

[6] Finitely many in view of Proposition 4.2.

the finite collection of points

$$\bigcup_{o\in\mathcal{O}_r(f)} \{(a^o, b^1_{a^o}), (a^o, b^2_{a^o}), \ldots, (a^o, b^{n(o)}_{a^o})\}$$

provides a base for the support of δ^f_r. Denote this base by $\mathcal{B}^f(\{a^o\})$ with $\{a^o\}$ the collection of elements a^o.

3. If for any $o \in \mathcal{O}_r(f)$ one chooses $b_o \in o$ and let $a^{b_o}_1, a^{b_o}_2, \ldots, a^{b_o}_{m(o)}$ be all critical values such that $\operatorname{supp} \delta^f_r \cap \mathbb{R}\times b_o = \{(a^{b_o}_1, b_o), (a^{b_o}_2, b_o), \ldots, (a^{b_o}_{m(o)}, b_o)\}$, then the collection of points

$$\bigcup_{o\in\mathcal{O}_r(f)} \{(a^{b_o}_1, b_o), (a^{b_o}_2, b_o), \ldots, (a^{b_o}_{m(o)}, b_o)\}$$

provides a base for the support of δ^f_r. Denote this base by $\mathcal{B}^f(\{b_o\})$.

4. Each element $(a, b) \in \mathcal{B}^f$ of a base provides a diagonal Δ_s with $s = b - a$ and each such diagonal Δ_s appears as many times as the number of pairs $\{(a, b) \in \mathcal{B}\colon b - a = s\}$. It is convenient to assign to Δ_s the number

$$\boldsymbol{\delta}^{\omega}_r(s) := \sum_{\substack{(a,b)\in B\\ b-a=s}} \delta^f_r(a, b)$$

which, by the first property above, is independent of the base $\mathcal{B}^f$.

The same definition can be made in case of γ^f and one can provide base $\mathcal{B}^f$ for $\operatorname{supp} \gamma^f_r$. The same observations remain valid when $\operatorname{supp} \delta^f_r \cap a\times\mathbb{R}$ and $\operatorname{supp} \delta^f_r \cap \mathbb{R}\times b$ are replaced by $\operatorname{supp} \gamma^f_r \cap a \times (a, \infty)$ and $\operatorname{supp} \gamma^f_r \cap (-\infty, b) \times b$, respectively.

The number assigned to Δ_s in the case of γ^f_r is

$$\boldsymbol{\gamma}^{\omega}_r(s) := \sum_{\substack{(a,b)\in\mathcal{B}^f\\ b-a=s}} \gamma^f_r(a, b)$$

which, by the first property, is independent of the base $\mathcal{B}^f$ and the lift f.

Given any lift f of a tame ω, $\boldsymbol{\delta}^{\omega}_r(s)$ and $\boldsymbol{\gamma}^{\omega}_r(s)$ can be calculated using either a base of type $\mathcal{B}(\{a^o\})$ or of type $\mathcal{B}(\{b_o\})$ and one obtains for any choice of a lift f and any choice $a^o \in o$ or $b_o \in o$, $o \in \mathcal{O}_r(f)$ the following formulae:

$$\boldsymbol{\delta}^{\omega}_r(s) = \sum_{o\in\mathcal{O}_r(f)} \delta^f_r(a^o, a^o + s) = \sum_{o\in\mathcal{O}_r(f)} \delta^f_r(b_o - s, b_o),$$

$$\boldsymbol{\gamma}^{\omega}_r(s) = \sum_{o\in\mathcal{O}_r(f)} \gamma^f_r(a^0, a^0 + s) = \sum_{o\in\mathcal{O}_r(f)} \gamma^f_r(b_o - s, b_o).$$

The items (i) and (ii) in Corollary 4.9 become

$$\beta^N_{\mathrm{top},r}(X;\omega) = \sum_{s\in\mathbb{R}} \boldsymbol{\delta}^\omega_r(s), \tag{5.1}$$

$$\sum_{o\in\mathcal{O}(f)} \dim H_r(\widetilde{X}_{a^o}, \widetilde{X}_{<a^o}) = \sum_{t\in\mathbb{R}} \boldsymbol{\delta}^\omega_r(t) + \sum_{t\in\mathbb{R}_+} \boldsymbol{\gamma}^\omega_r(t) + \sum_{t\in\mathbb{R}_+} \boldsymbol{\gamma}^\omega_{r-1}(t). \tag{5.2}$$

5.2 Derivation of the main results

Proof of Theorem 1.1 (i) follows from equality (5.1) combined with the equality of $\beta^N_{\mathrm{alg},r}(X;\xi(\omega)) = \beta^N_{\mathrm{top},r}(X;\omega)$ established by Theorem 5.2 below.
(ii) follows from equality (5.2).

□

Theorem 5.2 *If ω is a tame TC1-form then $\beta^N_{\mathrm{top},r}(X;\omega) = \beta^N_{\mathrm{alg},r}(X;\xi(\omega))$.*

Observe that in view of Proposition 3.1 for $x \in H_r(\widetilde{X})$ either there exists $a = \alpha(x) \in \mathbb{R}$ such that $x \in \mathbb{I}_a(r)\setminus\mathbb{I}_{<a}(r)$ or $x \in \bigcap_{t\in\mathbb{R}} \mathbb{I}_t(r) = \mathrm{Tor}\, H_r(\widetilde{X})$. Clearly, $x \in \mathbb{I}_a(r)\setminus\mathbb{I}_{<a}(r)$ implies $\hat{x}$, the image of x in $\mathbb{I}_a(r)/\mathbb{I}_{<a}(r)$, is not zero.

Observe that:

1. $\alpha(x) \in \mathrm{CR}(f)_r$,
2. $\alpha(g\cdot x) \equiv \alpha(\langle g\rangle(x)) = g + \alpha(x)$,
3. $\alpha(x+y) = \max\{\alpha(x), \alpha(y)\}$ if $\alpha(x) \neq \alpha(y)$ or if $\alpha(x) = \alpha(y)$ and $\hat{x} + \hat{y} \neq 0$,
4. $\alpha(x+y) < \alpha(x) = \alpha(y)$ if $\alpha(x) = \alpha(y)$ and $\hat{x} + \hat{y} = 0$.

Suppose that $e_1, e_2, \ldots, e_N$ is a base of $\mathcal{E} \subset H_r(\widetilde{X})$, a free $\kappa[\Gamma]$-submodule. Note that the multiplication with elements in Γ of any of e_i's does not change their status of remaining together a base for $\mathcal{E}$, but modifies $\alpha(e_i)$ as indicated in 2. above. In view of the above properties of $\alpha(x)$ one can modify this base into a base of $\mathcal{E}$ consisting of

$$\begin{array}{c} e_{1,1}, e_{1,2}, \ldots, e_{1,n_1}, \\ e_{2,1}, e_{2,2}, \ldots, e_{2,n_2}, \\ \ldots \\ e_{r,1}, e_{r,2}, \ldots, e_{r,n_r} \end{array}$$

with the following properties:

- $N = n_1 + n_2 + \cdots + n_r$,
- $\alpha(e_{i,j}) = a_i \in \mathrm{CR}_r(f)$,
- $a_1 > a_1 > \cdots > a_r$ with $a_i \in o_i$ different orbits of $\mathrm{CR}_r(f)/\Gamma = \mathcal{O}_r(f)$, i.e., Γ-independent.

First one observes that for any $i = 1, 2, \ldots, r$, $\hat{e}_{i,1}, \hat{e}_{i,2}, \ldots, \hat{e}_{i,n_i}$ are κ-linearly independent elements in $\mathbb{I}_{a_i}/\mathbb{I}_{<a_i}$. Indeed if for a fixed i one has $\sum_j \lambda_j \hat{e}_{i,j} = 0$, $\lambda_i \in \kappa$, then $\alpha\big(\sum_j \lambda_j e_{i,j}\big) < a_i$ by 4. above, and then $\sum_j \lambda_j e_{i,j} = \sum_j Q_j e_{i,j} +$

$\sum_{j,s\neq i} P_{s,j} e_{s,j}$ where $Q_j \in \kappa[\Gamma]$ contains only negative elements in Γ (i.e., in $\Gamma \cap (-\infty, 0)$). Then one obtains $\sum_j \lambda_j(1-Q_j)e_{i,j} - \sum_{j,s\neq i} P_{s,j}e_{s,j} = 0$, hence $\lambda_j(1-Q_j) = 0$, and because $1 - Q_j \neq 0$, $\lambda_j = 0$.

Second, in view of Γ-independence of a_i the entire collection $\{\hat{e}_{i,j}\}$ consists of elements $\kappa[\Gamma]$-linearly independent in the $\kappa[\Gamma]$-module $\bigoplus_{a\in\mathbb{R}} \mathbb{I}_a/\mathbb{I}_{<a}$, cf. Sect. 3.3. This implies $N \leqslant \beta^N_{\mathrm{top},r}(X;\omega)$ hence $\beta^N_{\mathrm{alg},r}(X;\xi(\omega)) \leqslant \beta^N_{\mathrm{top},r}(X;\omega)$. The inequality $\beta^N_{\mathrm{top},r}(X;\omega) \leqslant \beta^N_{\mathrm{alg},r}(X;\xi(\omega)$ follows from the injectivity of ${}^{\mathcal{S}}I(r)$ established in Proposition 4.4. provided that a Γ-compatible collection of splittings is chosen and such collection exists. A collection of splittings $i_{a,b}(r)\colon \hat{\delta}^f_r(a,b) \to \mathbb{F}_r(a,b) \subseteq H_r(\widetilde{X})$ is Γ-compatible if for any $g \in \Gamma$ one has

$$\langle g\rangle \cdot i_{a,b}(r) = i_{a+g,b+g}(r) \cdot \langle g\rangle_{a,b} \tag{5.3}$$

where the isomorphism $\langle g\rangle_{a,b}\colon \hat{\delta}^f_r(a,b) \to \hat{\delta}^f_r(a+g,b+g)$ is induced by the isomorphism $\langle g\rangle$. The choice of an arbitrary collection of splittings $i_{a,b}(r)$ for $(a,b) \in B^f$ a base for supp δ^f_r, which obviously exists, defines via formula (5.3) a family of compatible Γ-splittings. If the collection of splittings $\mathcal{S}$ is Γ-compatible then the κ-linear map ${}^{\mathcal{S}}I(r)$ is actually an injective $\kappa[\Gamma]$-linear map from the free $\kappa[\Gamma]$-module of rank $\dim \sum_{a,b\in B^f} \delta^f_r(a,b) = \beta^N_{\mathrm{top},r}(X;\omega)$ to $H_r(\widetilde{X})$, hence $\beta^N_{\mathrm{top},r}(X;\omega) \leqslant \beta^N_{\mathrm{alg},r}(X,\xi(\omega))$.

Theorems 1.2 and 1.3 in the generality stated will be proven in part II of this work. However in case ω is of degree of irrationality 0, hence $\Gamma = 0$ and then $\widetilde{X} = X$, they follow from [3, Theorems 5.2 and 5.3] in view of the fact that $\boldsymbol{\delta}^\omega(t) = \sum_a \delta^f_r(a, a+t)$. In case ω is of degree of irrationality 1 and the TC1-form determined by an angle-valued map $f\colon X \to \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ then they follow from [3, Theorems 5.5, 5.6, 6.3 and 6.4] by observing that $\boldsymbol{\delta}^\omega_r(t) = \sum_{z\in\mathbb{C}\setminus 0\colon \ln|z|=t} \delta^f_r(z)$ and $\boldsymbol{\gamma}^\omega_r(t) = \sum_{z\in\mathbb{C}\setminus 0\colon \ln|z|=t, |z|>1} \delta^f_r(z)$. If ω is of degree if irrationality 1 then $\Gamma \simeq Z$ with the positive generator, a real number $l \in \mathbb{R}_+$. One repeats the arguments in [3] with 2π replaced by l and one derives the two results from the same theorems in [3]. Reference [3] actually reproduces results in [5].

6 Appendix: Proof of Proposition 4.5

Consider commutative diagrams of κ-vector spaces

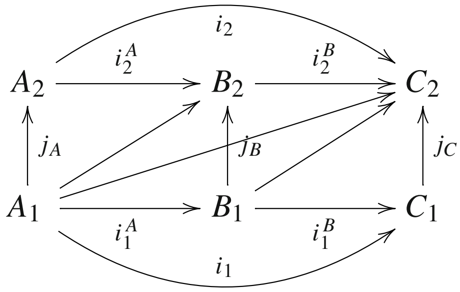
$$\mathbb{D} \;=\; \begin{array}{ccc} E_2 & \xrightarrow{\;i_2\;} & F_2 \\ {\scriptstyle j_E}\uparrow & \overset{u}{\nearrow} & \uparrow{\scriptstyle j_F} \\ E_1 & \xrightarrow[\;i_1\;]{} & F_1 \end{array}$$

which satisfy the following three properties:

1. j_E and j_F are injective,
2. $j_E:\ \ker i_1 \to \ker i_2$ is an isomorphism,
3. $\mathrm{img}(u) = \mathrm{img}(i_2) \cap \mathrm{img}(j_F) = \mathrm{img}(u)$,

and define $T(\mathcal{B}) := F_2/i_2(E_2) + j_F(F_1)$.

Consider the diagram



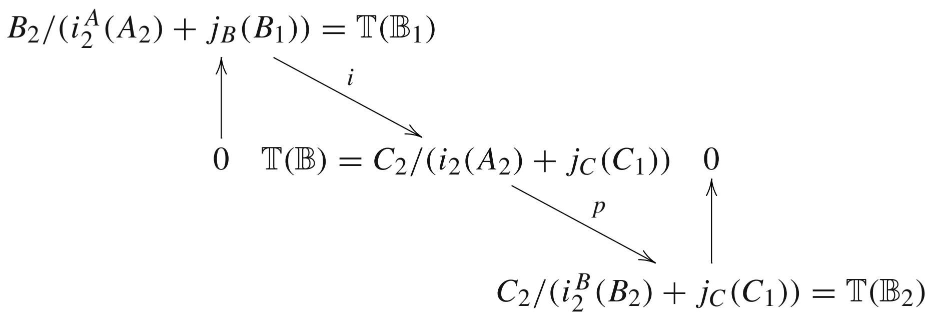
(6.1)

and observe that

O1: If each of the three diagrams $\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}$, associated with (6.1),

- $\mathbb{B}_1$ with vertices A_1, A_2, B_1, B_2,
- $\mathbb{B}_2$ with vertices B_1, B_2, C_1, C_2, and
- $\mathbb{B}$ with vertices A_1, A_2, C_1, C_2

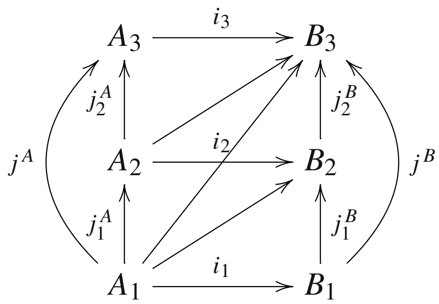
satisfy Properties 1–3 above then (6.1) induces the exact sequence



with i induced by i_2^B (well defined because $\mathrm{img}(i_2^B \cdot j_B) \subseteq \mathrm{img}(j_C)$), p induced by the inclusion $(i_2(A_2) + j_C(C_1)) \subseteq (i_2^B(B_2) + j_C(C_1))$.

Clearly p is surjective and $p \cdot i = 0$. Property 3 implies i injective. Properties 1–3 imply that the sequence is exact.

Similarly consider the diagram



(6.2)

and observe by the same arguments that

O2: If each of the three diagrams $\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}$, associated with (6.2),

- $\mathbb{B}_1$ with vertices A_2, A_3, B_2, B_3,
- $\mathbb{B}_2$ with vertices A_1, A_2, B_1, B_2, and
- $\mathbb{B}$ with vertices A_1, A_3, B_1, B_2

satisfy Properties 1–3 of the diagram $\mathbb{D}$, then (6.2) induces the exact sequence

$$\begin{array}{ccccc} B_2/(i_2(A_2)+j_1^B(B_1)) = \mathbb{T}(\mathbb{B}_1) & & & & \\ \uparrow & \overset{j}{\searrow} & & & \\ 0 & & \mathbb{T}(\mathbb{B}) = B_3/(i_3(A_3)+j^B(B_1)) & & 0 \\ & & & \overset{p}{\searrow} & \uparrow \\ & & & & B_3/(i_3(A_3)+j_2^B(B_2)) = \mathbb{T}(\mathbb{B}_2) \end{array}$$

Note that any ad-box $B = (a', a] \times (b', b]$, $a' < a < b \leqslant b' < b$, defines a diagram $\mathbb{D}$ as above with $E_2 = \mathbb{T}_r(a', b)$, $F_2 = \mathbb{T}_r(a, b)$, $E_1 = \mathbb{T}_r(a', b')$, $F_1 = \mathbb{T}_r(a, b')$ and i_1, i_2, j_E, j_F the induced linear maps. The ad-boxes $B_{12}, B_{22}, B_{\cdot 2}, B_{11}, B_{21}, B_{\cdot 1}$ and $B_{1,\cdot}, B_{2,\cdot}, B$ are in the situation provided by **O1**, and the boxes $B_{11}, B_{12}, B_{1,\cdot}$, $B_{21}, B_{22}, B_{2,\cdot}$ and $B_{\cdot 1}, B_{\cdot 2}, B$ are in the situation provided by **O2**. Consequently, Proposition 4.5 follows.

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