A RIEMANNIAN INVARIANT, EULER STRUCTURES AND SOME TOPOLOGICAL APPLICATIONS

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Contents

1. Introduction

This paper follows entirely the lecture the first author gave at Bedlewo's workshop in February 2004 and is a survey of some of the results in [BH04] and [B99]. We discuss in details two concepts, the invariant R which is a number associated with a Riemannian metric g, a vector field with isolated zeros X, and a closed one form ω, and the Euler resp. co-Euler structures which are affine versions of $H_1(M;\mathbb{Z})$ resp. $H^{n-1}(M;\mathcal{O}_M)$. They play an important role in our recent work about relating the topology of non-simply connected manifolds to the complex geometry/analysis of the variety of complex representations of their fundamental group.

Both concepts existed in literature prior to our work, cf. [BZ92] and [T90]. We have extended, generalized and Poincaré dualized them because of our needs, cf. [BH04], but we also believe that they have independent interest.

Euler and co-Euler structures represent the additional topological data needed to remove the geometric ambiguity from the Reidemeister torsion, resp. from the Ray–Singer torsion when extended to arbitrary representations, and provide genuine topological invariants. The invariant R, among other things, relates Euler and co-Euler structures.

We use the opportunity of having these two concepts presented in details to clarify the difference between the related concepts of *(combinatorial) torsion, Milnor metric* and *(modified) Ray–Singer metric* and to reformulate with their help the results of Bismut–Zhang, see [BZ92].

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The Bismut–Zhang theorem as formulated is about flat real vector bundles. The appendix completes the discussion with the case of flat complex vector bundles.

The invariant $R(X,g,\omega)$. Let M be a closed manifold and $\omega \in \Omega^1(M)$ a closed one form with real or complex coefficients.

(i) A pair of two Riemannian metrics g_1, g_2 determines the Chern–Simons class $\mathrm{cs}(g_1,g_2) \in \Omega^{n-1}(M;\mathcal{O}_M)/d\Omega^{n-2}(M;\mathcal{O}_M)$ and then the numerical invariant
$$R(g_1,g_2,\omega) := \int_M \omega \wedge \mathrm{cs}(g_1,g_2).$$

(ii) A pair of two vector fields without zeros X_1, X_2 determines a homology class $\mathrm{cs}(X_1,X_2) \in H_1(M;\mathbb{Z})$, see section 3 below, and then a numerical invariant
$$R(X_1,X_2,\omega) = \langle[\omega],\mathrm{cs}(X_1,X_2)\rangle. \tag{1}$$

(iii) A pair consisting of a vector field without zeros and a Riemannian metric g determines a degree $n-1$ form $X^*\Psi(g) \in \Omega^{n-1}(M;\mathcal{O}_M)$ and therefore a numerical invariant
$$R(X,g,\omega) = \int_M \omega \wedge X^*\Psi(g). \tag{2}$$

Here $\Psi(g) \in \Omega^{n-1}(TM \setminus 0_M;\mathcal{O}_M)$ is the global angular form cf. [BT82] also considered in [MQ86, section 7], cf. section 2 below.

One can extend the invariant (iii) to the case of vector fields with isolated zeros, not necessarily non-degenerate. Both smooth triangulations and Euler structures provide examples of such vector fields, cf. sections 4 and 5. If X has zeros then the integrand in (2) is defined only on $M \setminus \mathcal{X}$, $\mathcal{X}$ the set of zeros of X, and the integral might be divergent. Fortunately it can be regularized by a procedure we will refer to as *geometric regularization* as described in section 3 and this leads to the numerical invariant $R(X,g,\omega)$ from the title, cf. Theorem 1 below. This invariant for $X = -\operatorname{grad} f$, f a Morse function was considered in [BZ92] in terms of currents. One can also extend the invariant (ii) to vector fields with isolated zeros, cf. section 3.

A pleasant application of the invariant R and of the extension of (ii) is the extension of the Chern–Simons class from a pair of two Riemannian metrics g_1 and g_2 to a pair of two smooth triangulations τ_1 and τ_2 or to a pair of a Riemannian metric g and a smooth triangulation τ, cf. section 5. These classes permit to treat on "equal foot" a Riemannian metric and a smooth triangulation when comparing subtle invariants like "torsion" defined using a Riemannian metric, and using a triangulation, in analogy with the comparison of such invariants for two metrics or two triangulations.

Euler structures. Euler structures were introduced by Turaev cf. [T90] for manifolds M with vanishing Euler–Poincaré characteristic, $\chi(M) = 0$. We define the Euler structures for an arbitrary base pointed manifold (M,x_0) and show that the definition is independent of the base point provided $\chi(M) = 0$. The set of Euler structures $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ is an affine version of $H_1(M;\mathbb{Z})$ in the sense that $H_1(M;\mathbb{Z})$ acts freely and transitively on $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$. Similarly there is the set of Euler structures with real coefficients $\mathfrak{Eul}_{x_0}(M;\mathbb{R})$ which is an affine space over $H_1(M;\mathbb{R})$, and there is a homomorphism $\mathfrak{Eul}_{x_0}(M;\mathbb{Z}) \to \mathfrak{Eul}_{x_0}(M;\mathbb{R})$ which is affine over the homomorphism $H_1(M;\mathbb{Z}) \to H_1(M;\mathbb{R})$.

We also introduce the set $\mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ of co-Euler structures on which $H^{n-1}(M;\mathcal{O}_M)$ acts freely and transitively. The set of co-Euler structures is defined as the set of equivalence classes of pairs (g,α) where $\alpha\in\Omega^{n-1}(M\setminus x_0;\mathcal{O}_M)$ satisfies $d\alpha=E(g)$. Two pairs (g_1,α_1) and (g_2,α_2) are equivalent iff $\alpha_2-\alpha_1=\mathrm{cs}(g_1,g_2)$.

$\mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ represents a smooth version (deRham version) of a dual aspect of $\mathfrak{Eul}_{x_0}(M;\mathbb{R})$. In the case of a closed manifold M we show the existence of an affine version of Poincaré duality map $P:\mathfrak{Eul}^*_{x_0}(M;\mathbb{R})\to\mathfrak{Eul}_{x_0}(M;\mathbb{R})$. This can equivalently be described with the help of a coupling

$$\mathbb{T}:\mathfrak{Eul}_{x_0}(M;\mathbb{R})\times\mathfrak{Eul}^*_{x_0}(M;\mathbb{R})\to H_1(M;\mathbb{R})$$

based on a regularization very similar to the one for R, see section 3.[1]

Primarily, the interest of Euler and co-Euler structures comes from the following. Suppose F is a flat real or complex vector bundle, and let F_{x_0} denote the fiber over the base point x_0. A co-Euler structure $\mathfrak{e}^*\in\mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ removes the metric ambiguity of the Ray–Singer torsion and provides a Hermitian scalar product, *the analytic scalar product*, in the complex line:

$$\det H^*(M;F)\otimes(\det F_{x_0})^{-\chi(M)} \tag{3}$$

An Euler structure with real coefficients $\mathfrak{e}\in\mathfrak{Eul}_{x_0}(M;\mathbb{R})$ removes the triangulation ambiguity[2] and provides a Hermitian scalar product, *the combinatorial scalar product*, in the line (3), see also [FT00].

As an application of Euler and co-Euler structures we present a reformulation of a result of Bismut–Zhang, proved in [BZ92], referred to as the Bismut–Zhang theorem, see Theorem 3 in section 6. Precisely, the analytic scalar product associated to $\mathfrak{e}^*$ is the same as the combinatorial scalar product associated to $\mathfrak{e}$ multiplied by $e^{\langle(\log|\cdot|)_*\Theta_F,\mathbb{T}(\mathfrak{e},\mathfrak{e}^*)\rangle}$. Here $\Theta_F\in H^1(M;\mathbb{C}^*)$ is the cohomology class corresponding to $\det\circ\rho:H_1(M;\mathbb{Z})\to\mathbb{C}^*$, and $(\log|\cdot|)_*\Theta_F\in H^1(M;\mathbb{R})$ denotes its image under the homomorphism $(\log|\cdot|)_*:H^1(M;\mathbb{C}^*)\to H^1(M;\mathbb{R})$ which is induced from the homomorphism of coefficients $\log|\cdot|:\mathbb{C}^*\to\mathbb{R}$. This cohomology class is known as Kamber–Tondeur class.

The results. Suppose M is a closed manifold of dimension n. Given a Riemannian metric g denote by $E(g)\in\Omega^n(M;\mathcal{O}_M)$ the Euler form and by $\Psi(g)\in\Omega^{n-1}(TM\setminus M;\mathcal{O}_M)$ the Mathai–Quillen form associated to g. If X_1 and X_2 are two vector fields with isolated zeros we get an element

$$\mathrm{cs}(X_1,X_2)\in C_1(M;\mathbb{Z})/\partial(C_2(M;\mathbb{Z}))$$

whose boundary equals the zeros of X_1 and X_2, weighted with their indices, see section 2.

Theorem 1. *Let M be a closed connected manifold.*

[1] The concept of Euler and co-Euler structures can be extended from the tangent bundle to arbitrary rank k bundles. This is particularly easy if the Euler class of the bundle vanishes. The set of Euler structures of a vector bundle will be an affine version of $H_{n-k+1}(M;\mathbb{Z})$ resp. $H_{n-k+1}(M;\mathbb{R})$ and the set of co-Euler structures will be an affine version of $H^{k-1}(M;\mathcal{O}_E)$. There again is an affine version of Poincaré duality, based on a regularized integral. This permits to consider Euler and co-Euler structures as a functorial concept.

[2] and the additional ambiguity produced by the choice of a lift of each cell of the triangulation to the universal cover of the manifold

(i) *Suppose $\omega \in \Omega^1(M)$ is a real or complex valued closed one form, g a Riemannian metric and X a vector field with isolated zeros. Let f be a smooth real or complex valued function with $\omega = df$ in the neighborhood of the zero set $\mathcal{X}$ of X. Then the number*

$$R(X,g,\omega;f) := \int_{M\setminus\mathcal{X}} (\omega - df)\wedge X^*\Psi(g) - \int_M fE(g) + \sum_{x\in\mathcal{X}} \mathrm{IND}(x)f(x)$$

is independent of f and will therefore be denoted by $R(X,g,\omega)$.

(ii) *If g_1 and g_2 are two Riemannian metrics, then*

$$R(X,g_2,\omega) - R(X,g_1,\omega) = \int_M \omega\wedge \mathrm{cs}(g_1,g_2).$$

(iii) *If X_1 and X_2 are two vector fields with isolated zeros then*

$$R(X_2,g,\omega) - R(X_1,g,\omega) = \int_{\mathrm{cs}(X_1,X_2)} \omega.$$

(iv) *If ω_1 and ω_2 are two closed one forms so that $\omega_2 - \omega_1 = dh$ then*

$$R(X,g,\omega_2) - R(X,g,\omega_1) = -\int hE(g) + \sum_{x\in\mathcal{X}} \mathrm{IND}(x)h(x).$$

In section 3 we will verify statements (i) through (iv). More precisely they are the contents of Lemma 1, Proposition 1 and Proposition 2.

An Euler structure on a base pointed manifold (M, x_0) is an equivalence class of pairs (X, c), where X is a vector field with isolated singularities and c is a singular one chain with integral coefficients whose boundary equals $\sum_{x\in\mathcal{X}} \mathrm{IND}(x)x - \chi(M)x_0$, where $\mathcal{X}$ denotes the zero set of X. Two such pairs (X_1, c_1) and (X_2, c_2) are equivalent if c_2 differs from $c_1+\mathrm{cs}(X_1, X_2)$ by a boundary. We write $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ for the set of Euler structures based at x_0. This is an affine version of $H_1(M;\mathbb{Z})$ in the sense that $H_1(M;\mathbb{Z})$ acts freely and transitively on it. Considering chains c with real coefficients we get an affine version of $H_1(M;\mathbb{R})$ which we denote by $\mathfrak{Eul}_{x_0}(M;\mathbb{R})$.

The set $\mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ of co-Euler structures is defined as the set of equivalence classes of pairs (g,α) where $\alpha \in \Omega^{n-1}(M \setminus x_0; \mathcal{O}_M)$ satisfies $d\alpha = E(g)$. Two pairs (g_1, α_1) and (g_2, α_2) are equivalent iff $\alpha_2 - \alpha_1 = \mathrm{cs}(g_1, g_2)$. The cohomology $H^{n-1}(M;\mathcal{O}_M)$ acts on $\mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ freely and transitively by $[g,\alpha]+[\beta] := [g,\alpha-\beta]$.

Theorem 2. *Let (M, x_0) be a closed connected base pointed manifold.*

(i) *Let $\pi_0(\mathfrak{X}(M, x_0))$ denote the set of connected components of vector fields which vanish only at x_0 equipped with the C^∞ topology, or any C^r topology, $r \geq 0$. If $\dim M > 2$ then the assignment $[X] \mapsto [X, 0]$ defines a bijection:*

$$\pi_0(\mathfrak{X}(M, x_0)) \to \mathfrak{Eul}_{x_0}(M;\mathbb{Z}).$$

(ii) *Let $\pi_0(\mathfrak{X}_0(M))$ denote the set of connected components of nowhere vanishing vector fields equipped with the C^∞ topology, or any C^r topology, $r \geq 0$. If $\chi(M) = 0$ and $\dim M > 2$ then the assignment $[X] \mapsto [X, 0]$ defines a surjection:*

$$\pi_0(\mathfrak{X}_0(M)) \to \mathfrak{Eul}_{x_0}(M;\mathbb{Z}).$$

(iii) *There exists an isomorphism*

$$P : \mathfrak{Eul}^*_{x_0}(M;\mathbb{R}) \to \mathfrak{Eul}_{x_0}(M;\mathbb{R}),$$

which is affine over the Poincaré duality $\mathrm{PD} : H^{n-1}(M;\mathcal{O}_M) \to H_1(M;\mathbb{R})$. *That is* $P(\mathfrak{e}^* + \beta) = P(\mathfrak{e}^*) + \mathrm{PD}(\beta)$, *for all* $\beta \in H^{n-1}(M;\mathcal{O}_M)$.

(iv) *The assignment* $\mathbb{T}(\mathfrak{e},\mathfrak{e}^*) := P(\mathfrak{e}^*) - \mathfrak{e}$

$$\mathbb{T} : \mathfrak{Eul}_{x_0}(M;\mathbb{R}) \times \mathfrak{Eul}^*_{x_0}(M;\mathbb{R}) \to H_1(M;\mathbb{R})$$

is a corrected version of the invariant R. *More precisely, if* $\mathfrak{e} = [X, c]$, $\mathfrak{e}^* = [g, \alpha]$ *and* $[\omega] \in H^1(M;\mathbb{R})$ *we have*

$$\langle [\omega], \mathbb{T}(\mathfrak{e},\mathfrak{e}^*)\rangle = \int_M \omega \wedge (X^*\Psi(g) - \alpha) - \int_c \omega$$

where $\omega \in \Omega^1(M)$ *is any representative of* $[\omega]$ *which vanishes locally around* x_0 *and locally around the zeros of* X.

Statements (i) and (ii) are essentially due to Turaev and is the contents of Propositions 3 and 4 in section 4. The proof of (iii) and (iv) can be found at the end of section 4, cf. Proposition 5.

The theorem of Bismut–Zhang in our reformulation is contained in section 6 as Theorem 3.

2. Global angular form

Let $\pi : E \to M$ be a rank k real vector bundle, and let $\tilde{\nabla} := (\nabla, \mu)$ be a pair consisting of a connection ∇ and a parallel Hermitian structure, i.e. fiber wise scalar product, μ. Let $\mathcal{O}_E$ denote the orientation bundle of E, a flat real line bundle over M. There is a canonic $\mathrm{Vol} \in \Omega^k(E;\pi^*\mathcal{O}_E)$, which vanishes when contracted with horizontal vectors and which assigns to a k-tuple of vertical vectors *their volume times their orientation.* Moreover let ξ denote the Euler vector field on E which assigns to a point $x \in E$ the vertical vector $-x \in T_xE$. The differential form

$$\Psi(\tilde{\nabla}) := \frac{\Gamma(k/2)}{(2\pi)^{k/2}|\xi|^k} i_\xi \,\mathrm{Vol} \in \Omega^{k-1}(E \setminus M; \pi^*\mathcal{O}_E).$$

was probably first considered by Chern and can be found in [BT82] but see also [MQ86].

Clearly $\Psi(\tilde{\nabla})$ has the following properties which follow immediately from the definition.

(i) $\Psi(\tilde{\nabla})$ is the pullback of a form on $(E \setminus M)/\mathbb{R}_+$.

(ii) If $E(\tilde{\nabla}) \in \Omega^k(M;\mathcal{O}_E)$ denotes the Euler form of $\tilde{\nabla}$ then:

$$d\Psi(\tilde{\nabla}) = \pi^* E(\tilde{\nabla}). \tag{4}$$

(iii) If $\mathrm{cs}(\tilde{\nabla}_1, \tilde{\nabla}_2) \in \Omega^{k-1}(M;\mathcal{O}_E)/d(\Omega^{k-2}(M;\mathcal{O}_E))$ denotes the Chern–Simons class then:

$$\Psi(\tilde{\nabla}_2) - \Psi(\tilde{\nabla}_1) = \pi^* \,\mathrm{cs}(\tilde{\nabla}_1, \tilde{\nabla}_2) \tag{5}$$

(iv) Suppose $E = TM$ is equipped with a Riemannian metric g, $\nabla_g = (\tilde{\nabla}_g, g)$ is the Levi–Civita pair and X is a vector field with isolated zero x. Let B_ϵ denote the ball of radius ϵ around x, with respect to some chart. Then

$$\lim_{\epsilon \to 0} \int_{\partial(M \setminus B_\epsilon)} X^*\Psi(\tilde{\nabla}_g) = \mathrm{IND}(x), \tag{6}$$

where $\mathrm{IND}(x)$ denotes the Hopf index of X at x.

(v) For $M = \mathbb{R}^n$, $E := TM$ equipped with $g_{ij} = \delta_{ij}$, $\tilde{\nabla}_g$ the Levi–Civita pair and in the coordinates $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ one has:

$$\Psi(\tilde{\nabla}_g) = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \sum_{i=1}^{n} (-1)^i \frac{\xi_i}{(\sum \xi_i^2)^{n/2}} d\xi_1 \wedge \cdots \wedge \widehat{d\xi_i} \wedge \cdots \wedge d\xi_n.$$

Let $\xi : E \to M$ be a complex vector bundle equipped with a flat connection. Given two Hermitian structures μ_1 and μ_2 denote by $V(\mu_1, \mu_2)$ the positive real valued function given at $y \in M$ by the volume with respect to the scalar product defined by $(\mu_2)_y$ of a parallelepiped provided by an orthonormal frame with respect to $(\mu_1)_y$.

Suppose that the bundle ξ is equipped with a flat connection ∇. To any Hermitian structure in $\xi : E \to M$ following Kamber–Tondeur one associates the real valued closed (hence locally exact) differential form $\omega(\nabla, \mu) \in \Omega^1(M)$ defined as follows. For any $x \in M$ choose a contractible open neighborhood U, and denote by $\tilde{\mu}_x$ the Hermitian structure on $E|_U \to U$ obtained by parallel transport of μ_x. This Hermitian structure is well defined since U is one connected and the connection is flat.

Define $\omega(\nabla, \mu) := -\frac{1}{2} d \log V_x$ as being the logarithmic differential of the non-zero function $V_x : U \to \mathbb{R}$ defined by $V_x = V(\tilde{\mu}_x, \mu)$. The following property holds:

$$\omega(\nabla, \mu_2) - \omega(\nabla, \mu_1) = -\frac{1}{2} d \log(V(\mu_1, \mu_2))$$

3. The invariant $R(X, g, \omega)$. The geometric regularization

Suppose M is a closed manifold of dimension n, g a Riemannian metric and $X : M \to TM \setminus M$ a vector field without zeros. Suppose ω is a closed one form with real or complex coefficients. Define

$$R(X, g, \omega) := \int_M \omega \wedge X^* \Psi(g), \tag{7}$$

which will be a real or complex number. For every function h we have

$$R(X, g, \omega + dh) - R(X, g, \omega) = -\int_M h E(g),$$

and for any two Riemannian metrics g_1 and g_2 we have

$$R(X, g_2, \omega) - R(X, g_1, \omega) = \int_M \omega \wedge \mathrm{cs}(g_1, g_2).$$

These properties are straightforward consequences of (4), Stokes' theorem and (5).

Suppose X_1 and X_2 are two vector fields without zeros. Let $p : I \times M \to M$ denote the projection, where $I = [1, 2]$. Consider a section $\mathbb{X}$ of p^*TM which is transversal to the zero section and which restricts to X_i on $\{i\} \times M$, $i = 1, 2$. The zero set $\mathbb{X}^{-1}(0)$ is a closed one dimensional canonically oriented submanifold of $I \times M$. Hence it defines a homology class in $I \times M$, which turns out to be independent of the chosen homotopy $\mathbb{X}$. We thus define $\mathrm{cs}(X_1, X_2) := p_*(\mathbb{X}^{-1}(0)) \in H_1(M; \mathbb{Z})$. One can show that

$$R(X_2, g, \omega) - R(X_1, g, \omega) = \int_{\mathrm{cs}(X_1, X_2)} \omega$$

This property will be verified below in a slightly more general situation.

The above properties suggest the definition of the invariant R in the case X has isolated zeros even when the integral in (7) is divergent. This definition will be referred to as the *geometric regularization* of (7). We do not assume that the zeros of X are non-degenerate. Let $\mathcal{X}$ denote the zero set of X. Choose a function f so that $\omega' := \omega - df$ vanishes on a neighborhood of $\mathcal{X}$. Then

$$R(X, g, \omega; f) := \int_{M\setminus\mathcal{X}} \omega' \wedge X^*\Psi(g) - \int_M fE(g) + \sum_{x\in\mathcal{X}} \mathrm{IND}(x)f(x)$$

makes perfect sense. The next lemma establishes the proof of Theorem 1(i).

Lemma 1. *The quantity $R(X, g, \omega; f)$ does not depend on the choice of f.*

Proof. Suppose f_1 and f_2 are two functions such that $\omega_i' := \omega - df_i$, $i = 1, 2$ both vanish in a neighborhood U of $\mathcal{X}$, $i = 1, 2$. For every $x \in \mathcal{X}$ we choose a chart and let $B_\epsilon(x)$ denote the disk of radius ϵ around x. Put $B_\epsilon := \bigcup_{x\in\mathcal{X}} B_\epsilon(x)$.

For ϵ small enough $B_\epsilon \subset U$ and $f_2 - f_1$ is constant on each $B_\epsilon(x)$. Using (4), Stokes' theorem and (6) we get

$$\begin{aligned}
&R(X, g, \omega; f_2) - R(X, g, \omega; f_1) = \\
&= -\int_{M\setminus\mathcal{X}} d\big((f_2 - f_1) \wedge X^*(\Psi(g))\big) + \sum_{x\in\mathcal{X}} \mathrm{IND}(x)(f_2 - f_1)(x) \\
&= -\lim_{\epsilon\to 0} \int_{\partial(M\setminus B_\epsilon)} (f_2 - f_1) \wedge X^*\Psi(g) + \sum_{x\in\mathcal{X}} \mathrm{IND}(x)(f_2 - f_1)(x) \\
&= -\sum_{x\in\mathcal{X}} (f_2 - f_1)(x) \lim_{\epsilon\to 0} \int_{\partial(M\setminus B_\epsilon(x))} X^*\Psi(g) + \sum_{x\in\mathcal{X}} \mathrm{IND}(x)(f_2 - f_1)(x) \\
&= 0
\end{aligned}$$

and thus $R(X, g, \omega; f_1) = R(X, g, \omega; f_2)$. □

Definition 1. In view of the previous lemma we define $R(X, g, \omega) := R(X, g, \omega; f)$, where f is any function so that $\omega - df$ vanishes locally around $\mathcal{X}$.

¿From the very definition we immediately verify Theorem 1(iv) which we restate as

Proposition 1. *For every function h we have:*

$$R(X, g, \omega + dh) - R(X, g, \omega) = -\int_M hE(g) + \sum_{x\in\mathcal{X}} \mathrm{IND}(x)h(x) \tag{8}$$

For any vector field with isolated zeros $\mathcal{X}$ we set

$$e_X := \sum_{x\in\mathcal{X}} \mathrm{IND}(x)x,$$

a singular zero chain in M.

Suppose we have two vector fields X_1 and X_2 with non-degenerate zeros. Consider the vector bundle $p^*TM \to I \times M$, where $I := [1, 2]$ and $p : I \times M \to M$ denotes the natural projection. Choose a section $\mathbb{X}$ of p^*TM which is transversal to the zero section and which restricts to X_i on $\{i\} \times M$, $i = 1, 2$. The zero set of $\mathbb{X}$ is a canonically oriented one dimensional submanifold with boundary. Hence it defines a singular one chain which, when pushed forward via p, is a one chain $c(\mathbb{X})$ in M, satisfying

$$\partial c(\mathbb{X}) = e_{X_2} - e_{X_1}.$$

Suppose $\mathbb{X}_1$ and $\mathbb{X}_2$ are two non-degenerate homotopies from X_1 to X_2. Then certainly $\partial(c(\mathbb{X}_2) - c(\mathbb{X}_1)) = 0$, but we actually have

$$c(\mathbb{X}_2) - c(\mathbb{X}_1) = \partial\sigma, \tag{9}$$

for a two chain σ. Indeed, consider the vector bundle $q^*TM \to I \times I \times M$, where $q : I \times I \times M \to M$ denotes the natural projection. Choose a section of q^*TM which is transversal to the zero section, restricts to $\mathbb{X}_i$ on $\{i\} \times I \times M$, $i = 1, 2$ and which restricts to X_i on $\{s\} \times \{i\} \times M$ for all $s \in I$ and $i = 1, 2$. The zero set of such a section then gives rise to σ satisfying (9).

So for two vector fields with non-degenerate zeros this construction yields a one chain $\mathrm{cs}(X_1, X_2)$, well defined up to a boundary, satisfying $\partial\,\mathrm{cs}(X_1, X_2) = e_{X_2} - e_{X_1}$.

Let us extend this to vector fields with isolated singularities. Suppose X is a vector field with isolated singularities. For every zero $x \in \mathcal{X}$ we choose an embedded ball B_x centered at x, assuming all B_x are disjoint. Set $B := \bigcup_{x\in\mathcal{X}} B_x$. Choose a vector field with non-degenerate zeros X' that coincides with X on $M \setminus B$. Let $\mathcal{X}'$ denote its zero set. For every $x \in \mathcal{X}$ we have

$$\mathrm{IND}_X(x) = \sum_{y\in\mathcal{X}'\cap B_x} \mathrm{IND}_{X'}(y).$$

So we can choose a one chain $\tilde{c}(X, X')$ supported in B which satisfies $\partial\tilde{c}(X, X') = e_{X'} - e_X$. Since $H_1(B;\mathbb{Z})$ vanishes the one chain $\tilde{c}(X, X')$ is well defined up to a boundary.

Given two vector fields X_1 and X_2 with isolated zeros we choose perturbed vector fields X_1' and X_2' as above and set

$$\mathrm{cs}(X_1, X_2) := \tilde{c}(X_1, X_1') + \mathrm{cs}(X_1', X_2') - \tilde{c}(X_2, X_2').$$

Then obviously $\partial\,\mathrm{cs}(X_1, X_2) = e_{X_2} - e_{X_1}$. Using $H_1(B;\mathbb{Z}) = 0$ again, one checks that different choices for X_1' and X_2' yield the same $\mathrm{cs}(X_1, X_2)$ up to a boundary.

Summarizing, for every pair of vector fields X_1 and X_2 with isolated zeros we have constructed a one chain

$$\mathrm{cs}(X_1, X_2) \in C_1(M;\mathbb{Z})/\partial(C_2(M;\mathbb{Z})),$$

which satisfies $\partial\,\mathrm{cs}(X_1, X_2) = e_{X_2} - e_{X_1}$.

Definition 2. For two Riemannian metrics g_1, g_2 and a closed one form ω set

$$R(g_1, g_2, \omega) := \int_M \omega \wedge \mathrm{cs}(g_1, g_2). \tag{10}$$

For two vector fields X_1, X_2 and a closed one form ω set

$$R(X_1, X_2, \omega) := \int_{\mathrm{cs}(X_1, X_2)} \omega. \tag{11}$$

Remark 1. Even though $\mathrm{cs}(g_1, g_2)$ is only defined up to an exact form this ambiguity does not affect the integral (10). Similarly, even though $\mathrm{cs}(X_1, X_2)$ is only defined up to a boundary this ambiguity does not affect the integral (11).

The next proposition is a reformulation of Theorem 1(ii) and Theorem 1(iii).

Proposition 2. *Let M be a closed manifold, ω a closed one form, g, g_1, g_2 Riemannian metrics and let X, X_1, X_2 be vector fields with isolated zeros. Then*

$$R(X, g_2, \omega) - R(X, g_1, \omega) = R(g_1, g_2, \omega) \tag{12}$$

and

$$R(X_2, g, \omega) - R(X_1, g, \omega) = R(X_1, X_2, \omega). \tag{13}$$

Proof. Let us prove (12). Choose f so that $\omega' := \omega - df$ vanishes on a neighborhood of $\mathcal{X}$, the zero set of X. Using $X^*(\Psi(g_2) - \Psi(g_1)) = \mathrm{cs}(g_1, g_2)$ modulo exact forms, Stokes' theorem and $d\,\mathrm{cs}(g_1, g_2) = E(g_2) - E(g_1)$ we conclude

$$\begin{aligned}
R(X, g_2, \omega) &- R(X, g_1, \omega) = \\
&= \int_{M\setminus\mathcal{X}} \omega' \wedge X^*\big(\Psi(g_2) - \Psi(g_1)\big) - \int_M f\big(E(g_2) - E(g_1)\big) \\
&= \int_M \omega \wedge \mathrm{cs}(g_1, g_2) - \int_M df \wedge \mathrm{cs}(g_1, g_2) - \int_M f\big(E(g_2) - E(g_1)\big) \\
&= \int_M \omega \wedge \mathrm{cs}(g_1, g_2) \\
&= R(g_1, g_2, \omega).
\end{aligned}$$

Now let us turn to (13). Let $\mathcal{X}_i$ denote the zero set of X_i, $i = 1, 2$. Assume first that the vector fields X_1 and X_2 are non-degenerate and that there exists a non-degenerate homotopy $\mathbb{X}$ from X_1 to X_2 whose zero set is contained in a simply connected $I \times V \subseteq I \times M$. Choose a function f such that $\omega' := \omega - df$ vanishes on V. Then

$$R(X_1, X_2, \omega) = \int_{\mathbb{X}^{-1}(0)} p^* df = \sum_{x\in\mathcal{X}_2} \mathrm{IND}_{X_2}(x) f(x) - \sum_{x\in\mathcal{X}_1} \mathrm{IND}_{X_1}(x) f(x),$$

where $p : I \times M \to M$ denotes the natural projection. Let $\tilde{p} : p^*TM \to TM$ be the natural vector bundle homomorphism over p. Using the last equation, Stokes' theorem and $d(\mathbb{X}^*\tilde{p}^*\Psi(g)) = p^*E(g)$ we get:

$$\begin{aligned}
R(X_2, g, \omega) &- R(X_1, g, \omega) = \\
&= \int_{I\times(M\setminus V)} d\big(p^*\omega' \wedge \mathbb{X}^*\tilde{p}^*\Psi(g)\big) + R(X_1, X_2, \omega) \\
&= -\int_{I\times M} p^*(\omega' \wedge E(g)) + R(X_1, X_2, \omega) \\
&= R(X_1, X_2, \omega)
\end{aligned}$$

For the last equality note that $\omega' \wedge E(g) = 0$ for dimensional reasons.

Still assuming that X_1 and X_2 have non-degenerate zeros we next treat the case of a general non-degenerate homotopy $\mathbb{X}$, whose zero set is not necessarily contained in a simply connected subset. Perturbing the homotopy slightly we may assume that no component of its zero set lies in a single $\{s\} \times M$. Then we certainly find $0 = t_0, \dots, t_k = 1$ so that Y_{t_i}, the restriction of $\mathbb{X}$ to $\{t_i\} \times M$, is transversal to the zero section, and so that $\mathbb{X}^{-1}(0) \cap ([t_{i-1}, t_i] \times M)$ is contained in a simply connected subset for every $1 \le i \le k$. The previous paragraph tells us

$$R(Y_{t_i}, g, \omega) - R(Y_{t_{i-1}}, g, \omega) = R(Y_{t_{i-1}}, Y_{t_i}, \omega)$$

for every $1 \leq i \leq k$. Therefore:

$$R(X_2, g, \omega) - R(X_1, g, \omega) = \sum_{i=1}^{k} R(Y_{t_{i-1}}, Y_{t_i}, \omega) = R(X_1, X_2, \omega)$$

It remains to deal with vector fields having degenerate but isolated singularities. Let X be such a vector field and let X' denote a perturbation as used before. Let $\mathcal{X}$ and $\mathcal{X}'$ denote their zero sets, respectively. Choose a function f such that $\omega' := \omega - df$ vanishes on the set B. Recall that B was the union of small balls covering $\mathcal{X}$. Since X and X' agree on $M \setminus B$ we have

$$\begin{aligned} R(X', g, \omega) - R(X, g, \omega) &= \sum_{x \in \mathcal{X}'} \mathrm{IND}_{X'}(x) f(x) - \sum_{x \in \mathcal{X}} \mathrm{IND}_X(x) f(x) \\ &= \int_{\tilde{\mathrm{cs}}(X, X')} df \\ &= R(X, X', \omega). \end{aligned}$$

This completes the proof of (13). $\square$

Remark 2. A similar definition of $R(X, g, \omega)$ works for any vector field X with arbitrary singularity set $\mathcal{X} := \{x \in M \mid X(x) = 0\}$ provided ω is exact when restricted to a sufficiently small neighborhood of $\mathcal{X}$.

4. Euler and co-Euler structures

Let (M, x_0) be a base pointed closed connected manifold of dimension n. Let X be a vector field and let $\mathcal{X}$ denote its zero set. Suppose the zeros of X are isolated and define

$$e_X := \sum_{x \in \mathcal{X}} \mathrm{IND}(x) x \in C_0(M; \mathbb{Z}),$$

a singular zero chain. An Euler chain for X is a singular one chain $c \in C_1(M; \mathbb{Z})$ so that

$$\partial c = e_X - \chi(M) x_0.$$

Since $\sum_{x \in \mathcal{X}} \mathrm{IND}_X(x) = \chi(M)$ every vector field with isolated zeros admits Euler chains.

Consider pairs (X, c) where X is a vector field with isolated zeros and c is an Euler chain for X. We call two such pairs (X_1, c_1) and (X_2, c_2) equivalent if

$$c_2 = c_1 + \mathrm{cs}(X_1, X_2) \in C_1(M; \mathbb{Z}) / \partial(C_2(M; \mathbb{Z})).$$

For the definition of $\mathrm{cs}(X_1, X_2)$ see section 3. We will write $\mathfrak{Eul}_{x_0}(M; \mathbb{Z})$ for the set of equivalence classes as above and $[X, c] \in \mathfrak{Eul}_{x_0}(M; \mathbb{Z})$ for the element represented by the pair (X, c). Elements of $\mathfrak{Eul}_{x_0}(M; \mathbb{Z})$ are called (integral) Euler structures of M based at x_0. There is an obvious $H_1(M; \mathbb{Z})$ action on $\mathfrak{Eul}_{x_0}(M; \mathbb{Z})$ defined by

$$[X, c] + [\sigma] := [X, c + \sigma],$$

where $[\sigma] \in H_1(M; \mathbb{Z})$ and $[X, c] \in \mathfrak{Eul}_{x_0}(M; \mathbb{Z})$. Obviously this action is free and transitive. In this sense $\mathfrak{Eul}_{x_0}(M; \mathbb{Z})$ is an affine version of $H_1(M; \mathbb{Z})$.

Considering Euler chains with real coefficients one obtains in exactly the same way an affine version of $H_1(M; \mathbb{R})$ which we will denote by $\mathfrak{Eul}_{x_0}(M; \mathbb{R})$. There is an obvious map $\mathfrak{Eul}_{x_0}(M; \mathbb{Z}) \to \mathfrak{Eul}_{x_0}(M; \mathbb{R})$ which is affine over the homomorphism $H_1(M; \mathbb{Z}) \to H_1(M; \mathbb{R})$.

Remark 3. There is another way of understanding the $H_1(M;\mathbb{Z})$ action on $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$. Suppose $n>2$ and represent $[\sigma]\in H_1(M;\mathbb{Z})$ by a simple closed curve σ. Choose a tubular neighborhood N of S^1 considered as vector bundle $N\to S^1$. Choose a fiber metric and a linear connection on N. Choose a representative of $[X,c]\in\mathfrak{Eul}(M,x_0)$ such that $X|_N=\frac{\partial}{\partial\theta}$, the horizontal lift of the canonic vector field on S^1. Choose a function $\lambda:[0,\infty)\to[-1,1]$, which satisfies $\lambda(r)=-1$ for $r\leq\frac{1}{3}$ and $\lambda(r)=1$ for $r\geq\frac{2}{3}$. Finally choose a function $\mu:[0,\infty)\to\mathbb{R}$ satisfying $\mu(r)=r$ for $r\leq\frac{1}{3}$, $\mu(r)=0$ for $r\geq\frac{2}{3}$ and $\mu(r)>0$ for all $r\in(\frac{1}{3},\frac{2}{3})$. Now construct a new vector field $\tilde X$ on M by setting

$$\tilde X:=\begin{cases}X & \text{on } M\setminus N\\ \lambda(r)\frac{\partial}{\partial\theta}+\mu(r)\frac{\partial}{\partial r} & \text{on } N,\end{cases}$$

where $r:N\to[0,\infty)$ denotes the radius function determined by the fiber metric on N and $-r\frac{\partial}{\partial r}$ is the Euler vector field of N. This construction is known as Reeb surgery, see e.g. [N03]. If the zeros of X are all non-degenerate the homotopy $X_t:=(1-t)X+t\tilde X$ is a non-degenerate homotopy from $X_0=X$ to $X_1=\tilde X$ from which one easily deduces that

$$[\tilde X,c]=[X,c]+[\sigma].$$

Particularly all the choices that entered the Reeb surgery do not effect the outcoming Euler structure $[\tilde X,c]$.

Let us consider a change of base point. Let $x_0,x_1\in M$ and choose a path σ from x_0 to x_1. Define

$$\mathfrak{Eul}_{x_0}(M;\mathbb{Z})\to\mathfrak{Eul}_{x_1}(M;\mathbb{Z}),\quad [X,c]\mapsto[X,c-\chi(M)\sigma]. \tag{14}$$

This is an $H_1(M;\mathbb{Z})$ equivariant bijection but depends on the homology class of σ.

Remark 4. So the identification $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ with $\mathfrak{Eul}_{x_1}(M;\mathbb{Z})$ does depend on the choice of a homology class of paths from x_0 to x_1. However, different choices will give identifications which differ by the action of an element in $\chi(M)H_1(M;\mathbb{Z})$. So the quotient $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})/\chi(M)H_1(M;\mathbb{Z})$ does not depend on the base point. Particularly, if $\chi(M)=0$ then $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ does not depend on the base point.

Let $\mathfrak{X}(M,x_0)$ denote the space of vector fields which vanish at x_0 and are non-zero elsewhere. We equip this space with the C^∞ topology, or any C^r topology, $r\geq 0$. Let $\pi_0(\mathfrak{X}(M,x_0))$ denote the space of homotopy classes of such vector fields. If $X\in\mathfrak{X}(M,x_0)$ we will write $[X]$ for the corresponding class in $\pi_0(\mathfrak{X}(M,x_0))$. The following proposition (due to Turaev in the case $\chi(M)=0$) establishes the proof of Theorem 2(i).

Proposition 3. *Suppose $n>2$. Then there exists a natural bijection*

$$\pi_0(\mathfrak{X}(M,x_0))=\mathfrak{Eul}_{x_0}(M;\mathbb{Z}),\qquad [X]\mapsto[X,0]. \tag{15}$$

Proof. Clearly (15) is well defined. Let us prove that it is onto. So let $[X,c]$ represent an Euler class. Choose an embedded disk $D\subseteq M$ centered at x_0 which contains all zeros of X and the Euler chain c. For this we may have to change c, but without changing the Euler structure $[X,c]$. Choose a vector field X' which equals X on $M\setminus D$ and vanishes just at x_0. Since $H_1(D;\mathbb{Z})=0$ we clearly have $[X',0]=[X,c]\in\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ and thus (15) is onto.

Let us prove injectivity of (15). Let $X_1, X_2 \in \mathfrak{X}(M, x_0)$ and suppose $\operatorname{cs}(X_1, X_2) = 0 \in H_1(M;\mathbb{Z})$. Let $D \subseteq M$ denote an embedded open disk centered at x_0. Consider the vector bundle $p^*TM \to I \times M$ and consider the two vector fields as a nowhere vanishing section of p^*TM defined over the set $\partial I \times \dot{M}$, where $\dot{M} := M \setminus D$. We would like to extend it to a nowhere vanishing section over $I \times \dot{M}$. The first obstruction we meet is an element in

$$\begin{aligned} H^n(I \times \dot{M}, \partial I \times \dot{M}; \{\pi_{n-1}\}) &= H_1(I \times \dot{M}, I \times \partial D; \mathbb{Z}) \\ &= H_1(M, \bar{D}; \mathbb{Z}) \\ &= H_1(M; \mathbb{Z}) \end{aligned}$$

which corresponds to $\operatorname{cs}(X_1, X_2) = 0$. Here $\{\pi_{n-1}\}$ denotes the system of local coefficients determined by the sphere bundle of p^*TM with $\pi_{n-1} = \pi_{n-1}(S^{n-1})$. Since this obstruction vanishes by hypothesis the next obstruction is defined and is an element in:

$$\begin{aligned} H^{n+1}(I \times \dot{M}, \partial I \times \dot{M}; \{\pi_n\}) &= H_0(I \times \dot{M}, I \times \partial D; \pi_n(S^{n-1})) \\ &= H_0(M, \bar{D}; \pi_n(S^{n-1})) \\ &= 0 \end{aligned}$$

Since there is no other obstructions, obstruction theory, see e.g. [W78], tells us that we find a nowhere vanishing section of p^*TM defined over $I \times \dot{M}$, which restricts to X_i on $\{i\} \times \dot{M}$, $i = 1, 2$. Such a section can easily be extended to a globally defined section of $p^*TM \to I \times M$, which restricts to X_i on $\{i\} \times M$, $i = 1, 2$ and whose zero set is precisely $I \times \{x_0\}$. Such a section can be considered as homotopy from X_1 to X_2 showing $[X_1] = [X_2]$. Hence (15) is injective. □

Remark 5. If $n > 2$ Reeb surgery defines an $H_1(M;\mathbb{Z})$ action on $\pi_0(\mathfrak{X}(M, x_0))$ which via (15) corresponds to the $H_1(M;\mathbb{Z})$ action on $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$, cf. Remark 3.

Let $\mathfrak{X}_0(M)$ denote the space of nowhere vanishing vector fields on M equipped with the C^∞ topology, or any C^r topology, $r \geq 0$. Let $\pi_0(\mathfrak{X}_0(M))$ denote the set of its connected components. The next proposition is a restatement of Theorem 2(ii).

Proposition 4. *If $n > 2$ then we have a surjection:*

$$\pi_0(\mathfrak{X}_0(M)) \to \mathfrak{Eul}_{x_0}(M;\mathbb{Z}), \qquad [X] \mapsto [X, 0]. \tag{16}$$

Proof. The assignment (16) is certainly well defined. Let us prove surjectivity. Let $[X, c]$ be an Euler structure. Choose an embedded disk $D \subseteq M$ which contains all zeros of X and its Euler chain c, cf. proof of Proposition 3. Since $\chi(M) = 0$ the degree of $X : \partial D \to TD \setminus 0_D$ vanishes. Modifying X only on D we get a nowhere vanishing X' which equals X on $M \setminus D$. Certainly X' has an Euler chain c' which is also contained in D and satisfies $[X, c] = [X', c']$. Since X' has no zeros we get $\partial c' = 0$ and since $H_1(D;\mathbb{Z}) = 0$ we arrive at $[X, c] = [X', c'] = [X', 0]$ which proves that (16) is onto. □

We will now describe another approach to Euler structures which is in some sense Poincaré dual to the other approach. We still consider a closed connected n–dimensional manifold with base point (M, x_0). Consider pairs (g, α) where g is a Riemannian metric on M and $\alpha \in \Omega^{n-1}(M \setminus x_0; \mathcal{O}_M)$ with $d\alpha = E(g)$. Here

$E(g) \in \Omega^n(M; \mathcal{O}_M)$ denotes the Euler class of g which is a form with values in the orientation bundle $\mathcal{O}_M$. We call two pairs (g_1, α_1) and (g_2, α_2) equivalent if

$$\mathrm{cs}(g_1, g_2) = \alpha_2 - \alpha_1 \in \Omega^{n-1}(M \setminus x_0; \mathcal{O}_M)/d\Omega^{n-2}(M \setminus x_0; \mathcal{O}_M).$$

We will write $\mathfrak{Eul}^*_{x_0}(M; \mathbb{R})$ for the set of equivalence classes and $[g, \alpha]$ for the equivalence class represented by the pair (g, α). Elements of $\mathfrak{Eul}^*_{x_0}(M; \mathbb{R})$ are called co-Euler structures based at x_0. There is a natural $H^{n-1}(M; \mathcal{O}_M)$ action on $\mathfrak{Eul}^*_{x_0}(M; \mathbb{R})$ given by

$$[g, \alpha] + [\beta] := [g, \alpha - \beta]$$

with $[\beta] \in H^{n-1}(M; \mathcal{O}_M)$. Since $H^{n-1}(M; \mathcal{O}_M) = H^{n-1}(M \setminus x_0; \mathcal{O}_M)$ this action is obviously free and transitive.

For a pair (g, α) as above and a closed one form ω we define a regularization of $\int_M \omega \wedge \alpha$ as follows. Choose a function f such that $\omega' := \omega - df$ vanishes locally around the base point x_0 and set:

$$S(g, \alpha, \omega; f) := \int_M \omega' \wedge \alpha - \int_M f E(g) + \chi(M) f(x_0)$$

Lemma 2. *The quantity $S(g, \alpha, \omega; f)$ does not depend on the choice of f and will thus be denoted by $S(g, \alpha, \omega)$. If $[g_1, \alpha_1] = [g_2, \alpha_2] \in \mathfrak{Eul}^*_{x_0}(M; \mathbb{R})$ then*

$$S(g_2, \alpha_2, \omega) - S(g_1, \alpha_1, \omega) = \int_M \omega \wedge \mathrm{cs}(g_1, g_2). \tag{17}$$

Moreover, for a function h we have

$$S(g, \alpha, \omega + dh) - S(g, \alpha, \omega) = -\int_M h E(g) + \chi(M) h(x_0). \tag{18}$$

Proof. Suppose we have two functions f_1 and f_2 so that both $\omega_1' := \omega - df_1$ and $\omega_2' := \omega - df_2$ vanish locally around x_0. Let B_ϵ denote a ball of radius ϵ around x_0. Then $f_2 - f_1$ will be constant on B_ϵ for ϵ sufficiently small. Using Stokes' theorem, $d\alpha = E(g)$ and $\int_M E(g) = \chi(M)$ we get:

$$\begin{aligned}
&S(g, \alpha, \omega; f_2) - S(g, \alpha, \omega; f_1) = \\
&= \; -\int_{M \setminus \mathcal{X}} d((f_2 - f_1) \wedge \alpha) + \chi(M)(f_2 - f_1)(x_0) \\
&= \; -\lim_{\epsilon \to 0} \int_{\partial(M \setminus B_\epsilon)} (f_2 - f_1)\alpha + \chi(M)(f_2 - f_1)(x_0) \\
&= \; -(f_2 - f_1)(x_0) \lim_{\epsilon \to 0} \int_{\partial(M \setminus B_\epsilon)} \alpha + \chi(M)(f_2 - f_1)(x_0) \\
&= \; -(f_2 - f_1)(x_0) \lim_{\epsilon \to 0} \int_{M \setminus B_\epsilon} E(g) + \chi(M)(f_2 - f_1)(x_0) = 0
\end{aligned}$$

The second statement follows immediately from $\alpha_2 - \alpha_1 = \mathrm{cs}(g_1, g_2)$, Stokes' theorem and $d\,\mathrm{cs}(g_1, g_2) = E(g_2) - E(g_1)$. The last property is obvious. □

In view of (8), (12), (13), (17) and (18) the quantity

$$R(X, g, \omega) - S(g, \alpha, \omega) - \int_c \omega \tag{19}$$

does only depend on $[X,c] \in \mathfrak{Eul}_{x_0}(M;\mathbb{R})$, $[g,\alpha] \in \mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ and $[\omega] \in H^1(M;\mathbb{R})$. Thus (19) defines a coupling

$$\mathbb{T} : \mathfrak{Eul}_{x_0}(M;\mathbb{R}) \times \mathfrak{Eul}^*_{x_0}(M;\mathbb{R}) \to H_1(M;\mathbb{R}).$$

¿From the very definition we have

$$\langle [\omega], \mathbb{T}([X,c],[g,\alpha]) \rangle = \int_M \omega \wedge (X^*\Psi(g) - \alpha) - \int_c \omega, \tag{20}$$

where ω is any representative of $[\omega]$ which vanishes locally around the zeros of X and vanishes locally around the base point x_0. Moreover, we have

$$\mathbb{T}(\mathfrak{e} + \sigma, \mathfrak{e}^* + \beta) = \mathbb{T}(\mathfrak{e}, \mathfrak{e}^*) - \sigma + \mathrm{PD}(\beta) \tag{21}$$

for all $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M;\mathbb{R})$, $\mathfrak{e}^* \in \mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$, $\sigma \in H_1(M;\mathbb{R})$ and $\beta \in H^{n-1}(M;\mathcal{O}_M)$. Here PD is the Poincaré duality isomorphism $\mathrm{PD} : H^{n-1}(M;\mathcal{O}_M) \to H_1(M;\mathbb{R})$.

We have the following affine version of Poincaré duality, which establishes the proof of Theorem 2(iii) and (iv).

Proposition 5. *There is a natural isomorphism of affine spaces*

$$P : \mathfrak{Eul}^*_{x_0}(M;\mathbb{R}) \to \mathfrak{Eul}_{x_0}(M;\mathbb{R})$$

which is affine over the Poincaré duality $\mathrm{PD} : H^{n-1}(M;\mathcal{O}_M) \to H_1(M;\mathbb{R})$. *In other words, for every $\beta \in H^{n-1}(M;\mathcal{O}_M)$ and every $\mathfrak{e}^* \in \mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ we have*

$$P(\mathfrak{e}^* + \beta) = P(\mathfrak{e}^*) + \mathrm{PD}(\beta). \tag{22}$$

Moreover, $\mathbb{T}(\mathfrak{e}, \mathfrak{e}^*) = P(\mathfrak{e}^*) - \mathfrak{e}$.

Proof. Given $\mathfrak{e}^* = [g,\alpha] \in \mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ we choose a vector field X with isolated singularities $\mathcal{X}$. Then $X^*\Psi(g) - \alpha$ is closed and thus defines a cohomology class in $H^{n-1}(M \setminus (\mathcal{X} \cup \{x_0\});\mathcal{O}_M)$. We would like to define $P(\mathfrak{e}^*) := [X,c]$ where c be a representative of its Poincaré dual in $H_1(M, \mathcal{X} \cup \{x_0\};\mathbb{R})$. That is, we ask

$$\int_c \omega = \int_{M \setminus (\mathcal{X} \cup \{x_0\})} \omega \wedge (X^*\Psi(g) - \alpha)$$

to hold for every closed compactly supported one form ω on $M \setminus (\mathcal{X} \cup \{x_0\})$. In view of (20) this is equivalent to ask for $\mathbb{T}(P(\mathfrak{e}^*), \mathfrak{e}^*) = 0$. So we take the latter one as our definition of P. Because of (21) this has a unique solution. The equivariance property and the last equation follow at once. ☐

5. Smooth triangulations and extension of Chern–Simons theory

Smooth triangulations. Smooth triangulations provide a remarkable source of vector fields with isolated singularities.

To any smooth triangulation τ of the smooth manifold M one can associate a Lefschetz vector field X_τ called *Euler vector field*, with the following properties:

- P1: The zeros of X_τ are all non-degenerate and are exactly the barycenters x_σ of the simplexes σ.
- P2: For each zero x_σ the unstable set with respect to $-X_\tau$ coincides in a neighborhood of x_σ to the open simplex σ, consequently the zeros are hyperbolic. The Morse index of $-X_\tau$ at x_σ equals $\dim(\sigma)$ and the (Hopf) index of X_τ at x_σ equals $(-1)^{\dim(\sigma)}$.

P3: The piecewise differential function $f_\tau : M \to \mathbb{R}$ defined by $f_\tau(x_\sigma) = \dim(\sigma)$ and extended by linearity on M is a Lyapunov function for $-X_\tau$, i.e. strictly decreasing on non-constant trajectories of $-X_\tau$.

Such a vector field X_τ is unique up to an homotopy of vector fields which satisfy P1–P3. The convex combination provides the homotopy between any two such vector fields.

To construct such a vector field we begin with a standard simplex Δ_n of vectors $(t_0, \dots, t_n) \in \mathbb{R}^{n+1}$ satisfying $0 \le t_i \le 1$ and $\sum t_i = 1$.

(i) Let E_n denote the Euler vector field of the corresponding affine space ($\sum t_i = 1$) centered at the barycenter O (of coordinates $(1/(n+1), \dots, 1/(n+1))$ and restricted to Δ_n.
(ii) Let $e : \Delta_n \to [0, 1]$ denote the function which is 1 on the barycenter O and zero on all vertices.
(iii) Let $r : \Delta_n \setminus \{O\} \to \partial\Delta_n$ denote the radial retraction to the boundary.

Set $X'_n := e \cdot E_n$, which is a vector field on Δ_n.

By induction we will construct a canonical vector field X_n on Δ_n which at any point $x \in \Delta_n$ is tangent to the open face the point belongs to and vanishes only at the barycenter of each face. We proceed as follows:

Suppose we have constructed such canonical vector fields on all $\Delta(k)$, $k \le n-1$. Using the canonical vector fields X_{n-1} we define the vector field X_n on the boundary $\partial\Delta_n$ and extend it to the vector field X''_n by taking at each point $x \in \Delta_n$ the vector parallel to $X_n(r(x))$ multiplied by the function $(1-e)$ and at O the vector zero. Clearly such vector field vanishes on the radii $\overline{OP}$ (P the barycenter of any face). We finally put

$$X_n := X'_n + X''_n.$$

The vector field X_n is continuous and piecewise differential (actually Lipschitz) and has a well defined continuous flow.

Putting together the vector fields X_n on all simplexes (cells) we provide a piecewise differential (and Lipschitz) vector field X on any simplicial (cellular) complex or polyhedron and in particular on any smoothly triangulated manifold. The vector field X has a flow and f_τ is a Lyapunov function for $-X$. The vector field X is not necessary smooth but by a small (Lipschitz) perturbation we can approximate it by a smooth vector field X_τ which satisfies P1–P3. Any of the resulting vector fields is referred to as the Euler vector field of a smooth triangulation τ. It was pointed out to us that the vector field X_τ has first appeared in [S99].

Extension of Chern–Simons theory. Let M be a closed manifold of dimension n. We equip $\Omega^k(M;\mathbb{R})$ with the C^∞ topology. The continuous linear functionals on $\Omega^k(M;\mathbb{R})$ are called k currents and denoted by $\mathcal{D}_k(M)$. Consider $\delta : \mathcal{D}_k(M) \to \mathcal{D}_{k-1}(M)$ given by $(\delta\varphi)(\alpha) := \varphi(d\alpha)$. Clearly $\delta^2 = 0$.[3]

We have a morphism of chain complexes

$$C_*(M;\mathbb{R}) \to \mathcal{D}_*(M), \quad \sigma \mapsto \hat\sigma, \quad \hat\sigma(\alpha) := \int_\sigma \alpha.$$

[3] The chain complex $(\mathcal{D}_*(M), \delta)$ computes the homology of M with real coefficients.

Her $C_*(M;\mathbb{R})$ denotes the space of singular chains with real coefficients. Moreover, we have a morphism of chain complexes

$$\Omega^{n-*}(M;\mathcal{O}_M)\to\mathcal{D}_*(M),\quad \beta\mapsto\hat\beta,\quad \hat\beta(\alpha):=(-1)^{\frac{1}{2}|\alpha|(|\alpha|+1)}\int_M\alpha\wedge\beta.$$

Here $|\alpha|$ denotes the degree of α. The sign is necessary so that this mappings actually intertwines the two differentials d and δ.

Every vector field with isolated singularities X gives rise to a zero chain e_X, cf. section 4. Via the first morphism we get a zero current $\hat E(X)$. More explicitly $(\hat E(X))(h)=\sum_{x\in\mathcal{X}}\mathrm{IND}(x)h(x)$ for a function $h\in\Omega^0(M;\mathbb{R})$.

A Riemannian metric g has an Euler form $E(g)\in\Omega^n(M;\mathcal{O}_M)$. Via the second morphism we get a zero current $\hat E(g)$. More explicitly $(\hat E(g))(h)=\int_M hE(g)$ for a function $h\in\Omega^0(M;\mathbb{R})$.

Let $\mathcal{Z}^k(M;\mathbb{R})\subseteq\Omega^k(M;\mathbb{R})$ denote the space of closed k forms on M equipped with the C^∞ topology. The continuous linear functionals on $\mathcal{Z}^k(M;\mathbb{R})$ are referred to as *k currents rel. boundary* and identify to $\mathcal{D}_k(M)/\delta(\mathcal{D}_{k+1}(M))$. The two chain morphisms provide mappings

$$C_k(M;\mathbb{R})/\partial(C_{k+1}(M;\mathbb{R}))\to\mathcal{D}_k(M)/\delta(\mathcal{D}_{k+1}(M))\tag{23}$$

and

$$\Omega^{n-k}(M;\mathcal{O}_M)/d(\Omega^{n-k-1}(M;\mathcal{O}_M))\to\mathcal{D}_k(M)/\delta(\mathcal{D}_{k+1}(M))\tag{24}$$

For two vector fields with isolated zeros X_1 and X_2 we have constructed $\mathrm{cs}(X_1,X_2)\in C_1(M;\mathbb{Z})/\partial(C_2(M;\mathbb{Z}))$, cf. section 3. This gives rise to $\mathrm{cs}(X_1,X_2)\in C_1(M;\mathbb{R})/\partial(C_2(M;\mathbb{R}))$, and via (23) we get a one current rel. boundary which we will denote by $\hat{\mathrm{cs}}(X_1,X_2)$. More precisely, $(\hat{\mathrm{cs}}(X_1,X_2))(\omega)=\int_{\mathrm{cs}(X_1,X_2)}\omega$ for a closed one form $\omega\in\mathcal{Z}^1(M;\mathbb{R})$. Recall that $\mathrm{cs}(X_2,X_1)=-\mathrm{cs}(X_1,X_2)$, $\mathrm{cs}(X_1,X_3)=\mathrm{cs}(X_1,X_2)+\mathrm{cs}(X_2,X_3)$, $\partial\,\mathrm{cs}(X_1,X_2)=e_{X_2}-e_{X_1}$ and thus

$$\begin{aligned}\hat{\mathrm{cs}}(X_2,X_1)&=-\hat{\mathrm{cs}}(X_1,X_2)\\ \hat{\mathrm{cs}}(X_1,X_3)&=\hat{\mathrm{cs}}(X_1,X_2)+\hat{\mathrm{cs}}(X_2,X_3)\\ \delta\hat{\mathrm{cs}}(X_1,X_2)&=\hat E(X_2)-\hat E(X_1).\end{aligned}$$

For two Riemannian metrics g_1 and g_2 we have the Chern–Simons form $\mathrm{cs}(g_1,g_2)\in\Omega^{n-1}(M;\mathcal{O}_M)/d(\Omega^{n-2}(M;\mathcal{O}_M))$. Via (24) we get a one current rel. boundary which we will denote by $\hat{\mathrm{cs}}(g_1,g_2)$. More precisely $(\hat{\mathrm{cs}}(g_1,g_2))(\omega)=-\int_M\omega\wedge\mathrm{cs}(g_1,g_2)$ for a closed one form $\omega\in\mathcal{Z}^1(M;\mathbb{R})$. Recall that $\mathrm{cs}(g_2,g_1)=-\mathrm{cs}(g_1,g_2)$, $\mathrm{cs}(g_1,g_3)=\mathrm{cs}(g_1,g_2)+\mathrm{cs}(g_2,g_3)$, $d\,\mathrm{cs}(g_1,g_2)=E(g_2)-E(g_1)$ and thus

$$\begin{aligned}\hat{\mathrm{cs}}(g_2,g_1)&=-\hat{\mathrm{cs}}(g_1,g_2)\\ \hat{\mathrm{cs}}(g_1,g_3)&=\hat{\mathrm{cs}}(g_1,g_2)+\hat{\mathrm{cs}}(g_2,g_3)\\ \delta\hat{\mathrm{cs}}(g_1,g_2)&=\hat E(g_2)-\hat E(g_1)\end{aligned}$$

Suppose X is a vector field with isolated zeros and g is a Riemannian metric. We define one currents rel. boundary by $(\hat{\mathrm{cs}}(g,X))(\omega):=R(X,g,\omega)$ and $\hat{\mathrm{cs}}(X,g):=-\hat{\mathrm{cs}}(g,X)$. Proposition 1 and Proposition 2 tell that

$$\begin{aligned}\delta\hat{\mathrm{cs}}(g,X)&=\hat E(X)-\hat E(g)\\ \hat{\mathrm{cs}}(g_1,X)&=\hat{\mathrm{cs}}(g_1,g_2)+\hat{\mathrm{cs}}(g_2,X)\\ \hat{\mathrm{cs}}(g,X_2)&=\hat{\mathrm{cs}}(g,X_1)+\hat{\mathrm{cs}}(X_1,X_2)\end{aligned}$$

We summarize these observations in

Proposition 6. *Let any of the symbols x, y, z denote either a Riemannian metric g or a vector field with isolated zeros. Then one has:*

(i) $\hat{\mathrm{cs}}(y,x) = -\hat{\mathrm{cs}}(x,y)$
(ii) $\hat{\mathrm{cs}}(x,z) = \hat{\mathrm{cs}}(x,y) + \hat{\mathrm{cs}}(y,z)$
(iii) $\delta\hat{\mathrm{cs}}(x,y) = \hat{E}(y) - \hat{E}(x)$.

Suppose τ is a smooth triangulation. We define its Euler current by $\hat{E}(\tau) := \hat{E}(X_\tau)$, where X_τ is the Euler vector field. Similarly for two triangulations τ_1 and τ_2 we define a one current rel. boundary by $\hat{\mathrm{cs}}(\tau_1,\tau_2) := \hat{\mathrm{cs}}(X_{\tau_1}, X_{\tau_2})$.

Corollary 1. *Let any of the symbols x, y, z denote either a Riemannian metric g or a smooth triangulation. Then one has:*

(i) $\hat{\mathrm{cs}}(y,x) = -\hat{\mathrm{cs}}(x,y)$
(ii) $\hat{\mathrm{cs}}(x,z) = \hat{\mathrm{cs}}(x,y) + \hat{\mathrm{cs}}(y,z)$
(iii) $\delta\hat{\mathrm{cs}}(x,y) = \hat{E}(y) - \hat{E}(x)$.

6. Theorem of Bismut–Zhang

Let (M, x_0) be a closed connected manifold with base point. Let $\mathbb{K}$ be a field of characteristic zero, and suppose F is a flat $\mathbb{K}$ vector bundle over M, that is F is equipped with a flat connection ∇. Let F_{x_0} denote the fiber over the base point x_0. Holonomy at the base point provides a right $\pi_1(M,x_0)$ action on F_{x_0} and when composed with the inversion in $\mathrm{GL}(F_{x_0})$ a representation $\rho_F : \pi_1(M,x_0) \to \mathrm{GL}(F_{x_0})$. So we get a homomorphism $\det \circ \rho_F : \pi_1(M,x_0) \to \mathbb{K}^*$ which descends to a homomorphism $H_1(M;\mathbb{Z}) \to \mathbb{K}^*$ and thus determines a cohomology class $\Theta_F \in H^1(M;\mathbb{K}^*)$.

Suppose we have a smooth triangulation τ of M. It gives rise to a cellular complex $C^*_\tau(M;F)$ which computes the cohomology $H^*(M;F)$. Let $\mathcal{X}_\tau$ denote the set of barycenters of τ. For a cell σ of τ we let x_σ denote the barycenter of σ. Let X_τ denote the Euler vector field of τ, cf. section 5. Then $\mathcal{X}_\tau$ is the zero set of X_τ. Moreover, for a cell σ we have $\mathrm{IND}_{X_\tau}(\sigma_x) = (-1)^{\dim\sigma}$. As a graded vector space we have $C^k_\tau(M;F) = \bigoplus_{\dim\sigma=k} F_{x_\sigma}$. So we get a canonical isomorphism of $\mathbb{K}$ vector spaces:

$$\det C^*_\tau(M;F) = \det H(C^*_\tau(M;F)) = \det H^*(M;F) \tag{25}$$

Recall that the determinant line of a vector space W is by definition $\det W := \Lambda^{\dim W} W$. For a $\mathbb{Z}$ graded vector space V^* one sets $V^{\mathrm{even}} := \bigoplus_{k \text{ even}} V^k$, $V^{\mathrm{odd}} := \bigoplus_{k \text{ odd}} V^k$ and defines its determinant line by $\det V^* := \det V^{\mathrm{even}} \otimes (\det V^{\mathrm{odd}})^*$.

Suppose we have given an Euler structure $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M;\mathbb{Z})$. For every $x \in \mathcal{X}_\tau$ choose a path π_x from x_0 to x, so that with $c := \sum_{x\in\mathcal{X}_\tau} \mathrm{IND}_{X_\tau}(x_\sigma)\pi_x$ we have $\mathfrak{e} = [X_\tau, c]$.[4] Let f_0 be a non-zero element in $\det F_{x_0}$. Note that a frame (basis) in F_{x_0} determines such an element in $\det F_{x_0}$. Using parallel transport along π_x we get a non-zero element in every $\det F_{x_\sigma}$. If the barycenters x_σ where ordered we would get a well defined non-zero element in $\det C^*_\tau(M;F)$.

Suppose $\mathfrak{o}$ is a cohomology orientation of M, i.e. an orientation of $\det H^*(M;\mathbb{R})$. We say an ordering of the zeros x_σ is compatible with $\mathfrak{o}$ if the non-zero element in $\det C^*_\tau(M;\mathbb{R})$ provided by this ordered base is compatible with the orientation $\mathfrak{o}$ via the canonic isomorphism

$$\det C^*_\tau(M;\mathbb{R}) = \det H(C^*_\tau(M;\mathbb{R})) = \det H^*(M;\mathbb{R}).$$

[4] Such a representative for the Euler structure is called spray or Turaev spider.

So an integral Euler structure $\mathfrak{e}$, a cohomology orientation $\mathfrak{o}$ and an element $f_0 \in \det F_{x_0}$ provide a non-zero element in $\det C^*_\tau(M;F)$ which corresponds to a non-zero element in $\det H^*(M;F)$ via (25). We thus get a mapping

$$\det F_{x_0} \setminus 0 \to \det H^*(M;F) \setminus 0. \tag{26}$$

This mapping is obviously homogeneous of degree $\chi(M)$. A straight forward calculation shows that it does not depend on the choice of π_x. As a matter of fact this mapping does not depend on τ either, only on the Euler structure $\mathfrak{e}$ and the cohomology orientation $\mathfrak{o}$. This is a non-trivial fact, and its proof is contained in [M66] and [T86] for acyclic case and implicit in the existing literature cf. [FT00] and [B99]. We define the *combinatorial torsion* to be the element

$$\tau^{\mathrm{comb}}_{F,\mathfrak{e},\mathfrak{o}} \in \det H^*(M;F) \otimes (\det F_{x_0})^{-\chi(M)}$$

corresponding to the homogeneous mapping (26). Note that we also have

$$\tau^{\mathrm{comb}}_{F,\mathfrak{e}+\sigma,\mathfrak{o}} = \tau^{\mathrm{comb}}_{F,\mathfrak{e},\mathfrak{o}} \cdot \langle \Theta_F, \sigma\rangle^{-1},$$

for all $\sigma \in H_1(M;\mathbb{Z})$. Here $\langle\cdot,\cdot\rangle$ denotes the natural pairing of homology with integer coefficients and cohomology with coefficients in the Abelian group $\mathbb{K}^*$. Moreover

$$\tau^{\mathrm{comb}}_{F,\mathfrak{e},-\mathfrak{o}} = (-1)^{\mathrm{rank}\, F}\tau^{\mathrm{comb}}_{F,\mathfrak{e},\mathfrak{o}}.$$

Clearly, if $\chi(M) = 0$ then $\tau^{\mathrm{comb}}_{F,\mathfrak{e},\mathfrak{o}} \in \det H^*(M;F)$.

Now consider the case when $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. Let μ be a Hermitian structure, i.e. fiber wise Hermitian scalar product, on F. It induces a scalar product on $\det C^*_\tau(M;F)$ and via (25) a scalar product $||\cdot||^{\mathcal{M}}_{F,\tau,\mu}$ on the line $\det H^*(M;F)$. This is exactly what is called *Milnor metric* in [BZ92]. The Hermitian structure μ also defines a scalar product on $(\det F_{x_0})^{-\chi(M)}$ which we will denote by $||\cdot||_{\mu_{x_0}}$. Moreover, μ gives rise to a closed one form $\omega(\nabla,\mu)$, where ∇ is the flat connection of F, see [BZ92] and section 2. For its cohomology class we have $[\omega(\nabla,\mu)] = (\log|\cdot|)_*\Theta_F$. Here $(\log|\cdot|)_* : H^1(M;\mathbb{C}^*) \to H^1(M;\mathbb{R})$ in the complex case, and $(\log|\cdot|)_* : H^1(M;\mathbb{R}^*) \to H^1(M;\mathbb{R})$ in the real case. Given an Euler structure with real coefficients $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M;\mathbb{R})$ we choose an Euler chain c so that $[X_\tau, c] = \mathfrak{e}$, and define a metric on $\det H^*(M;F) \otimes (\det F_{x_0})^{-\chi(M)}$ by:

$$||\cdot||^{\mathrm{comb}}_{F,\mathfrak{e}} := ||\cdot||^{\mathcal{M}}_{F,\tau,\mu} \otimes ||\cdot||_{\mu_{x_0}} \cdot e^{\int_c \omega(\nabla,\mu)} \tag{27}$$

As the notation indicates this does not depend on the cohomology orientation, is independent of μ and does only depend on the Euler structure $\mathfrak{e} = [X, c]$. This follows from known anomaly formulas for the Milnor torsion, implicit in [BZ92], or can be seen as a consequence of (28) and (29) below. Note that

$$||\cdot||^{\mathrm{comb}}_{F,\mathfrak{e}+\sigma} = ||\cdot||^{\mathrm{comb}}_{F,\mathfrak{e}} \cdot e^{\langle(\log|\cdot|)_*\Theta_F,\sigma\rangle} \tag{28}$$

for all $\sigma \in H_1(M;\mathbb{R})$. For an integral Euler structure $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ we have

$$||\tau^{\mathrm{comb}}_{F,\mathfrak{e},\mathfrak{o}}||^{\mathrm{comb}}_{F,\mathfrak{e}} = 1. \tag{29}$$

Here, abusing notation, $\mathfrak{e}$ at the same time denotes its image in $\mathfrak{Eul}_{x_0}(M;\mathbb{R})$.

Now let g be a Riemannian metric on M. Then we also have the Ray–Singer metric $||\cdot||^{\mathrm{RS}}_{F,g,\mu}$ on $\det H^*(M;F)$, cf. [BZ92]. Let $\mathfrak{e}^* \in \mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ and suppose $[g,\alpha] = \mathfrak{e}^*$, i.e. $d\alpha = E(g)$. Define a metric on $\det H^*(M;F) \otimes (\det F_{x_0})^{-\chi(M)}$ by:

$$||\cdot||^{\mathrm{an}}_{F,\mathfrak{e}^*} := ||\cdot||^{\mathrm{RS}}_{F,g,\mu} \otimes ||\cdot||_{\mu_{x_0}} \cdot e^{-S(g,\alpha,\omega(\nabla,\mu))} \tag{30}$$

We call this metric the *modified Ray–Singer metric.*

The known anomaly formulas for the Ray–Singer torsion, see [BZ92], imply that this is independent of μ and only depends on the co-Euler structure $\mathfrak{e}^*$. Note that

$$||\cdot||^{\mathrm{an}}_{F,\mathfrak{e}^*+\beta} = ||\cdot||^{\mathrm{an}}_{F,\mathfrak{e}^*} \cdot e^{\langle(\log|\cdot|)_*\Theta_F, \mathrm{PD}(\beta)\rangle} \tag{31}$$

for all $\beta \in H^{n-1}(M;\mathcal{O}_M)$. The main theorem of Bismut–Zhang, see [BZ92], can now be reformulated as follows:

Theorem 3 (Bismut–Zhang)**.** *Suppose (M, x_0) is a closed connected manifold with base point and F a flat real or complex vector bundle over M. Let $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M;\mathbb{R})$ be an Euler structure with real coefficients, and let $\mathfrak{e}^* \in \mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ be a co-Euler structure, both based at x_0. Then one has:*

$$||\cdot||^{\mathrm{an}}_{F,\mathfrak{e}^*} = ||\cdot||^{\mathrm{comb}}_{F,\mathfrak{e}} \cdot e^{\langle(\log|\cdot|)_*\Theta_F, \mathbb{T}(\mathfrak{e},\mathfrak{e}^*)\rangle}$$

Particularly, if $\mathfrak{e} = P(\mathfrak{e}^*)$ then $||\cdot||^{\mathrm{an}}_{F,\mathfrak{e}^*} = ||\cdot||^{\mathrm{comb}}_{F,\mathfrak{e}}$.

For an alternative proof of the (original) Bismut–Zhang theorem see also [BFK01].

Appendix A. Complex versus real torsion

Suppose V is a finite dimensional complex vector space. Let $V_{\mathbb{R}}$ denote the vector space V considered as real vector space. We have a mapping

$$\begin{aligned} \theta_V : \det V &\to \det(V_{\mathbb{R}}) \\ v_1 \wedge v_2 \wedge \cdots \wedge v_n &\mapsto v_1 \wedge iv_1 \wedge v_2 \wedge iv_2 \wedge \cdots \wedge v_n \wedge iv_n. \end{aligned}$$

It has the property

$$\theta_V(z\alpha) = |z|^2\theta_V(\alpha),$$

for all $z \in \mathbb{C}$ and $\alpha \in \det V$. If $f : V \to W$ is a complex linear mapping then the following diagram commutes:

$$\begin{tikzcd} \det V \arrow[r, "\theta_V"] \arrow[d, "\det f"'] & \det(V_{\mathbb{R}}) \arrow[d, "\det(f_{\mathbb{R}})"] \\ \det W \arrow[r, "\theta_W"] & \det(W_{\mathbb{R}}) \end{tikzcd}$$

After identifying $\det\mathbb{C} = \mathbb{C}$ and $\det(\mathbb{C}_{\mathbb{R}}) = \Lambda^2\mathbb{R}^2 = \mathbb{R}$ we have

$$\theta_{\mathbb{C}} : \mathbb{C} \to \mathbb{R}, \qquad \theta_{\mathbb{C}}(z) = |z|^2.$$

Suppose L is a complex line, R a real line and $\theta : L \to R$ a mapping which satisfies

$$\theta(z\lambda) = |z|^2\theta(\lambda), \tag{32}$$

for all $z \in \mathbb{C}$ and all $\lambda \in L$. If L' is another complex line, R' another real line and $\theta' : L' \to R'$ another mapping which satisfies (32) we can define

$$\theta \otimes \theta' : L \otimes L' \to R \otimes R', \qquad (\theta \otimes \theta')(\lambda \otimes \lambda') := \theta(\lambda) \otimes \theta'(\lambda')$$

which again satisfies (32) Note that

$$\begin{tikzcd} L \otimes \mathbb{C} \arrow[r, equal] \arrow[d, "\theta\otimes\theta_{\mathbb{C}}"'] & L \arrow[d, "\theta"] \\ R \otimes \mathbb{R} \arrow[r, equal] & R \end{tikzcd}$$

commutes. If $0 \to V \to W \to U \to 0$ is a short exact sequence of complex vector spaces we have a commutative diagram:

$$\begin{array}{ccc}
\det V \otimes \det U & = & \det W \\
{\scriptstyle \theta_V \otimes \theta_U}\downarrow & & \downarrow{\scriptstyle \theta_W} \\
\det(V_{\mathbb{R}}) \otimes \det(U_{\mathbb{R}}) & = & \det(W_{\mathbb{R}})
\end{array}$$

Note that for a complex vector space V we have a canonic isomorphism

$$(V^*)_{\mathbb{R}} = (V_{\mathbb{R}})^*, \qquad \varphi \mapsto \Re \circ \varphi.$$

Using this identification we get a commutative diagram:

$$\begin{array}{ccccc}
\det V \otimes \det(V^*) & = & \det V \otimes (\det V)^* & = & \mathbb{C} \\
{\scriptstyle \theta_V \otimes \theta_{V^*}}\downarrow & & & & \downarrow{\scriptstyle \theta_{\mathbb{C}}} \\
\det(V_{\mathbb{R}}) \otimes \det((V^*)_{\mathbb{R}}) & = & \det(V_{\mathbb{R}}) \otimes (\det(V_{\mathbb{R}}))^* & = & \mathbb{R}
\end{array}$$

Putting all this together we obtain

Proposition 7. *Let C^* be a finite dimensional chain complex over $\mathbb{C}$. Let $C^*_{\mathbb{R}}$ denote the same chain complex viewed as chain complex over $\mathbb{R}$. Then $H(C^*_{\mathbb{R}}) = H(C^*)_{\mathbb{R}}$, and we have a commutative diagram:*

$$\begin{array}{ccc}
\det C^* & = & \det H(C^*) \\
{\scriptstyle \theta_{C^*}}\downarrow & & \downarrow{\scriptstyle \theta_{H(C^*)}} \\
\det(C^*_{\mathbb{R}}) & = & \det H(C^*_{\mathbb{R}})
\end{array}$$

Now suppose F is a flat complex vector bundle over a closed manifold (M, x_0) with base point. Let $F_{\mathbb{R}}$ denote the vector bundle F considered as real bundle. Recall the mappings (26) from section 6. Clearly $H^*(M;F)_{\mathbb{R}} = H^*(M;F_{\mathbb{R}})$. Let $A := \theta_{H^*(M;F)} \otimes (\theta_{F_{x_0}})^{-\chi(M)}$ denote the canonical mapping:

$$\det H^*(M;F) \otimes (\det F_{x_0})^{-\chi(M)} \xrightarrow{A} \det H^*(M;F_{\mathbb{R}}) \otimes (\det(F_{\mathbb{R}})_{x_0})^{-\chi(M)}$$

In this situation we obviously we have

Proposition 8. *a) For an integral Euler structure $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ and a cohomology orientation $\mathfrak{o}$ we have $A(\tau^{\text{comb}}_{F,\mathfrak{e},\mathfrak{o}}) = \tau^{\text{comb}}_{F_{\mathbb{R}},\mathfrak{e},\mathfrak{o}}$. b) For an Euler structure with real coefficients $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M;\mathbb{R})$ we have $||\cdot||^{\text{comb}}_{F_{\mathbb{R}},\mathfrak{e}} \circ A = (||\cdot||^{\text{comb}}_{F,\mathfrak{e}})^2$. c) For a co-Euler structure $\mathfrak{e}^* \in \mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ we have $||\cdot||^{\text{an}}_{F_{\mathbb{R}},\mathfrak{e}^*} \circ A = (||\cdot||^{\text{an}}_{F,\mathfrak{e}^*})^2$.*

Note that the previous proposition and the real version of Theorem 3 imply the complex version of Theorem 3.

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