

VIRTUALLY SMALL SPECTRAL PACKAGE OF A RIEMANNIAN MANIFOLD

DAN BURGHELEA, YOONWEON LEE

ABSTRACT. For a Morse function f on a closed orientable Riemannian manifold (M, g) one introduces the *virtually small spectral package*, an analytic object consisting of a finite number of analytic quantities derived from (g, f) which, in principle, can be calculated. One shows that they determine the *Torsion* of the underlying space M , a parallel to the result that the dimensions of the spaces of harmonic forms calculate the Euler-Poincaré characteristic of M and extend the Poincaré duality between harmonic forms and between Betti numbers for a closed oriented Riemannian manifold .

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1. INTRODUCTION

For a compact ANR X , the integral homology $H_r(X; Z)$ is a finitely generated abelian group of a finite rank $\beta_r(X)$ whose set of finite order elements has a finite cardinality $\text{Tor}_i(X)$. The following two numbers are remarkable topological invariants

$$\chi(X) := \sum (-1)^i \beta_i(X) \quad \text{and} \quad \text{Tor}(X) := \prod (\text{Tor}_i(X))^{(-1)^i}.$$

If M^n is an n -dimensional closed orientable (topological) manifold, the Poincaré duality implies $\beta_r(M) = \beta_{n-r}(M)$ and $\text{Tor}_r(M) = \text{Tor}_{n-1-r}(M)$ and therefore for n odd $\chi(M^n) = 0$ and for n even $\text{Tor}(M^n) = 1$.

If (M, g) is an orientable closed Riemannian manifold and $\mathcal{H}^r(M, g) \subset \Omega^r(M)$ denotes the space of harmonic forms of degree r and $\mathcal{H}_0^r \subset \mathcal{H}^r$ the subspace of integral forms, then

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$T_r(M, g) := \mathcal{H}^r(M)/\mathcal{H}_0^r(M)$ is the compact torus of dimension $\beta_r(M)$ ¹ as a Riemannian manifold with a metric induced from the scalar product on $\Omega^r(M)$ provided by g . Let $V_r(M, g) := \text{Vol}(T_r(M, g))$ and $\mathbb{V}(M, g) := \prod (V_r(M, g))^{(-1)^r} \in \mathbb{R}_{>0}$. This is a Riemannian invariant. In view of the Hodge-de Rham theorem $\chi(M)$ is also a Riemannian invariant.

If $f : M \rightarrow \mathbb{R}$ is a smooth real-valued function, f defines a deformation (parametrization) $(\Omega^*(M), d^*(t))$ of the de Rham complex $(\Omega^*(M), d^*)$ with $d^*(t) := e^{-tf}d^*e^{tf} = d^* + tdh \wedge \cdot$, and therefore the one parameter family of second order elliptic differential operators $\Delta_{g,f}^q(t) : \Omega^q(M) \rightarrow \Omega^q(M)$ with

$$\Delta_{g,f}^q(t) := d^{q-1}(t) \cdot \delta_g^q(t) + \delta_g^{q+1}(t) \cdot d^q(t) = \Delta_g^q + t(L_X^q + \mathcal{L}_X^q) + t^2\|X\|^2,$$

where

- (1) $\delta_g^q(t) := (-1)^{n(q-1)+1} \star_g^{n-q+1} \cdot e^{tf}d^{n-q}e^{-tf} \cdot \star_g^q$,
- (2) $X = -\text{grad}_g f$, L_X the Lie derivative in direction X , $\mathcal{L}_X^q := (-1)^{(n+1)q+1} \star^{n-q} \cdot L_X^{n-q} \cdot \star^q$,
- (3) $\|X\| : M \rightarrow \mathbb{R}_{\geq 0}$ the length of the vector $X(x) = -\text{grad}_g f(x)$.

These operators remain self-adjoint, nonnegative elliptic differential operators on $L_2(\Omega^q(M))$, the L_2 -completion of $\Omega^q(M)$ with $\Delta_{g,f}^q(t)$ a zero order perturbation of the standard Laplace-Beltrami operator Δ_g^q ².

When both g and f are implicit from the context, one abbreviates $\Delta_{g,f}^q(t), \Delta_g^q, \delta_g^q, \star_g^q$ to $\Delta^q(t), \Delta^q, \delta^q, \star^q$ for simplicity in writing.

The operators $\Delta^q(t)$, referred below as Witten Laplacians, provide a *holomorphic family of type A* of self-adjoint operators in the sense of Kato cf [8] and therefore, in view of a theorem of Rellich-Kato, cf. Theorem 3.9 chapter 7 in [8], one has:

Theorem 1.1. (*Rellich - Kato*) *There exist a collection of non-negative real-valued functions $\lambda_\alpha^q(t)$, unique up to permutation, and a collection of norm one q -differential form-valued maps $\omega_\alpha^q(t) \in \Omega^q(M)$, analytic in $t \in \mathbb{R}$, indexed by $\alpha \in \mathcal{A}^q$, \mathcal{A}^q a countable set, each with holomorphic extension to a neighborhood of the real line $\mathbb{R} \subset \mathbb{C}$ ³ such that:*

- (1) $\Delta^q(t)\omega_\alpha^q(t) = \lambda_\alpha^q(t)\omega_\alpha^q(t)$,
- (2) *for any t the collections $\lambda_\alpha^q(t)$ represent all repeated eigenvalues of $\Delta^q(t)$ and the collection $\omega_\alpha^q(t)$ form a complete orthonormal family of associated eigenvectors for the operator $\Delta^q(t)$,*
- (3) *exactly $\beta_q = \dim H^q(M; \mathbb{R})$ eigenvalue functions $\lambda_\alpha^q(t)$ are identically zero and all others are strictly positive.*

These analytic maps $\lambda_\alpha^q(t)$ and $\omega_\alpha^q(t)$ are called branches, eigenvalue branch and eigenform branch respectively, with extensions to holomorphic maps in the neighborhood of $\mathbb{R} \subset \mathbb{C}$. The maps $\omega_\alpha^q(t)$ have \pm ambiguity and if the branch $\lambda_\alpha^q(t)$ has multiplicity ≥ 2 , i.e. $\lambda_\alpha^q(t) = \lambda^q(t)$

¹in view of the Hodge de-Rham theorem

² L_X and \mathcal{L}_X are order one differential operators but $L_X + \mathcal{L}_X$ is of order zero as well as the multiplication by the smooth function $\|X\|^2$

³holomorphic extension means extensions $\lambda^q(z) \in \mathbb{C}$, $\omega^q(z) \in \Omega(M) \otimes \mathbb{C}$ for z in a neighborhood of \mathbb{R} in \mathbb{C} which for $t \in \mathbb{R}$ is a real number and $\omega^q(t) \in \Omega(M) \otimes 1$

for a finite set of indices α , then the family of finite dimensional vector spaces spanned by the corresponding $\omega_\alpha^q(t)$ is unique and of course analytic in t .

If f is a Morse function with c_q critical points of Morse index q , then in view of a result of Witten [9], cf [5] Proposition 5.2 or [4] Theorem 2.8 for details, for any q exactly c_q eigenvalue branches of $\Delta^q(t)$ go exponentially fast to zero and all others go at least linearly fast to ∞ . Moreover each eigenvalue branch which converges to zero corresponds to a critical point and its corresponding eigenform branch concentrates to this critical point. We index these analytic functions as $\lambda_x^{\text{ind } x}(t)$ and $\omega_x^{\text{ind } x}(t)$ and refer to the finite collection of branches

$$\boxed{\{\lambda_x^{\text{ind } x}(t), \omega_x^{\text{ind } x}(t), x \in Cr(f)\}}$$

as the *virtually small spectral package* of (M, g, f) and to the finite set

$$\boxed{\{\lambda_x^{\text{ind } x}(0), \omega_x^{\text{ind } x}(0), x \in Cr(f)\}}$$

the subset of the infinite set $\{\lambda_\alpha^q, \omega_\alpha^q\}$, the spectral package of (M, g) , as the *virtually small spectral package* of (M, g) determined by f .

Note that for t large enough, in view of the *spectral gap theorem*, cf Theorem ?? stated in Section 3, the eigenvalues $\lambda_x^q(t)x \in Cr_q(f)$, exhaust the first $c_q = \sharp Cr_q(f)$, possibly repeated, smallest eigenvalues of $\Delta^q(t)$, however this is not true for $t = 0$ as the Example in section 3 shows. This explains the name *virtually small* for the collection $\{\lambda_x^{\text{ind } x}(0), \omega_x^{\text{ind } x}(0), x \in Cr(f)\}$.

We denote by $\Omega_{v_s}^*(M)(t)$ the span of the eigenforms $\omega_x^{\text{ind } x}(t)x \in Cr(f)$ inside $\Omega^*(M)$. They generate a finite dimensional sub-complex $(\Omega_{v_s}^*(M)(t), d^*(t))$ of $(\Omega^*(M), d^*(t))$ with the q -component of dimension c_q , which is an analytic family of cochain complexes with cohomology of constant dimension $\beta_r(M)$. Since f is a Morse function, for any $x \in Cr(f)$ the stable/unstable set W_x^\pm of the vector field $-\text{grad}_g f$ are submanifolds diffeomorphic to $\mathbb{R}^{n-\text{ind } x}/\mathbb{R}^{\text{ind } x}$. For any x choose an orientation \mathcal{O}_x for W_x^- and for any $x, y \in Cr_q(f)$ consider the integral

$$A^q(x, y)(t) := \int_{W_y^-} e^{tf} \omega_x^q(t), \quad (1.1)$$

which a priori might not be convergent but when convergent for any $x, y \in Cr_q(f)$ provide the non-negative number

$$a^q(M, g, f)(t) := |\det |||A^q(x, y)(t)||| | \geq 0,$$

where $|||A^q(x, y)(t)|||$ is a $c_q \times c_q$ matrix with $c_q = \sharp Cr_q(f)$. Changing of the orientation \mathcal{O}_y changes the sign of the integrals $A^q(x, y)$ for all x but leave $a^q(M, g, f)$ unchanged when defined.

Proposition 1.2.

- (1) *If the vector field $-\text{grad}_g f$ is Morse-Smale, then the integral (1.1) is uniformly convergent and both $A^q(x, y)(t)$ as well as $a^q(M, g, f)(t)$ are analytic in t , with the last being independent on the choice of $\omega_x^q(t)$ and the orientations \mathcal{O}_x .*
- (2) *The analytic function $a^q(M, g, f)(t)$ is non-negative with at most finitely many zeros in any interval $[T, \infty)$.*

In particular

$$a(M, g, f)(t) := \prod (a^q(M, g, f)(t))^{(-1)^q}$$

is a priori a nonnegative meromorphic function in t with at most finitely many zeros and poles in any interval $[T, \infty)$.

The main result of this Note is the following theorem:

Theorem 1.3. *Suppose that the vector field $-\text{grad}_g f$ is Morse-Smale. Then the following holds true.*

- (1) *The meromorphic map $a(M, g, f)(t)$ is strictly positive and has no zeros and no poles.*
- (2) *The virtually small spectral package determined by f together with the numbers $a(M, g, f)$ and $\mathbb{V}(M, g)$ are all analytic invariants and determine the topological invariant $\mathbb{T}or(M)$ by the formula:*

$$\log \mathbb{T}or(M) = 1/2 \sum_q (-1)^{q+1} q \left(\sum_{\alpha \in \mathcal{A}_{vs,+}^q} \log \lambda_\alpha^q(0) \right) + \log a(M, g, f) - \log \mathbb{V}(M, g),$$

where $a(M, g, f) := a(M, g, f)(0)$ and $\mathcal{A}_{vs,+}^q \subset \mathcal{A}^q$ is the set of indices α 's such that $\lambda_\alpha^q(t)$ belongs to the virtually small spectral package with $\lambda_\alpha^q(t) > 0$.

Conjecture 1: *The statement remains true without the hypothesis that " $-\text{grad}_g f$ is Morse-Smale".*

Conjecture 2: *Under the hypothesis that " $-\text{grad}_g f$ is Morse-Smale" one has $a^q(M, g, f)(t) \neq 0$.*

Note that if Conjecture 2 holds true for $t = 0$, then it can be shown that the Morse complex defined by (g, f) can be canonically realized as a sub-complex of the de Rham complex equipped with the scalar product defined by the metric g . Recall that Hodge-de Rham theorem implies that the complex $(H^r(M), 0)$ can be realized in this way as the sub-complex of harmonic forms.

As shown in Section 4, for an oriented closed Riemannian manifold the Hodge star operator

$$\star : \Omega^q(M) \rightarrow \Omega^{n-q}(M)$$

identifies the virtually small q -spectral package of (M, g, f) to the the virtually small $(n - q)$ -spectral package of $(M, g, -f)$. This can be viewed as an extension of Poincaré duality.

2. PROOF OF PROPOSITION 1.2 AND THEOREM 1.3

Proof of Proposition 1.2:

One says that the vector field $X = -\text{grad}_g f$ is *Morse-Smale* if for any $x, y \in Cr(f)$ the unstable set W_x^- and the stable set W_y^+ are transversal, which implies that $\mathcal{T}(x, y) = (W_x^- \cap W_y^+)/\mathbb{R}$ ⁴, the space of trajectories from x to y , is a manifold of dimension $\text{ind}(x) - \text{ind}(y) - 1$. Under the hypothesis that X is Morse-Smale, it is shown in [2] or [6] that the embedding

⁴ \mathbb{R} acts freely by translation along the flow defined by $-\text{grad}_g f$

$i_x : W_x^- \rightarrow M$ extends to a smooth map $\hat{i}_x : \hat{W}_x^- \rightarrow M$, where \hat{W}_x^- is a compact smooth manifold with corners whose interior is W_x^- . Hence, it follows that

$$\int_{W_y^-} e^{tf} \omega_x^q(t) = \int_{\hat{W}_y^-} (\hat{i}_x)^*(e^{tf} \omega_x^q(t)).$$

This implies the uniform convergence of the integral (1.1) and the analyticity of $A^q(x, y)(t)$ and of $a^q(M, g, f)(t)$. To conclude the independence on the choices of $\omega_\alpha^q(t)$ it suffices to note that the matrices $\|A^q(\dots)(t)\|$ for two choices differ one from the other by composition by an orthogonal matrix, hence have the same determinant up to sign. The results in [2] shows also that the partition $M = \bigcup_x W_x^-$ provides a CW structure of M with open cells W_x^- and if one equips each cell W_x^- with the orientation O_x and one denotes by C^q the \mathbb{R} -vector space $Maps(C_q(f), \mathbb{R})$, then $Int^q(t) : \Omega^q(M) \rightarrow C^q$ defined by

$$Int^q(t)(\omega)(x) = \int_{W_x^-} \omega$$

provides a linear map, and the collection of the linear maps $Int^q(t) : \Omega^q(M) \rightarrow C^q$ define a map of cochain complexes $Int^*(t) : (\Omega^*(M), d^*(t)) \rightarrow (C^*, \partial^*)$, which by de Rham theorem is a quasi-isomorphism. Of course this involves the explicit description of the corner structures of \hat{W}_x^- . The cochain complex morphism $Int^*(t)$ is clearly analytic in t and restricts to $(\Omega^*(M), d^*(t))$ an analytic family of quasi-isomorphisms $(\Omega_{vs}^*(M), d^*(t)) \rightarrow (C^*, \partial^*)$.

If one considers $\omega_x^q(t)$'s as a basis for $\Omega_{vs}^q(M)(t)$ and the characteristic functions of the set $Cr_q(f)$ as a basis for C^q , then one realizes that the matrix representation of $Int^q(t)$ is exactly the matrix $\|A^q(x, y)(t)\|$.

For t large enough Witten-Helffer-Sjöstrand results imply that $Int^q(t)$ restricted to $\Omega_{vs}^q(M)(t)$ is an isomorphism, (for details [5] theorem 5.5. item 5, or [1] theorem 3.1), which shows that, for t large enough, $a^q(M, g, f)(t) \neq 0$. In view of analyticity item 2 follows as stated.

Proof of Theorem (1.3):

First observe the following facts.

- (1) For an isomorphism $\varphi : (V, \langle \cdot, \cdot \rangle_V) \rightarrow (W, \langle \cdot, \cdot \rangle_W)$ between two finite dimensional vector spaces equipped with scalar product, let $\text{Vol}(\varphi) := \sqrt{\det(\varphi^\sharp \cdot \varphi)^{1/2}} = \sqrt{\det(\varphi \cdot \varphi^\sharp)^{1/2}}$ with φ^\sharp the adjoint of φ .
- (2) If $\varphi(t) : (V(t), \langle \cdot, \cdot \rangle_{V(t)}) \rightarrow (W(t), \langle \cdot, \cdot \rangle_{W(t)})$ is a continuous/analytic family of isomorphisms between finite dimensional vector spaces equipped with scalar products,⁵ then the function $\text{Vol}(\varphi(t))$ is continuous/analytic in t .
- (3) For a cochain complex $\mathcal{C} = (C^*, d^*)$ of finite dimensional vector spaces equipped with scalar products

$$\mathcal{C} : 0 \longrightarrow (C^0, \langle \cdot, \cdot \rangle_0) \xrightarrow{d^0} (C^1, \langle \cdot, \cdot \rangle_1) \xrightarrow{d^1} \dots \quad (C^n, \langle \cdot, \cdot \rangle_n) \longrightarrow 0,$$

⁵for example $V(t)$ resp. $W(t)$ appear as images in \mathcal{V} resp. \mathcal{W} , of an analytic/continuous family of bounded projectors $P(t) : \mathcal{V} \rightarrow \mathcal{V}$ resp. $Q(t) : \mathcal{W} \rightarrow \mathcal{W}$ for \mathcal{V} resp \mathcal{W} topological vector spaces; this give meaning to "analytic family"

one denotes by $\Delta_{\mathcal{C}}^q := \delta^{q+1} \cdot d^q + d^{q-1} \cdot \delta^q$, δ the adjoint of d , and by $\det' \Delta_{\mathcal{C}}^q \neq 0$ the product of nonzero eigenvalues of $\Delta_{\mathcal{C}}^q$. The product

$$T(\mathcal{C}) := \prod (\det' \Delta_{\mathcal{C}}^q)^{\frac{1}{2}q(-1)^{q+1}}$$

is referred to as the torsion of \mathcal{C} . Here $\det \Delta'$ denotes the product of nonzero eigenvalues of Δ . For a continuous/analytic family of cochain complexes $\mathcal{C}(t) = (C^*(t), d^*(t))$ such that $\dim C^q(t)$ and $\dim H^q(\mathcal{C}(t))$ are constant in t for any q , the function $T(\mathcal{C}(t))$ is continuous/analytic in t .

The verifications of items (2) and (3) above are straightforward from definitions.

- (4) Suppose that $\varphi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is a morphism of cochain complexes of finite dimensional vector spaces with scalar products, where $\mathcal{C}_i = (C_i^*, d_i^*)$, $i = 1, 2$, and $\varphi = \{\varphi^q : C_1^q \rightarrow C_2^q\}$. Suppose that φ^q is an isomorphism for any q . Then φ induces the isomorphism $H^q(\varphi) : H^q(\mathcal{C}_1) \rightarrow H^q(\mathcal{C}_2)$ between vector spaces equipped with induced scalar product. Let

$$\text{Vol}(\varphi) := \prod (\text{vol}(\varphi^q))^{(-1)^q}$$

and

$$\text{Vol}(H(\varphi)) := \prod (\text{vol}(H^q(\varphi)))^{(-1)^q}.$$

As verified in [4] Proposition 2.5 one has

$$T(\mathcal{C}_2)/T(\mathcal{C}_1) = \text{Vol}(H(\varphi))/\text{Vol}(\varphi). \quad (2.1)$$

- (5) For a continuous/analytic family of isomorphisms $\varphi(t) : \mathcal{C}_1(t) \rightarrow \mathcal{C}_2(t)$, $t \in \mathbb{R}$, with $\dim C_1^q(t) = \dim C_2^q(t)$ and $\dim H^q(\mathcal{C}_1(t)) = \dim H^q(\mathcal{C}_2(t))$ constant in t , the real-valued functions $T(\mathcal{C}_1(t))$, $T(\mathcal{C}_2(t))$, $\text{Vol}(\varphi(t))$, $\text{Vol}(H(\varphi(t)))$ are nonzero and continuous/analytic.

We consider $\varphi(t) = \text{Int}^*(t) : (\Omega_{vs}^*(M)(t), d^*(t)) \rightarrow (C^*, \partial^*)$ with $* = 0, 1, \dots, \dim M$. The first cochain complex is equipped with the scalar products defined by the metric g , and the second with the unique scalar product which makes the characteristic functions of the critical points orthonormal. In view of item (4) the function

$$\frac{T(\Omega_{vs}^*(M)(t), d^*(t)) \cdot \text{Vol}(H(\varphi(t)))}{T(C^*, \partial^*)}$$

is a strictly positive analytic function and in view of (4) agrees with $a(t)$ for all t but the finite collection which might be a zero or a pole for $a(t)$. Hence the meromorphic function $a(t)$ has no zeros and no poles. This establishes Theorem 1.3 part 1. Together with (2.1) it also implies

$$\frac{T(\Omega_{vs}^*(M)(t), d^*(t))}{a(t)} \cdot \text{Vol}(H(\varphi(t))) = T(C^*, \partial^*).$$

Evaluation at $t = 0$ combined with the observation that $\text{Tor}(M) = T(C^*, \partial^*)$ implies

$$\frac{T(\Omega_{vs}^*(M), d^*)}{a(0)} \cdot \text{Vol}(H(\varphi(0))) = \text{Tor}(M).$$

Taking "log", one derives Theorem 1.3 part 2.

3. POINCARÉ DUALITY FOR THE VIRTUALLY SMALL SPECTRAL PACKAGE

For a closed oriented Riemannian manifold (M, g) and smooth function $f : M \rightarrow \mathbb{R}$ one has the Hodge star operator $\star^q : \Omega^q(M) \rightarrow \Omega^{n-q}(M)$ which satisfies the following properties.

- (1) $\star^{n-q} \cdot \star^q = (-1)^{q(n-q)} \text{Id}$,
- (2) $(-1)^{q(n-q)} \star^q \Delta^q \star^{n-q} = \Delta^{n-q}$,
- (3) $(-1)^{q(n-q)} \star^q \Delta_{g,f}^q(t) \star^{n-q} = \Delta_{g,-f}^{n-q}(t)$,
- (4) $\Delta_{g,f}^q(-t) = \Delta_{g,-f}^q(t)$.

As a consequence, the Hodge operator \star^q identifies the q -spectral package of (M, g, f) with the $(n - q)$ -spectral package of $(M, g, -f)$ and the q -virtually small spectral package

$$\left\{ \lambda_{g,f,x}^q(t), \omega_{g,f,x}^q(t), x \in Cr_q(f) \right\}$$

of (M, g, f) with the $(n - q)$ -virtually small spectral package

$$\left\{ \lambda_{g,-f,x}^{n-q}(t), \omega_{g,-f,x}^{n-q}(t), x \in Cr_{n-q}(-f) \right\}$$

of $(M, g, -f)$. More precisely, it holds that for $x \in Cr_q(f) = Cr_{n-q}(-f)$,

$$\lambda_{g,-f,x}^{n-q}(t) = \lambda_{g,f,x}^q(t), \quad \omega_{g,-f,x}^{n-q}(t) = \star^q \omega_{g,f,x}^q(t).$$

Poincaré duality

The above identification can be regarded as a refinement of the Poincaré duality which states that $\beta^q(M)$ viewed as the multiplicity of the eigenvalue 0 of $\Delta_{g,f}^q(0)$ is equal to $\beta^{n-q}(M)$ viewed as the multiplicity of the eigenvalue 0 of $\Delta_{g,-f}^{n-q}(0)$.

4. VIRTUALLY SMALL EIGENVALUES VERSUS THE SMALLEST EIGENVALUES

We are going to show that the virtually small eigenvalues may not be equal to the smallest eigenvalues by giving an example. For $\mathbb{S}^1 := \mathbb{R}/(2\pi\mathbb{Z})$, we consider a torus $M = \mathbb{S}^1 \times \mathbb{S}^1$ equipped with the flat metric g_0 induced from the canonical metric on $\mathbb{R} \times \mathbb{R}$. The function $f(\theta_1, \theta_2) = \sin(2\theta_1) + \sin(2\theta_2)$, $(\theta_1, \theta_2) \in \mathbb{S}^1 \times \mathbb{S}^1$, is a Morse function on M having four critical points of index 0, four critical points of index 2 and eight critical points of index 1. The sequence of eigenvalues of $\Delta_{g_0}^0$ in increasing order is $0 \leq 1 \leq 1 \leq 1 \leq 1 \leq 4 \dots$. The virtually small 0-eigenvalues consist of four real numbers. The first one is $\lambda_1(0) = 0$, the next two are $\lambda_2(0) = \lambda_3(0) \geq 1$ and the final one is $\lambda_4(0) = 2\mu_2(0)$, where $\mu_2(0) \geq 1$. This shows that the virtually small eigenvalues are not the same as the smallest eigenvalues.

Proof. We first observe that $h : \mathbb{S}^1 \rightarrow \mathbb{R}$ given by $h(\theta) = \sin 2\theta$ is a Morse function on \mathbb{S}^1 , whose Witten Laplacian $\Delta^0(t)$ is

$$\Delta^0(t) = -\partial^2/\partial\theta^2 + 4t \sin 2\theta + 4t^2(\cos 2\theta)^2.$$

Then, $\Delta^0(t)$ has two virtually small eigenvalue branches. One of them is $\mu_1(t) \equiv 0$ and the other is $\mu_2(t) > 0$ because $\beta^0(\mathbb{S}^1) = 1$ and h has two critical points of index 0. Observe that the eigenvalues of $\Delta^0(0)$ are

$$0, 1, 2^2, 3^2, \dots, n^2, \dots,$$

where 0 has the multiplicity 1 and all others have multiplicity 2. Since $\mu_1(0)$ and $\mu_2(0)$ are among the above eigenvalues, one has $\mu_2(0) \geq 1$. In view of the definition of $f(\theta_1, \theta_2)$ and that of $\Delta^0(t)$ for $(\mathbb{S}^1 \times \mathbb{S}^1, g_0, f)$, the four virtually small eigenvalue branches $\lambda_1(t), \lambda_2(t), \lambda_3(t), \lambda_4(t)$ are of the form $\mu_i(t) + \mu_j(t)$ with $i, j \in \{1, 2\}$, and hence the virtually small 0-eigenvalues are $\lambda_1(0) = 0, \lambda_2(0) = \lambda_3(0) = \mu_1(0)$ and $\lambda_4(0) = 2\mu_1(0)$. \square

When M^2 is a 2-dimensional oriented closed Riemannian manifold, the above example shows that the virtually small spectral package of Δ^0 is not the same as the collection of the smallest eigenvalues. By the Poincaré duality, the nonzero eigenvalues of Δ^1 on M^2 are two times of the nonzero eigenvalues of Δ^0 or Δ^2 , and hence the result remains the same for Δ^1 .

REFERENCES

- [1] D. Burghilea, *lectures on Witten Helper Sjöstrand theory* arXiv:1101.0778 *On the Space of Trajectories of a Generic Vector Field* Analele Universitatii de Vest din Timisoara, seria matematica-informatica, vol XLVIII Fasc 1,2, 2010 pp 45-126 arXiv:1101.0778
- [2] D. Burghilea, L.Friedlander and T. Kappeler, *On the Space of Trajectories of a Generic Vector Field* Analele Universitatii de Vest din Timisoara, seria matematica-informatica, vol XLVIII Fasc 1,2, 2010 pp 45-126 arXiv:1101.0778
- [3] D. Burghilea, L.Friedlander and T. Kappeler, *Deformation of the de Rham complex* (book in preparation)
- [4] D. Burghilea, L.Friedlander and T. Kappeler, *Torsion for Manifolds with Boundary and Glueing Formulas*, Math. Nachr. 208 (1999), 31-91.
- [5] D. Burghilea, L. Friedlander, T. Kappeler and P. McDonald, *Analytic and Reidemeister torsion for representations in finite type Hilbert modules*, Geom. Funct. Anal. 6(1996), 751-859.
- [6] D. Burghilea and S. Haller, *On the topology and analysis of closed one form. I (Novikov theory revisited)*, Monogr. Enseign. Math. 38(2001), 133-175.
- [7] Jeff Cheeger, *Analytic torsion and heat equation*, Annals of Mathematics, 109 (1979), 259-322.
- [8] T. Kato, *Perturbation theory for linear operators*. Second edition. Grundlehren der Mathematischen Wissenschaften 132. Springer-Verlag, Berlin-New York, 1976.
- [9] E.Witten, *Supersymmetry and Morse theory*, J. of Diff. Geom. 17 (1982) 661-692.

Dan Burghilea
 Department of Mathematics,
 Ohio State University, Columbus, OH 43210, USA

Yoonweon Lee
 Department of Mathematics Education,
 Inha University, Incheon, 22212, Korea