Topological Persistence for Circle Valued Maps

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Abstract

We study *circle valued maps* and consider the *persistence of the homology of their fibers*. The outcome is a finite collection of computable invariants which answer the basic questions on persistence and in addition encode the topology of the source space and its relevant subspaces. Unlike persistence of real valued maps, circle valued maps enjoy a different class of invariants called *Jordan cells* in addition to bar codes. We establish a relation between the homology of the source space and of its relevant subspaces with these invariants and provide a new algorithm to compute these invariants from an input matrix that encodes a circle valued map on an input simplicial complex.

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1 Introduction

Data analysis provides plenty of scenarios where one ends up with a nice space, most often a simplicial complex, a smooth manifold, or a stratified space equipped with a real valued or a circle valued map. The persistence theory, introduced in [13], provides a great tool for analyzing real valued maps with the help of homology. Similar theory for circle valued maps has not yet been developed in the literature. The work in [18] brings the concept of circle valued maps in the context of persistence by deriving a circle valued map for a given data using the existing persistence theory. In contrast, we develop a persistence theory for circle valued maps.

One place where circle valued maps appear naturally is the area of dynamics of vector fields. The measurements in the dynamics described by a curl free vector field [1] can be interpreted as 1-cocycles which are intimately connected to circle valued maps as shown in [1]. Consequently, a notion of persistence for circle valued maps also provides a notion of persistence for 1-cocycles which appear in some data analysis problems [19, 20]. In summary, persistence theory for circle valued maps promises to play the role for some vector fields as does the standard persistence theory for the scalar fields [5, 6, 13, 17].

One of the main concepts of the persistence theory is the notion of *bar codes* [17]–invariants that characterize a real valued map at the homology level. The angle (circle) valued maps, when characterized at homology level, require a new invariant called *Jordan cells* in addition to the bar codes.

The standard persistence [13, 17] which we refer as *sublevel persistence* deals with the change in the homology of the sublevel sets which can not make sense for a circle valued map. However the change in the homology of the level sets can be considered for both real and circle valued maps. The notion of persistence, when considered for the level sets of a real valued map is referred here as *level persistence*. It refines the sublevel persistence. The zigzag persistence introduced in [4] provides complete invariants (bar codes) for level persistence of (tame) real valued map. They are defined using representation theory for linear quivers.

The change in homology of the level sets of a (tame) circle valued map is more complicated because of the *return* of the level to itself when one goes along the circle. It turns out that representation theory of cyclic quivers provides the complete invariants for persistence in the homology of the level sets of the circle valued maps. This notion of persistence is called here *persistence for circle valued maps* and its invariants, *bar codes* and *Jordan cells* are also effectively computable.

Our results include a derivation of the homology for the source space and its relevant subspaces in terms of the invariants (Theorem 3.1 and 3.2). The result also applies to real valued maps as they are special cases of the circle valued maps. This leads to a result (Corollary 3.4) which to our knowledge has not yet appeared in the literature [2]. A number of important topological results which can not be derived from any of the previously defined persistence theories are described in [3] providing additional motivation for this work.

After developing the results on invariants, we propose a new algorithm to compute the bar codes and Jordan cells. For a simplicial complex, the entire computation can be done by manipulating the original matrix that encodes the input complex and the map. The algorithm first builds a block matrix from the original incidence matrix which encodes linear maps induced in homology among regular and critical level sets, more precisely the quiver representations ρ_r described in section 4. Next, it iteratively reduces this new matrix eliminating and hence computing the bar codes. The resulting matrix which is invertible can be further processed to Jordan canonical form [8] providing Jordan cells. The algorithm for zigzag persistence [4] when applied to what we refer in section 3 as the *infinite cyclic covering map* $\tilde{f}$ can compute bar codes but not Jordan cells. In contrast, our method can compute the bar codes and Jordan cells simultaneously by

[1] It is not necessary to be curl free; it suffices to have a Lyapunov closed one form.

[2] it was brought to our attention by David Cohen-Steiner that the extended persistence proposed in [6] allows similar connections between homology of source spaces and persistence.

manipulating matrices and can be also used as an alternative algorithm to calculate the bar codes in zig-zag persistence.

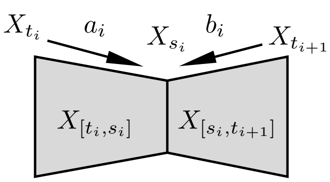
2 Definitions and background

We begin with the technical definition of tameness of a map.

For a continuous map $f : X \to Y$ between two topological spaces X and Y, let $X_U = f^{-1}(U)$ for $U \subseteq Y$. When $U = y$ is a single point, the set X_y is called a *fiber* over y and is also commonly known as the level set of y. We call the continuous map $f : X \to Y$ *good* if every $y \in Y$ has a contractible neighborhood U so that the inclusion $X_y \to X_U$ is a homotopy equivalence. The continuous map $f : X \to Y$ is a *fibration* if each $y \in Y$ has a neighborhood U so that the maps $f : X_U \to U$ and $pr : X_y \times U \to U$ are fiber wise homotopy equivalent. This means that there exist continuous maps $l : X_U \to X_y \times U$ with $pr|_U \cdot l|_U = f|_U$ which, when restricted to the fiber for any $z \in U$, are homotopy equivalences. In particular, f is good.

Definition 2.1 *A proper continuous map $f : X \to Y$ is* tame *if it is good, and for some discrete closed subset $S \subset Y$, the restriction $f : X \setminus f^{-1}(S) \to Y \setminus S$ is a fibration. The points in $S \subset Y$ which prevent f to be a fibration are called* critical values*.*

If $Y = \mathbb{R}$ and X is compact or $Y = \mathbb{S}^1$, [3] then the set of critical values is finite, say $s_1 < s_2 < \cdots s_k$. The fibers above them, X_{s_i}, are referred to as *singular fibers*. All other fibers are called *regular*. In the case of $\mathbb{S}^1$, s_i can be taken as angles and we can assume that $0 < s_i \leq 2\pi$. Clearly, for the open interval (s_{i-1}, s_i) the map $f : f^{-1}(s_{i-1}, s_i) \to (s_{i-1}, s_i)$ is a fibration which implies that all fibers over angles in (s_{i-1}, s_i) are homotopy equivalent with a fixed regular fiber, say X_{t_i}, with $t_i \in (s_{i-1}, s_i)$.



In particular, there exist maps $a_i : X_{t_i} \to X_{s_i}$ and $b_i : X_{t_{i+1}} \to X_{s_i}$, unique up to homotopy defined as follows: If t_i and t_{i+1} are contained in $U_i \subset Y$ with $X_{s_i} \subset X_{U_i}$ a homotopy equivalence with a homotopy inverse $r_i : X_{U_i} \to X_{s_i}$, then a_i and b_i are the restrictions of r_i to X_{t_i} and $X_{t_{i+1}}$ respectively. If not, in view of the tameness of f, one can find t'_i and t'_{i+1} in U_i so that X_{t_i} and $X_{t_{i+1}}$ are homotopy equivalent to $X_{t'_i}$ and $X_{t'_{i+1}}$ respectively and compose the restrictions of r_i with these homotopy equivalences. These maps determine homotopically $f : X \to Y$, when $Y = \mathbb{R}$ or $\mathbb{S}^1$. For simplicity in writing, when $Y = \mathbb{R}$ we put $t_{k+1} \in (s_k, \infty)$ and $t_1 \in (-\infty, s_1)$ and when $Y = \mathbb{S}^1$ we put $t_{k+1} = t_1 \in (s_k, s_1 + 2\pi)$. All scalar or circle valued simplicial maps on a simplicial complex, and smooth maps with generic isolated critical points on a smooth manifold or stratified space are tame. In particular, Morse maps are tame.

2.1 Persistence and invariants for real valued maps

Since our goal is to extend the notion of persistence from real valued maps to circle valued maps, we first summarize the questions that the persistence answers when applied to real valued maps, and then develop a notion of persistence for circle valued maps which can answer similar questions and more. We fix a field κ and write $H_r(X)$ to denote the homology vector space of X in dimension r with coefficients in a field κ.

Sublevel persistence. The persistent homology introduced in [13] and further developed in [17] is concerned with the following questions:

Q1. Does the class $x \in H_r(X_{(-\infty,t]})$ originate in $H_r(X_{(-\infty,t'']})$ for $t'' < t$? Does the class $x \in H_r(X_{(-\infty,t]})$ vanish in $H_r(X_{(-\infty,t']})$ for $t < t'$?

[3] since the map f is proper and $\mathbb{S}^1$ compact, so is X

Q2. What are the smallest t' and largest t'' such that this happens?

This information is contained in the linear maps $H_r(X_{(-\infty,t]}) \to H_r(X_{(-\infty,t']})$ where $t' \geq t$ and is known as persistence. Since the involved subspaces are sublevel sets, we refer to this persistence as *sublevel persistence*. When f is tame, the persistence for each $r = 0, 1, \cdots \dim X$, is determined by a finite collection of invariants referred to as *bar codes* [17]. For sublevel persistence the bar codes are a collection of *closed intervals* of the form $[s, s']$ or $[s, \infty)$ with s, s' being the critical values of f. From these bar codes one can derive the Betti numbers of $X_{(-\infty,a]}$, the dimension of $\mathrm{img}(\mathrm{H_r}(\mathrm{X}_{(-\infty,t]}) \to \mathrm{H_r}(\mathrm{X}_{(-\infty,t']}))$ and get the answers to questions Q1 and Q2. For example, the number of r-bar codes which contain the interval $[a, b]$ is the dimension of $\mathrm{img}(\mathrm{H_r}(\mathrm{X}_{(-\infty,a]}) \to \mathrm{H_r}(\mathrm{X}_{(-\infty,b]}))$. The number of r-bar codes which identify to the interval $[a, b]$ is the maximal number of linearly independent homology classes born exactly in $X_{(-\infty,a]}$ but not before and die exactly in $H_r(X_{-\infty,b]})$ but not before.

Level persistence. Instead of sublevels, if we use levels, we obtain what we call level persistence. The level persistence was first considered in [12] but was better understood computationally when the zigzag persistence was introduced in [4]. Level persistence is concerned with the homology of the fibers $H_r(X_t)$ and addresses questions of the following type.

Q1. Does the image of $x \in H_r(X_t)$ vanish in $H_r(X_{[t,t']})$, where $t' > t$ or in $H_r(X_{[t'',t]})$, where $t'' < t$?

Q2. Can x be detected in $H_r(X_{t'})$ where $t' > t$ or in $H_r(X_{t''})$ where $t'' < t$? The precise meaning of detection is explained below.

Q3. What are the smallest t' and the largest t'' for the answers to Q1 and Q2 to be affirmative?

To answer such questions one has to record information about the following maps:

$$H_r(X_t) \to H_r(X_{[t,t']}) \leftarrow H_r(X_{t'}).$$

The *level persistence* is the information provided by this collection of vector spaces and linear maps for all t, t'.

We say that $x \in H_r(X_t)$ is dead in $H_r(X_{[t,t']}),\ t' > t$, if its image by $H_r(X_t) \to H_r(X_{[t,t']})$ vanishes. Similarly, x is dead in $H_r(X_{[t'',t]}),\ t'' < t$, if its image by $H_r(X_t) \to H_r(X_{[t'',t]})$ vanishes.

We say that $x \in H_r(X_t)$ is detected in $H_r(X_{t'}),\ t' > t$, (resp. $t'' < t$), if its image in $H_r(X_{[t,t']})$ (resp. in $H_r(X_{[t'',t]})$) is nonzero and is contained in the image of $H_r(X_{t'}) \to H_r(X_{[t,t']})$ (resp. $H_r(X_{t"}) \to H_r(X_{[t",t]})$). In Figure 1, the class consisting of the sum of two circles at level t is not detected on the right, but is detected at all levels on the left up to (but not including) the level t'. In case of a tame map the collection of the vector spaces and linear maps is determined up to coherent isomorphisms by a collection of invariants called *bar codes for level persistence* which are intervals of the form $[s, s'], (s, s'), (s, s'], [s, s')$ with s, s' critical values as opposed to the *barcodes for sublevel persistence* which are intervals of the form $[s, s'], [s, \infty)$ with s, s' critical values. These bar codes are called *invariants* because two tame maps $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ which are fiber wise homotopy equivalent have the same associated bar codes. In the case of level persistence the open end of an interval signifies the death of a homology class at that end (left or right) whereas a closed end signifies that a homology class cannot be detected beyond this level (left or right). In the case of the sublevel persistence the left end signifies *birth* while the right *death*. Level persistence provides considerably more information than the sub level persistence. The bar codes of the sub level persistence (for a tame map) can be recovered from the ones of level persistence. Precisely a level bar code $[s, s']$ gives a sublevel bar code $[s, \infty)$ and a level bar code $[s, s')$ gives a sublevel bar code $[s, s']$; the sublevel persistence does not see any of the level bar codes (s, s') or $(s, s']$. It turns out that the bar codes of the level persistence can also be recovered from the bar codes of the sub level persistence of f and additional maps canonically associated to f.

In Figure 1, we indicate the bar codes both for sub level and level persistence for some simple map $f : X \to \mathbb{R}$ in order to illustrate their differences. The space X is the one end open tube and f is the height function laid horizontally.

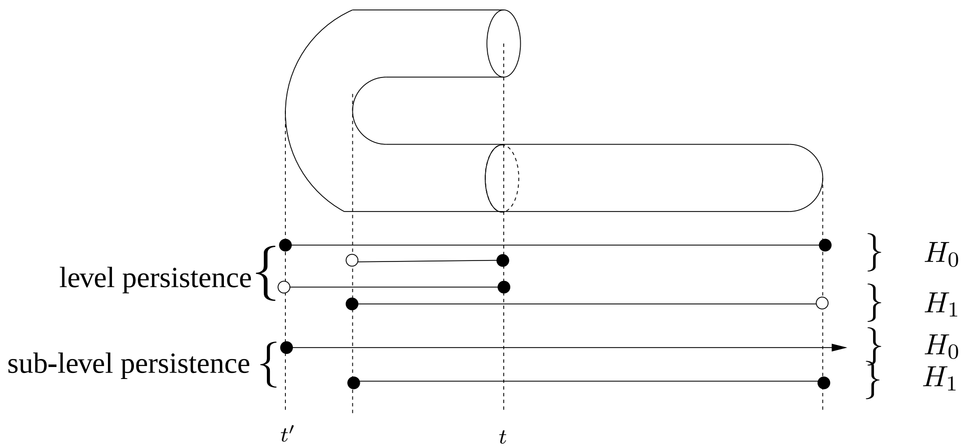


Figure 1: Bar codes for level and sub-level persistence.

3 Persistence for circle valued maps

Let $f : X \to \mathbb{S}^1$ be a circle valued map. The sublevel persistence for such a map cannot be defined since circularity in values prevents defining sub-levels. Even level persistence cannot be defined as per se since the intervals may repeat over values. To overcome this difficulty we associate the infinite cyclic covering map $\tilde{f} : \tilde{X} \to \mathbb{R}$ for f. It is defined by the commutative diagram:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \mathbb{R} \\ \psi \big\downarrow & & p \big\downarrow \\ X & \xrightarrow{f} & \mathbb{S}^1 \end{array}$$

The map $p : \mathbb{R} \to \mathbb{S}^1$ is the universal covering of the circle (the map which assigns to the number $t \in \mathbb{R}$ the angle $\theta = t(mod\ 2\pi)$ and ψ is the pull back of p by the map f which is an infinite cyclic covering. Notice that $X_\theta = \tilde{X}_t$ if $p(t) = \theta$. If $x \in H_r(X_\theta) = H_r(\tilde{X}_t),\ p(t) = \theta$, the questions Q1, Q2, Q3 for f and X can be formulated in terms of the level persistence for $\tilde{f}$ and $\tilde{X}$.

Suppose that $x \in H_r(\tilde{X}_t) = H_r(X_\theta)$ is detected in $H_r(\tilde{X}_{t'})$ for some $t' \geq t + 2\pi$. Then, in some sense, x returns to $H_r(X_\theta)$ going along the circle $\mathbb{S}^1$ one or more times. When this happens, the class x may change in some respect . This gives rise to new questions that were not encountered in sublevel or level persistence.

Q4. When $x \in H_r(X_\theta)$ returns, how does the “returned class” compare with the original class x? It may disappear after going along the circle a number of times, or it might never disappear and if so how does this class change after its return.

To answer Q1-Q4 one has to record information about $H_r(X_\theta) \to H_r(X_{[\theta,\theta']}) \leftarrow H_r(X_{\theta'})$ for any pair of angles θ and θ' which differ by at most 2π. This information is referred to as the *persistence for the circle valued map* f.

When f is tame, this is again completely determined up to coherent isomorphisms by a finite collection of invariants. However, unlike sublevel and level persistence for real valued maps, the invariants include structures other than bar codes called *Jordan cells*. Specifically, for any $r = 0, 1, \cdots, \dim(X)$ we have two types of invariants:

- *bar codes*: intervals with ends s, s' $0 < s \leq 2\pi,\ s \leq s' < \infty$, that are closed or open at s or s', precisely of the form $[s, s'], (s, s'], [s, s'), (s, s')$. These intervals can be geometerized as "spirals" with equations 1. For any interval $\{s, s'\}$ the spiral is the plane curve (see Figure 3 in section 4)

$$\begin{array}{l} x(\theta) = (\theta + 1 - s)\cos\theta \\ y(\theta) = (\theta + 1 - s)\sin\theta \end{array} \qquad \theta \in \{s, s'\}. \tag{1}$$

- *Jordan cells*. A Jordan cell is a pair (λ, k), $\lambda \in \overline{\kappa}\backslash 0,\ \ k \in \mathbb{Z}_{>0}$, where $\overline{\kappa}$ denotes the algebraic closure of the field κ. It corresponds to a $k \times k$ matrix of the form

$$\begin{pmatrix} \lambda & 1 & 0 \ldots & 0 \\ 0 & \lambda & 1 \ldots & 0 \\ \vdots & & & \\ 0 & \ldots & \lambda & 1 \\ 0 & \ldots & 0 & \lambda \end{pmatrix}. \tag{2}$$

The bar codes for f can be inferred from $\tilde{f} : \tilde{X}_{[a,b]} \to \mathbb{R}$ with $[a, b]$ being any large enough interval. Specifically, the bar codes of $f : X \to \mathbb{S}^1$ are among the ones of $\tilde{f} : \tilde{X}_{[a,b]} \to \mathbb{R}$ for $(b - a)$ being at most $\sup_\theta \dim H_r(X_\theta)$.

The Jordan cells can not be derived from $\tilde{f} : \tilde{X} \to \mathbb{R}$ or any of its truncations $\tilde{f} : \tilde{X}_{[a,b]} \to \mathbb{R}$ unless additional information, like the deck transformation of $\tilde{X}$, is provided. The collection of bar codes and Jordan cells for each $r = 0, 1, 2, \cdots \dim X$ constitute the r-*invariants* of the map f which we define precisely in the next section using quiver representations. They can be defined without quiver representation theory, based on the concepts of *death* and *detectability* applied to $\tilde{f}$ in conjunction with the deck transformation induced on $\tilde{X}$, an approach which will be considered in further work and partially in [9]. The end points of any bar code for f correspond to critical angles, that is, s and $s'(mod\ 2\pi)$ of a bar code interval $\{s, s'\}$ are critical angles for f. One can recover the following information from the bar codes and Jordan cells:

1. The Betti numbers of each fiber,
2. The Betti numbers of the source space X, and
3. The dimension of the kernel and the image of the linear map induced in homology by the inclusion $X_\theta \subset X$ as well as other additional topological invariants not discussed here [3].

Theorems 3.1 and 3.2 make the above statement precise. Let B be a bar code described by a spiral (eqn. 1) and θ be any angle. Let $n_\theta(B)$ denote the cardinality of the intersection of the spiral with the ray originating at the origin and making an angle θ with the x-axis. For the Jordan cell $J = (\lambda, k)$, let $n(J) = k$ and $\lambda(J) = \lambda$. Furthermore, let $\mathcal{B}_r$ and $\mathcal{J}_r$ denote the set of bar codes and Jordan cells for r-dimensional homology. We have the following results.

Theorem 3.1 $\dim H_r(X_\theta) = \sum_{B \in \mathcal{B}_r} n_\theta(B) + \sum_{J \in \mathcal{J}_r} n(J)$.

Theorem 3.2 $\dim H_r(X) = \sharp\{B \in \mathcal{B}_r | \text{both ends closed}\} + \sharp\{B \in \mathcal{B}_{r-1} | \text{both ends open}\} + \sharp\{J \in \mathcal{J}_r | \lambda(J) = 1\} + \sharp\{J \in \mathcal{J}_{r-1} | \lambda(J) = 1\}$.

Using the same arguments as in the proof of the above Theorems one can derive:

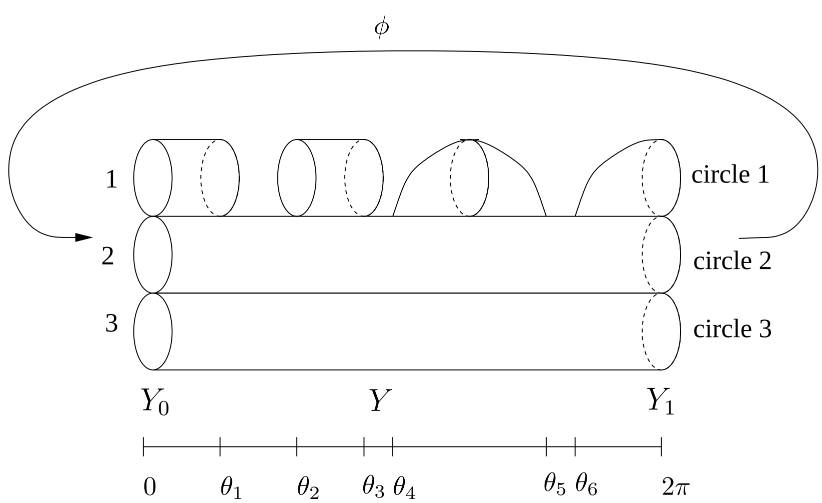
Proposition 3.3 $\dim \mathrm{img}(\mathrm{H_r}(\mathrm{X}_\theta) \to \mathrm{H_r}(\mathrm{X})) = \sharp\{\mathrm{B} \in \mathcal{B}_\mathrm{r} | \mathrm{n}_\theta(\mathrm{B}) \neq 0 \text{ and both ends closed}\} + \sharp\{\mathrm{J} \in \mathcal{J}_\mathrm{r} | \lambda = 1\}$

A real valued tame map $f : X \to \mathbb{R}$ can be regarded as a circle valued tame map $f' : X \to \mathbb{S}^1$ by identifying $\mathbb{R}$ to $(0, 2\pi)$ with critical values $t_1, \cdots, t_m$ becoming the critical angles $\theta_1, \cdots, \theta_m$ where $\theta_i = 2 \arctan t_i + \pi$. The map f' in this case will not have any Jordan cells and the bar codes will be the same as level persistence bar codes. We have the following corollary:

Corollary 3.4 $\dim H_r(X_\theta) = \sum_{B \in \mathcal{B}_r} n_\theta(B)$ *and*
$\dim H_r(X) = \sharp\{B \in \mathcal{B}_r | \text{both ends closed}\} + \sharp\{\mathrm{B} \in \mathcal{B}_{\mathrm{r}-1} | \text{both ends open}\}$.

Theorem 3.1 is quite intuitive and is in analogy with the derived results for sublevel and level persistence [4, 17]. Theorem 3.2 is more subtle. Its counterpart for real valued function (Corollary 3.4) has not yet appeared in the literature though a related result for homology of source space can be derived from extended persistence [6]. The proofs of these results require the definition of the bar codes and Jordan cells which appear in the next section. The proofs are sketched in Section 5.

The Questions Q1-Q3 can be answered using the bar codes. The question Q4 about returned homology can be answered using the bar codes and Jordan cells.



map ϕ	r-invariants		
	dimension	bar codes	Jordan cells
circle 1: 3 times around 2, 2 times around 3	0		$(1,1)$
circle 2: 3 times around 1, one time around 2, 3 times around 3	1	$(\theta_6, \theta_1 + 2\pi]$	$(3,2)$
circle 3: 1 time around 3		$[\theta_2, \theta_3]$	
		(θ_4, θ_5)	

Figure 2: Example of r-invariants for a circle valued map

Figure 2 indicates a tame map $f : X \to \mathbb{S}^1$ and the corresponding invariants, bar codes, and Jordan cells. The space X is obtained from Y in the figure by identifying its right end Y_1 (a union of three circles) to the left end Y_0 (again a union of three circles) following the map $\phi : Y_1 \to Y_0$. The map $f : X \to \mathbb{S}^1$ is induced by the projection of Y on the interval $[0, 2\pi]$. We have $H_1(Y_1) = H_1(Y_0) = \kappa \oplus \kappa \oplus \kappa$ and ϕ

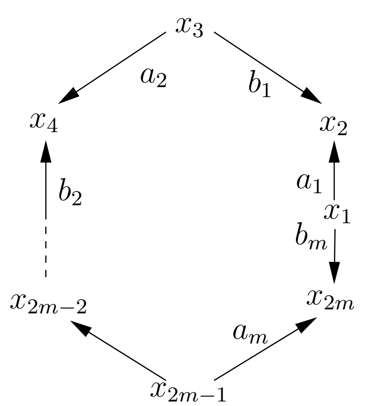
induces a linear map in 1-homology represented by the matrix

$$\begin{pmatrix} 0 & 3 & 0 \\ 3 & 2 & 0 \\ 2 & 3 & 1 \end{pmatrix}.$$

The first generator (circle 1) of $H_1(\tilde{X}_{2\pi})$ is dead in $H_1(\tilde{X}_{[\theta,2\pi]})$ for $\theta \leq \theta_6$ but not for $\theta \in (\theta_6, 2\pi]$ and is detected in $H_1(\tilde{X}_{2\pi+\theta})$ for $0 \leq \theta \leq \theta_1$ but not for $\theta > \theta_1$. It generates a bar code $(\theta_6, 2\pi + \theta_1]$. The other two (circle 2 and 3) never die and provide a Jordan cell $(1, 2)$. In Appendix it will be shown how to use the algorithm to calculate the bar codes and Jordan cells in the example above.

4 Representation theory and r-invariants

The invariants for the circle valued map are derived from the representation theory of quivers. The quivers are directed graphs. The representation theory of simple quivers such as paths with directed edges was described by Gabriel [10] and is at the heart of the derivation of the invariants for zigzag and then level persistence in [4]. For circle valued maps, one needs representation theory for circle graphs with directed edges. This theory appears in the work of Nazarova [15], and Donovan and Ruth-Freislich [11].



Let G_{2m} be a directed graph with $2m$ vertices, $x_1, x_1, \cdots x_{2m}$. Its underlying undirected graph is a simple cycle. The directed edges in G_{2m} are of two types: *forward* $a_i : x_{2i-1} \to x_{2i}$, $1 \leq i \leq m$, and *backward* $b_i : x_{2i+1} \to x_{2i}$, $1 \leq i \leq m-1$, $b_m : x_1 \to x_{2m}$.

We think of this graph as being located on the unit circle centered at the origin o in the plane.

A representation ρ on G_{2m} is an assignment of a vector space V_x to each vertex x and a linear map $V_e : V_x \to V_y$ for each oriented edge $e = \{x, y\}$. Two representations ρ and ρ' are isomorphic if for each vertex x there exists an isomorphism from the vector space V_x of ρ to the vector space V'_x of ρ', and these isomorphisms intertwine the linear maps $V_x \to V_y$ and $V'_x \to V'_y$. A *non-trivial* representation assigns at least one vector space which is not zero-dimensional. A representation is *indecomposable* if it is not isomorphic to the sum of two nontrivial representations. It is not hard to observe that each representation has a decomposition as a sum of indecomposable representations unique up to isomorphisms.

We provide a description of indecomposable representations of the quiver G_{2m}. For any triple of integers $\{i, j, k\}$, $1 \leq i, j \leq m$, $k \geq 0$, one may have any of the four representations, $\rho^I([i, j]; k)$, $\rho^I((i, j]; k)$, $\rho^I([i, j); k)$, and $\rho^I((i, j); k)$ defined below. The exponent I suggests that this representation is associated to an interval (bar code).

Suppose that the evenly indexed vertices $\{x_2, x_4, \cdots x_{2m}\}$ of G_{2m} which are the targets of the directed arrows correspond to the angles $0 < s_1 < s_2 < \cdots < s_m \leq 2\pi$. To construct the representation $\rho^I(\{i, j\}; k)$, draw the spiral curve given by equation (1) for the interval $\{s_i, s_j + 2k\pi\}$; refer to Figure 3.

For each x_i, let $\{e_i^1, e_i^2, \cdots\}$ denote the ordered set (possibly empty) of intersection points of the ray ox_i with the spiral. While considering these intersections, it is important to realize that the point $(x(s_i), y(s_i))$ (resp. $(x(s_j + 2k\pi), y(s_j + 2k\pi))$) does not belong to the spiral (eqn. 1) if $\{i, j\}$is open at i (resp. j). For example, in Figure 3, the last circle on the ray ox_{2j} is not on the spiral since $[i, j)$ in $\rho^I([i, j); 2)$ is open at right.

Let V_{x_i} denote the vector space generated by the base $\{e_i^1, e_i^2, \cdots\}$. Furthermore, let $\alpha_i : V_{x_{2i-1}} \to V_{x_{2i}}$ and $\beta_i : V_{x_{2i+1}} \to V_{x_{2i}}$ be the linear maps defined on bases and extended by linearity as follows: assign the vector $e_{2i}^h \in V_{x_i}$ to $e_{2i\pm1}^\ell$ if e_{2i}^h is an adjacent intersection point to the points $e_{2i\pm1}^\ell$ on the spiral. If e_{2i}^h does not exist, assign zero to $e_{2i\pm1}^\ell$. If $e_{2i\pm1}^\ell$ do not go to zero, h has to be l, $l-1$, or $l+1$.

The construction above provides a representation on G_{2m} which is indecomposable. One can also think these representations as the bar codes $[s_i, s_j + 2k\pi]$, $(s_i, s_j + 2k\pi]$, $[s_i, s_j + 2k\pi)$, and $(s_i, s_j + 2k\pi)$.

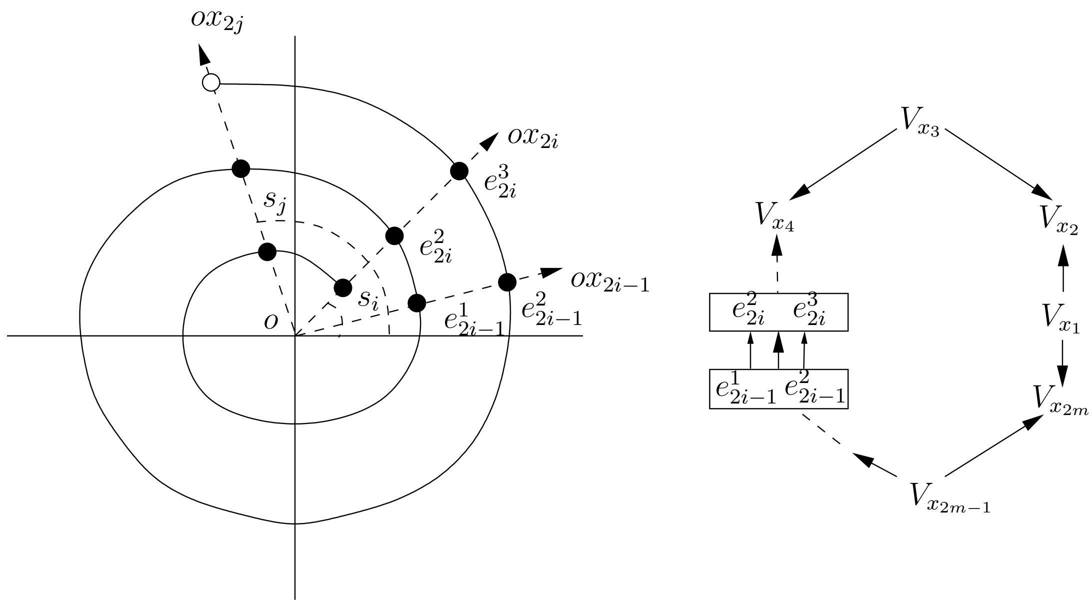


Figure 3: The spiral for $[s_i, s_j + 4\pi)$.

For any Jordan cell (λ, k) we associate a representation $\rho^J(\lambda, k)$ defined as follows. Assign the vector space with base $e_1, e_2, \cdots, e_k$ to each x_i and take all linear maps α_i but one (say α_1) and β_i the identity. The linear map α_1 is given by the Jordan matrix defined by (λ, k). The exponent J suggest that this representation is associated to a Jordan cell. Again this representation is indecomposable.

It follows from the work of [11, 15] that bar codes and Jordan cells as constructed above are all and only indecomposable representations of the quiver G_{2m}. We record a representation ρ as a collection of matrices with entries in a given field κ, $\{\alpha_i \in \mathbb{M}^{n_{2i}\times n_{2i-1}}, \beta_i \in \mathbb{M}^{n_{2i}\times n_{2i+1}}, 1 \le i \le m\}$ with the convention $n_{2m+1} = n_1$.

Observation 4.1 *A representation ρ has no indecomposable representations of type ρ^I in its decomposition iff all linear maps α_i's and β_i's are isomorphisms. For such a representation, starting with an index i, consider the linear isomorphism*

$$T_i = \beta_i^{-1} \cdot \alpha_i \cdot \beta_{i-1}^{-1} \cdot \alpha_{i-1} \cdots \beta_2^{-1} \cdot \alpha_2 \cdot \beta_1^{-1} \cdot \alpha_1 \cdot \beta_m^{-1} \cdot \alpha_m \cdot \beta_{m-1}^{-1} \cdot \alpha_{m-1} \cdots \beta_{i+1}^{-1} \cdot \alpha_{i+1}.$$

The Jordan canonical form [8] of the isomorphism T_i is independent of i and is a block diagonal matrix with the diagonal consisting of Jordan cells (λ, k)s. Clearly, ρ is the direct sum of $\rho^J(\lambda, k)$s with each (λ, k) being a Jordan cell of T_i.

The r-invariants. Let f be a circle valued tame map defined on a topological space X. For f with m critical angles $0 < s_1 < s_2, \cdots s_m \le 2\pi$, consider the quiver G_{2m} with the vertices x_{2i} identified with the angles s_i and the vertices x_{2i-1} identified with the angles t_i that satisfy $0 < t_1 < s_1 < t_2 < s_2, \cdots t_m < s_m$.

For any r, consider the representation ρ_r of G_{2m} with $V_{x_i} = H_r(X_{x_i})$ and the linear maps α_is and β_is induced in the r-homology by maps $a_i : X_{x_{2i-1}} \to X_{x_{2i}}$ and $b_i : X_{x_{2i+1}} \to X_{x_{2i}}$ described in section

2. The bar codes (with the ends expressed in terms of s_i's) and the Jordan cells of this representation are independent of the choice of t_is and are the r-*invariants* of f.

5 Proof of the main results

A careful look at Figure 2 and the bar codes indicate why a semi-closed (open one end closed the other) bar code does not contribute to the homology of the total space X and why a closed r-bar code (both ends closed) contributes one unit while an open (both end open) $(r-1)$-bar code contributes one unit to the r-homology of the total space. The lack of contribution of a Jordan cell with $\lambda \neq 1$ and the contribution of a r-Jordan cell with $\lambda = 1$ of one unit to both r and $r+1$ dimensional homology of the total space should not be a surprise for the reader familiar with the calculation the homology of mapping torus. Below we will explain rigorously but schematically the arguments for the proof of Theorems 3.1, 3.2, and Corollary 3.4.

First recall that a representation ρ of the graph G_{2m} is indicated by the vector spaces $V_{x_{2i-1}}, V_{x_{2i}}$ and the linear maps α_i and β_i. To such representation ρ we associate the block matrix $M_\rho : \bigoplus_{1\leq i\leq m} V_{x_{2i-1}} \to \bigoplus_{1\leq i\leq m} V_{x_{2i}}$ defined by:

$$\begin{pmatrix} \alpha_1 & -\beta_1 & 0 & \dots & \dots & 0 \\ 0 & \alpha_2 & -\beta_2 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & \dots & \dots & \dots & \dots \alpha_{m-1} & -\beta_{m-1} \\ -\beta_m & \dots & \dots & \dots & \dots & \alpha_m \end{pmatrix}$$

For this representation we define the "dimension" as the $2m$-tuple of positive integers $\dim(\rho) = (n_1, r_1 \cdots n_m, r_m)$ with $n_i = \dim V_{x_{2i-1}}$ and $r_i = \dim V_{x_{2i}}$ and the numbers $\ker(\rho) = \dim \ker M_\rho$ and $coker(\rho) = \dim coker M_\rho$. For the sum of two such representations $\rho = \rho_1 \oplus \rho_2$ we have:

Proposition 5.1

1. $\dim(\rho_1 \oplus \rho_2) = \dim(\rho_1) + \dim(\rho_2)$,
2. $\ker(\rho_1 \oplus \rho_2) = \ker(\rho_1) + \ker(\rho_2)$,
3. $coker(\rho_1 \oplus \rho_2) = coker(\rho_1) + coker(\rho_2)$.

The explicit description of the representations corresponding to the bar codes permits explicit calculations.

Proposition 5.2

1. *If $i \leq j$ then*
   - (a) $\dim \rho^I([i,j];k)$ *is given by:*
     $n_l = k+1$ *if* $(i+1) \leq l \leq j$ *and* k *otherwise,*
     $r_l = k+1$ *if* $i \leq l \leq j$ *and* k *otherwise*
   - (b) $\dim \rho^I((i,j];k)$ *is given by:*
     $n_l = k+1$ *if* $(i+1) \leq l \leq j$ *and* k *otherwise,*
     $r_l = k+1$ *if* $(i+1) \leq l \leq j$ *and* k *otherwise,*
   - (c) $\dim \rho^I([i,j);k)$ *is given by:*
     $n_l = k+1$ *if* $(i+1) \leq l \leq j$ *and* k *otherwise,*
     $r_l = k+1$ *if* $i \leq l \leq (j-1)$ *and* k *otherwise,*

(d) $\dim \rho^I((i,j);k)$ *is given by:*
$n_l = k+1$ *if* $(i+1) \leq l \leq j$ *and* k *otherwise,*
$r_l = k+1$ *if* $(i+1) \leq l \leq (j-1)$ *and* k *otherwise*

2. *If* $i > j$ *then similar statements hold.*

(a) $\dim \rho^I([i,j];k)$ *is given by:*
$n_l = k$ *if* $(j+1) \leq l \leq i$ *and* $k+1$ *otherwise;*
$r_l = k$ *if* $(j+1) \leq l \leq (i-1)j$ *and* $k+1$ *otherwise*

(b) $\dim \rho^I((i,j];k)$ *is given by:*
$n_l = k$ *if* $(j+1) \leq l \leq i$ *and* $k+1$ *otherwise.*
$r_l = k$ *if* $(j+1) \leq l \leq i$ *and* $k+1$ *otherwise,*

(c) $\dim \rho^I([i,j);k)$ *is given by:*
$n_l = k$ *if* $(j+1) \leq l \leq i$ *and* $k+1$ *otherwise;*
$r_l = k$ *if* $j \leq l \leq (i-1)$ *and* $k+1$ *otherwise,*

(d) $\dim \rho^I((i,j);k)$ *is given by:*
$n_l = k$ *if* $(j+1) \leq l \leq i$ *and* $k+1$ *otherwise;*
$r_l = k$ *if* $j \leq l \leq i$ *and* $k+1$ *otherwise.*

3. $\dim \rho^J(\lambda, k)$ *is given by* $n_i = r_i = k$

Proposition 5.3

1. $\ker \rho^I([i,j];k) = 0$, $coker \rho^I([i,j];k) = 1$,
2. $\ker \rho^I([i,j);k) = 0$, $coker \rho^I([i,j);k) = 0$,
3. $\ker \rho^I((i,j];k) = 0$, $coker \rho^I((i,j];k) = 0$,
4. $\ker \rho^I((i,j);k) = 1$, $coker \rho^I((i,j);k) = 0$,
5. $\ker \rho^J(\lambda, k) = 0$ (resp. 1) if $\lambda \neq 1$ (resp. 1),
6. $coker \rho^J(\lambda, k) = 0$ (resp. 1) if $\lambda \neq 1$ (resp. 1).

The proof of Theorem 3.1 is a consequence of Propositions 5.1 and 5.2. The proof of Theorem 3.2 goes on the following lines. First observe that, up to homotopy, the space X can be regarded as the iterated mapping torus $\mathcal{T}$ described below. Consider the collection of spaces and continuous maps:

$$X_m = X_0 \overset{b_0 = b_m}{\longleftarrow} R_1 \overset{a_1}{\longrightarrow} X_1 \overset{b_1}{\longleftarrow} R_2 \overset{a_2}{\longrightarrow} X_2 \cdots X_{m-1} \overset{b_{m-1}}{\longleftarrow} R_m \overset{a_m}{\longrightarrow} X_m$$

with $R_i := X_{t_i}$ and $X_i := X_{s_i}$ and denote by $\mathcal{T} = T(\alpha_1 \cdots \alpha_m; \beta_1 \cdots \beta_m)$ the space obtained from the disjoint union

$$(\bigsqcup_{1 \leq i \leq m} R_i \times [0,1]) \sqcup (\bigsqcup_{1 \leq i \leq m} X_i)$$

by identifying $R_i \times \{1\}$ to X_i by α_i and $R_i \times \{0\}$ to X_{i-1} by β_{i-1}. Denote by $f^{\mathcal{T}} : \mathcal{T} \to [0,m]$ where $f^{\mathcal{T}} : R_i \times [0,1] \to [i-1,i]$ is the projection on $[0,1]$ followed by the translation of $[0,1]$ to $[i-1,i]$. This map is a homotopical reconstruction of $f : X \to \mathbb{S}^1$ provided that, with the choice of angles t_i, s_i and maps a_i b_i described in section 2, $X_i := f^{-1}(s_i), R_i := f^{-1}(t_i)$.

Let $\mathcal{P}'$ denote the space obtained from the disjoint union

$$(\bigsqcup_{1\le i\le m} R_i \times (\epsilon, 1]) \sqcup (\bigsqcup_{1\le i\le m} X_i)$$

by identifying $R_i \times \{1\}$ to X_i by α_i, and $\mathcal{P}''$ denote the space obtained from the disjoint union

$$(\bigsqcup_{1\le i\le m} R_i \times [0, 1-\epsilon) \sqcup (\bigsqcup_{1\le i\le m} X_i)$$

by identifying $R_i \times \{0\}$ to X_{i-1} by β_{i-1}.

Let $\mathcal{R} = \bigsqcup_{1\le i\le m} R_i$ and $\mathcal{X} = \bigsqcup_{1\le i\le m} X_i$. Then, one has:

1. $\mathcal{T} = \mathcal{P}' \cup \mathcal{P}''$,
2. $\mathcal{P}' \cap \mathcal{P}'' = (\bigsqcup_{1\le i\le m} R_i \times (\epsilon, 1-\epsilon)) \sqcup \mathcal{X}$, and
3. the inclusions $(\bigsqcup_{1\le i\le m} R_i \times \{1/2\}) \sqcup \mathcal{X} \subset \mathcal{P}' \cap \mathcal{P}''$ as well as the obvious inclusions $\mathcal{X} \subset \mathcal{P}'$and $\mathcal{X} \subset \mathcal{P}''$ are homotopy equivalences.

The Mayer-Vietoris long exact sequence leads to the diagram

$$\begin{array}{ccccccccc}
 & & H_r(\mathcal{R}) & \xrightarrow{M_r(\alpha,\beta)} & H_r(\mathcal{X}) & & & & \\
 & \nearrow & \uparrow{\scriptstyle pr_1} & & \uparrow{\scriptstyle (Id,-Id)} & \searrow & & & \\
\cdots \longrightarrow H_{r+1}(\mathcal{T}) & \xrightarrow{\partial_{r+1}} & H_r(\mathcal{R}) \oplus H_r(\mathcal{X}) & \xrightarrow{N} & H_r(\mathcal{X}) \oplus H_r(\mathcal{X}) & \xrightarrow{(i^r,-i^r)} & H_r(\mathcal{T}) & \longrightarrow & \\
 & & \uparrow{\scriptstyle in_2} & & \uparrow{\scriptstyle \Delta} & & & & \\
 & & H_r(\mathcal{X}) & \xrightarrow{Id} & H_r(\mathcal{X}) & & & &
\end{array}$$

Here Δ denotes the diagonal, in_2 the inclusion on the second component, pr_1 the projection on the first component, i^r the linear map induced in homology by the inclusion $\mathcal{X} \subset \mathcal{T}$, and $M_r(\alpha, \beta)$ the map given by matrix

$$\begin{pmatrix}
\alpha_1^r & -\beta_1^r & 0 & \ldots & \ldots & 0 \\
0 & \alpha_2^r & -\beta_2^r & \ldots & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & \ldots & \ldots & \ldots & \ldots \alpha_{m-1}^r & -\beta_{m-1}^r \\
-\beta_m^r & \ldots & \ldots & \ldots & \ldots & \alpha_m^r
\end{pmatrix}$$

with $\alpha_i^r : H_r(R_i) \to H_r(X_i)$ and $\beta_i^r : H_r(R_{i+1}) \to H_r(X_i)$ induced by the maps α_i and β_i, and N defined by

$$\begin{pmatrix}
\alpha^r & Id \\
-\beta^r & Id
\end{pmatrix}$$

where α^r and β^r are the matrices

$$\begin{pmatrix}
\alpha_1^r & 0 & \ldots & \ldots & 0 \\
0 & \alpha_2^r & \ldots & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & \alpha_{m-1}^r
\end{pmatrix}$$

$$\begin{pmatrix} 0 & \beta_1^r & 0 & \dots & 0 \\ 0 & 0 & \beta_2^r & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & \beta_{m-1}^r \\ \beta_m^r & 0 & \dots & 0 & 0 \end{pmatrix}.$$

From the diagram above we retain only the long exact sequence

$$\dots \to H_r(\mathcal{R}) \xrightarrow{M(\rho_r)} H_r(\mathcal{X}) \to H_r(\mathcal{T}) \to H_{r-1}(\mathcal{R}) \xrightarrow{M(\rho_{r-1})} H_{r-1}(\mathcal{X}) \to \cdots \quad (3)$$

from which we derive the short exact sequence

$$0 \to coker M(\rho_r) \to H_r(\mathcal{T}) \to \ker M(\rho_{r-1}) \to 0 \quad (4)$$

and then

$$H_r(\mathcal{T}) = \mathrm{cokerM_r}(\alpha,\beta) \oplus \ker \mathrm{M_{r-1}}(\alpha,\beta) \quad (5)$$

Theorem 3.1 is rather straightforward. Theorem 3.2 follows from Propositions 5.1, 5.3 and the equation (5) above. A specified decomposition of ρ_r and ρ_{r-1} in indecomposable representations and a splitting in the sequence (4) provide specified elements in $H_r(X_\theta)$ and $H_r(\mathcal{T})$ which can be compared. This leads to the verification of Proposition 3.3.

6 Algorithm

Given a circle valued tame map $f : X \to \mathbb{S}^1$, we now present an algorithm to compute the bar codes when X is a finite simplicial complex, and f is generic and linear. This makes the map tame. Genericity means that f is injective on vertices. To explain linearity we recall that, for any simplex $\sigma \in X$, the restriction $f|_\sigma$ admits liftings $\hat{f} : \sigma \to \mathbb{R}$, i.e. $\hat{f}$ is a continuous map which satisfies $p \cdot \hat{f} = f|_\sigma$. The map $f : X \to \mathbb{S}^1$ is called *linear* if for any simplex σ, at least one of the liftings (and then any other) is linear.

First our algorithm takes the simplicial complex X equipped with the map f as input and computes a matrix form of the quiver representation $\rho = \rho_r$ for f. A quiver representation ρ for f is indicated by the vector spaces $V_{x_{2i-1}} = H_r(X_{x_{2i-1}}), V_{x_{2i}} = H_r(X_{x_{2i}})$ and the linear maps α_i and β_i. The matrix form of ρ is the block matrix $M_\rho : \bigoplus_{1\leq i\leq m} V_{x_{2i-1}} \to \bigoplus_{1\leq i \leq m} V_{x_{2i}}$ described in the previous section. We augment it by a row and two columns for convenience in computation, and thus write it as:

$$\begin{pmatrix} \alpha_m & -\beta_m & 0 & 0 & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & \alpha_1 & -\beta_1 & 0 & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & \alpha_2 & -\beta_2 & \dots & \dots & \dots & \dots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & & \dots & \dots & \dots \alpha_{m-1} & -\beta_{m-1} & 0 \\ 0 & -\beta_m & \dots & \dots & & \dots & \dots & \dots & \alpha_m & -\beta_m \end{pmatrix}$$

Next, from the augmented block matrix above one computes the bar codes and then the Jordan cells. The algorithm consists of three steps. We describe the first and second steps in sufficient details. The third step is a routine application of Observation 4.1 and is accomplished by standard algorithms in linear algebra (reduction of the matrix to the canonical Jordan form).

Step 1. Compute the matrix form of the quiver representation ρ_r.

Step 2. Process the matrix of ρ_r to derive the Jordan cells ending up with a matrix whose all α_i and β_i are invertible matrices.

Step 3 Compute the Jordan cells

Step 1. Step 1 uses a subroutine that takes following as input: the incidence matrix of a cell complex R which is not necessarily simplicial, and the submatrix of a subcomplex $A \subset R$. Assume that the ordering of the cells is compatible with the face incidences, that is, σ comes before σ' if σ is a face of σ'. Also, assume that all cells of A precede those of $R \setminus A$ in this order. This means that the submatrix for A appears in the upper left corner of the matrix for R. Run the persistence algorithm [7, 17] on the incidence matrix for A to compute a base of the homology group $H_r(A)$. Continue the procedure by adding the columns and rows of the matrix for R to obtain a base of $H_r(R)$. It is straightforward to compute a matrix representation of the linear map $H_r(A) \to H_r(R)$ w.r.t. the bases computed by the persistence algorithm.

In Step 1 we begin with the incidence matrix of the input simplicial complex X equipped with the map $f : X \to \mathbb{S}^1$. Let the angles $0 \leq s_1 < s_2 \cdots s_m \leq 2\pi$ be the critical values of f. Choose a collection of regular angles $0 < t_1 < t_2 \cdots t_m < 2\pi$ with $t_i < s_i < t_{i+1} < s_{i+1}$. Consider a canonical subdivision of X into a cell complex so that $X_{[t_i,t_{i+1}]}$, and X_{t_i} are subdivided into subcomplexes R_i and X_i as follows. For any open simplex σ we associate the open cells :

1. $\sigma(i) := \sigma \cap X_{t_i}$ with $\dim(\sigma(i)) = \dim \sigma - 1$ if the intersection is nonempty
2. $\sigma\langle i\rangle := \sigma \cap X_{(t_i,t_{i+1})}$ with $\dim \sigma\langle i\rangle = \dim \sigma$ if the intersection is nonempty.

The cells of X_i are exactly of the form $\sigma(i)$ and their incidences are given as $I(\sigma(i), \tau(i)) = I(\sigma, \tau)$ where $I(\sigma, \tau) = 0, +1,$ or -1 depending on whether τ is a coface of σ and whether their orientations match or not. The cells of R_i consist of cells of X_i, X_{i+1}, and all cells of the form $\sigma\langle i\rangle$. The incidences are given as $I(\sigma\langle i\rangle, \tau\langle i\rangle) = I(\sigma, \tau)$, $I(\sigma(i), \sigma\langle i\rangle) = 1$, and $I(\sigma(i+1), \sigma\langle i\rangle) = -1$. All other incidences are zero. Assume that we are given a total order for the simplices of X that is compatible with f and also the incidence relations. This induces a total order for the cells in X_i and X_{i+1} and also the cells in $R'_i = R_i \setminus X_i \sqcup X_{i+1}$ for any $1 \leq i \leq m$ with $X_{m+1} := X_1$. Impose a total order on R_i by juxtaposing the total orders of X_i, X_{i+1}, and R'_i in this sequence. Clearly, the incidence matrix for R_i can be derived from the incidence matrix of X. Apply the persistence algorithm to $R := R_i$ and $A = X_i \sqcup X_{i+1}$ as described before. We obtain the matrices for the linear maps $\alpha_i : H_r(X_{t_i}) \to H_r(X_{s_i})$ and $\beta_i : H_r(X_{t_{i+1}}) \to H_r(X_{s_i})$.

Step 2. Step 2 takes the matrix representation M_ρ constructed out of matrices α_i, β_i computed in step 1, and uses four elementary operations $T_1, T_2, T_3,$ and T_4 defined below to transform M_ρ to a matrix $M_{\rho'}$ whose total number of rows and columns is strictly smaller than that of M_ρ. The elementary operations modify submatrices by computing echelon forms and thus implicitly modify the representation ρ to ρ'. Each operation (modification) keeps the Jordan cells unchanged while eliminating or modifying the bar codes. When no such operation is applicable, the algorithm terminates with all α_i and β_i being necessarily invertible matrices. At this point the bar codes can be reconstructed by listing the eliminations/modifications performed and the Jordan cells can be obtained proceeding with Step 3.

To describe how Step 2 works consider the minors B_{2i-1} and B_{2i}, $i = 1, \cdots, m$ of M_ρ defined by

$$B_{2i-1} = \begin{pmatrix} \alpha_i & -\beta_i \\ 0 & \alpha_{i+1} \end{pmatrix} \quad B_{2i} = \begin{pmatrix} -\beta_i & 0 \\ \alpha_{i+1} & -\beta_{i+1} \end{pmatrix} \tag{6}$$

with $\alpha_{m+1} := \alpha_1, \beta_{m+1} := \beta_1, \alpha_0 = \alpha_m, \beta_0 = \beta_m$.

When $m > 1$ the algorithm iterates over the minors in multiple passes. In a single pass, it processes the minors $B_1, B_2, \ldots, B_{2m}$ in this order. When it considers B_i to process, it checks if any of the four elementary transformations is applicable to B_i. If so, it applies the respective transformation and updates B_i. These updates necessarily include deletion of some rows and columns that are made "irrelevant" by the respective transformation. These deletions of rows and columns correspond to shrinking of bar codes from their end points leading to eliminations in some cases. The algorithm terminates either when there is no rows and columns to operate or when no elementary transformation is applicable to any B_i in a pass. Clearly, the algorithm is guaranteed to terminate as each transformation necessarily decreases the number of rows and columns.

When $m = 1$, the operations on above minors are not well defined. In this case we extend the quiver G_2 to G_4 ($m = 2$) by adding fake levels t_2, s_2 where $H_r(X_{t_2}) = H_r(X_{s_2}) = H_r(X_{s_1})$ and α_2, β_2 are identities[4]. The algorithm works on input M_ρ as follows:

The main loop

Loop: for $j := 1$ to $2m$ do

1. if $j = 2i - 1$ is odd
   - i. if α_{i+1} is not injective, update $B_{2i-1} := T_2(B_{2i-1})$.
   - ii. if α_i is not surjective, update $B_{2i-1} := T_3(B_{2i-1})$.
   - iii. delete any rows and columns rendered irrelevant.
2. if $j = 2i$ is even
   - i. if β_{i+1} is not surjective, update $B_{2i} := T_4(B_{2i})$.
   - ii. if β_i is not injective, update $B_{2i} := T_1(B_{2i})$.
   - iii. delete any rows and columns rendered irrelevant.

endfor

if M_ρ has been updated and not empty, go to Loop. Otherwise, output M_ρ.

At termination, all α_i and β_i become isomorphisms because otherwise one of the transformations would be applicable. Each transformation effectively shrinks some bar codes even eliminating some of them. The bar codes can be recovered by keeping track of all eliminations of the bar codes after each elementary transformation. A bar code which is not eliminated in a pass gets shrunk by exactly two units, that is, a bar code $\{i, j\}$ shrinks to $\{i + 1, j - 1\}$ by exactly two distinct elementary transformations. The type of bar codes does not change by elementary transformations. When a barcode $[i, i]$ is eliminated, say, in the kth pass, we know that it corresponds to a bar code $[i-k+1, i+k-1]$ in the original representation. Similarly, other bar codes of type $\{i, i+1\}$ eliminated at the kth pass correspond to the bar code $\{i-k+1, i+k\}$ for f. In both cases, the multiplicity of the barcodes can be determined from the multiplicity of the eliminated bar codes thanks to Proposition 6.1 below. The Appendix provides an illustration on how the algorithm works on the example described in section 3.

Step 3. At termination, all α_i and β_i become isomorphisms because otherwise one of the transformations would be applicable. The Jordan cells can be recovered from the Jordan decomposition of the matrix

$$T = \beta_{i-1}^{-1} \cdot \alpha_{i-1} \cdot \beta_{i-2}^{-1} \cdots \beta_1^{-1} \cdot \alpha_1 \cdot \beta_m^{-1} \cdot \alpha_m \cdots \beta_{i+1}^{-1} \cdot \alpha_{i+1} \cdot \beta_i^{-1} \cdot \alpha_i \quad \text{for any } i.$$

Standard linear algebra routines permit the calculation of the Jordan cells for familiar algebraic closed fields. Note that if κ is not algebraically closed Step 1 and Step 2 can still be performed and the matrix T can be obtained. In this case we might not want to decompose the matrix T in Jordan cells since this requires

[4]Other easier methods can also be used in this case

passing to the algebraic closure of κ but only to decompose the matrix T up to conjugacy as a sum of indecomposable invertible matrices, remaining in the world of matrices with coefficients in the field κ. This is the case of the field $\kappa = \mathbb{Z}_2$ of particular interest in Computer Sciences. Such linear algebra algorithms exists too.

Elementary transformations. In our algorithm, we used four elementary transformations T_1, T_2, T_3, and T_4 that modify the matrix form of a quiver representation. We elaborate on these four transformations here. First, we interpret these transformations in terms of the quiver representations, and then give their implementations in terms of matrices.

Each transformation takes an index i and a representation

$$\rho = \{V_j | 1 \leq j \leq 2m,\ \alpha_s : V_{2s-1} \to V_{2s},\ \beta_s : V_{2s+1} \to V_{2s} | 1 \leq s \leq m\}$$

and produces a new representation

$$\rho' = \{V'_j | 1 \leq j \leq 2m,\ \alpha'_s : V'_{2s-1} \to V'_{2s},\ \beta'_s : V'_{2s+1} \to V'_{2s} | 1 \leq s \leq m\}$$

as follows:

1. If $\rho' = T_1(\rho, i)$ then $V'_{2i-1} = V_{2i-1}/\ker(\beta_{i-1})$, $V'_{2i} = V_{2i}/\alpha_i(\ker(\beta_{i-1}))$, $V'_j = V_j$ for $j \neq \{2i-1, 2i\}$ with α'_s, β'_s being induced from α_s, β_s for $s \in [1, m]$.

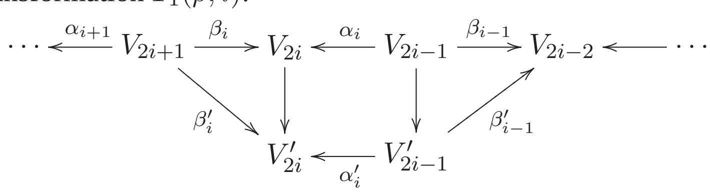
2. If $\rho' = T_2(\rho, i)$ then $V'_{2i+1} = V_{2i+1}/\ker(\alpha_{i+1})$, $V'_{2i} = V_{2i}/\beta_i(\ker \alpha_{i+1})$, $V'_j = V_j$ for $j \neq \{2i+1, 2i\}$ with α'_s, β'_s being induced from α_s, β_s for $s \in [1, m]$.

3. If $\rho' = T_3(\rho, i)$ then $V'_{2i} = \alpha_i(V_{2i-1})$, $V'_{2i+1} = \beta_i^{-1}(\alpha_i(V_{2i-1}))$, $V'_j = V_j$ for $j \neq \{2i, 2i+1\}$ with α'_s, β'_s being the restrictions of α_s, β_s for $s \in [1, m]$.

4. If $\rho' = T_4(\rho, i)$ then $V'_{2i} = \beta_i(V_{2i+1})$, $V'_{2i-1} = \alpha_i^{-1}(\beta_i(V_{2i+1}))$, $V'_j = V_j$ for $j \neq \{2i, 2i-1\}$ with α'_s, β'_s being the restrictions of α_s, β_s for $s \in [1, m]$.

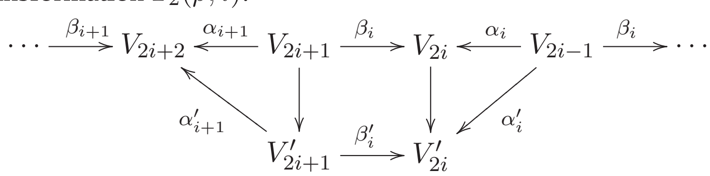
The following diagrams [5] indicate the constructions described above. The indices increase from right to left to signify that the vector spaces are laid counterclockwise with increasing indices around a quiver.

Transformation $T_1(\rho, i)$:



$$V'_{2i-1} = V_{2i-1}/\ker(\beta_{i-1}) \quad V'_{2i} = V_{2i}/\alpha_i(\ker(\beta_{i-1}))$$

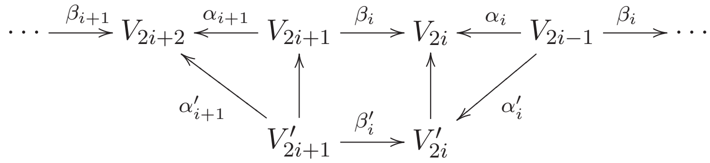
Transformation $T_2(\rho, i)$:



[5] in the diagrams $V_0 = V_{2m}$ and $\beta_0 = \beta_m$

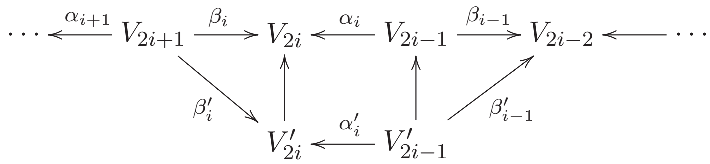
$$V'_{2i+1} = V_{2i+1}/\ker(\alpha_{i+1}), \quad V'_{2i} = V_{2i}/\beta_i(\ker(\alpha_{i+1}))$$

Transformation $T_3(\rho, i)$:



$$V'_{2i} = \alpha_i(V_{2i-1}) \quad V'_{2i+1} = \beta_i^{-1}(\alpha_i(V_{2i-1}))$$

Transformation $T_4(\rho, i)$:



$$V'_{2i} = \beta_i(V_{2i+1}) \quad V'_{2i-1} = \alpha_i^{-1}(\beta_i(V_{2i+1}))$$

The following observations follow straightforwardly from the definitions.

$T_1(\rho, i)$ eliminates all bar codes of the form $(i-1, i)$ and $(i-1, i]$, shrinks each bar code of the form $(i-1, k]$ and $(i-1, k)$, $k \geq (i+2)$, into bar codes $(i, k]$ and (i, k) respectively with the convention that $(i-1, k\}$ is $(m, m+k\}$ when $i = 1$, and leaves all other barcodes and Jordan cells unchanged.

$T_2(\rho, i)$ eliminates all bar codes of the form $(i, i+1)$ and $[i, i+1)$, shrinks each bar code of the form $[l, i+1)$ and $(l, i+1)$, $l \leq i-1$, into bar codes $[l, i)$ and (l, i) respectively, and leaves any other barcodes and Jordan cells unchanged.

Type $T_3(\rho, i)$ eliminates all bar codes of the form $[i, i]$ and $[i, i+1)$,shrinks each bar code of the forms $[i, k)$ and $[i, k]$, $k \geq i+1$, into the bar codes $[i+1, k)$ and $[i+1, k]$ respectively, and leaves all other type of barcodes and Jordan cells unchanged.

Type $T_4(\rho, i)$ eliminates all bar codes of the form $[i, i]$ and $(i-1, i]$, shrinks each bar code of the forms $(l, i]$ and $[l, i]$, $l \leq i-1$, into the bar codes $(l, i-1]$ and $[l, i-1]$ respectively with the convention that $\{l, 0\}$ is identified to $\{l+m, m\}$, and leaves all other type of barcodes and Jordan cell unchanged.

Let $\sharp\{i, j\}_\rho$ denote the number of bar codes of type $\{i, j\}$ [6] for a representation ρ. We have the following:

Proposition 6.1

1. $\sharp(i, i+1)_\rho = \dim\ker\beta_i \cap \ker\alpha_{i+1}$
2. $\sharp[i, i]_\rho = \dim(V_{2i}/((\beta_i(V_{2i+1}) + \alpha_i(V_{2i-1}))$
3. $\sharp(i, i+1]_\rho = \dim(\beta_i(V_{2i+1}) + \alpha_i(\ker\beta_{i-1})) - \dim(\beta_i(V_{2i+1}))$
4. $\sharp[i, i+1)_\rho = \dim(\alpha_i(V_{2i-1}) + \beta_i(\ker\alpha_{i+1})) - \dim(\alpha_i(V_{2i-1}))$

which can be easily derived from the matrices α_i and β_i.

[6] we use "$\{$" resp. for "$\}$" as a common notation for "$[$" and "$($" resp. "$]$" and "$)$".

Implementing the elementary transformations. Elementary transformations are implemented with algorithms known to produce the so-called *echelon form* of a matrix. We briefly describe the concept of echelon form for completeness. Let A be any $m \times n$ matrix. For any column j in A, denote by $r(j) \in \{1, \cdots, m\}$ the first index so that $a_{r(j),j} \neq 0$, i.e., $a_{s,j} = 0$ if $s < r(j)$. We say that A is in column- echelon form if:

1. $a_{r(j),j} = 1$
2. $r(j) < r(j+1)$
3. $a_{r(j),s} = 0$ for $s < j$

and in row echelon form if the transpose A^* is in column echelon form. The following is well-known in linear algebra.

Observation 6.2 *Given a matrix A,*

1. *there exists an $m \times m$ matrix $L(A)$ so that $L(A) \cdot A$ is in row echelon form;*
2. *there exists an $n \times n$ matrix $R(A)$ so that $A \cdot R(A)$ is in column echelon form;*
3. *the rank and the dimension of the kernel of A can be read off from its echelon form.*

There are well known algorithms to obtain the matrices $L(A)$ and $R(A)$ to produce the echelon forms.

Now we describe how we take advantage of the echelon forms to implement the elementary transformations. In transformations T_2 and T_3, we replace the matrix

$$\begin{pmatrix} \alpha_i & -\beta_i \\ 0 & \alpha_{i+1} \end{pmatrix} \text{ first by } \begin{pmatrix} \bar{\alpha}_i & -\bar{\beta}_i \\ 0 & \bar{\alpha}_{i+1} \end{pmatrix} \text{ and then by } \begin{pmatrix} \bar{\bar{\alpha}}_i & -\bar{\bar{\beta}}_i \\ 0 & \bar{\bar{\alpha}}_{i+1} \end{pmatrix}$$

while in transformations T_1 and T_4 we replace

$$\begin{pmatrix} -\beta_i & 0 \\ \alpha_{i+1} & -\beta_{i+1} \end{pmatrix} \text{ first by } \begin{pmatrix} -\bar{\beta}_i & 0 \\ \bar{\alpha}_{i+1} & -\bar{\beta}_{i+1} \end{pmatrix} \text{ and then by } \begin{pmatrix} -\bar{\bar{\beta}}_i & 0 \\ \bar{\bar{\alpha}}_{i+1} & -\bar{\bar{\beta}}_{i+1} \end{pmatrix}.$$

Because of that we will also write

$$T_1(B_{2i}),\ T_4(B_{2i}),\ T_2(B_{2i-1}),\ T_3(B_{2i-1})$$

for

$$T_1(\rho, i+1),\ T_4(\rho, i+1),\ T_2(\rho, i),\ T_3(\rho, i).$$

The blocks (submatrices) in the second and the third matrices represent the same linear transformations as the ones in the first one but with respect to different bases of the same vector spaces V_is. The columns and rows which became zero by passing to the second and to third forms are referred to as “irrelevant” and are removed without damaging the Jordan cells but implying elimination or changes in the bar codes as specified earlier.

Transformation $T_2(\rho, i) = T_2(B_{2i-1}) = T_2 \begin{pmatrix} \alpha_i & -\beta_i \\ 0 & \alpha_{i+1} \end{pmatrix}$:

The matrices below summarize the modifications carried out for this transformation.

$$\left(\begin{array}{c|c} \alpha_i & \beta_i \\ \hline 0 & \alpha_{i+1} \end{array}\right) \Rightarrow \left(\begin{array}{c|c|c} \bar{\alpha}_i = \alpha_i & \bar{\beta}_{i,1} & \bar{\beta}_{i,2} \\ \hline 0 & \bar{\alpha}_{i+1,1} & 0 \end{array}\right) \Rightarrow \left(\begin{array}{c|c|c} \bar{\bar{\alpha}}_{i,1} & \bar{\bar{\beta}}_{i,1,1} & \bar{\bar{\beta}}_{i,2} \\ \hline \bar{\bar{\alpha}}_{i,2} & \bar{\bar{\beta}}_{i,1,2} & 0 \\ \hline 0 & \bar{\bar{\alpha}}_{i+1,1} & 0 \end{array}\right)$$

First take

$$\begin{aligned} \bar{\alpha}_i &= \alpha_i \\ \bar{\beta}_i &= \beta_i \cdot R(\alpha_{i+1}) = (\bar{\beta}_{i,1},\ \bar{\beta}_{i,2}) \\ \bar{\alpha}_{i+1} &= \alpha_{i+1} \cdot R(\alpha_{i+1}) = (\bar{\alpha}_{i+1,1},\ 0) \end{aligned}$$

and then take

$$\begin{aligned} \bar{\bar{\alpha}}_i &= L(\bar{\beta}_{i,2}) \cdot \bar{\alpha}_i = \begin{pmatrix} \bar{\bar{\alpha}}_{i,1} \\ \bar{\bar{\alpha}}_{i,2} \end{pmatrix} \\ \bar{\bar{\beta}}_{i,1} &= L(\bar{\beta}_{i,2}) \cdot \bar{\beta}_{i,1} = \begin{pmatrix} \bar{\bar{\beta}}_{i,1,1} \\ \bar{\bar{\beta}}_{i,1,2} \end{pmatrix} \\ \bar{\bar{\beta}}_{i,2} &= L(\bar{\beta}_{i,2}) \cdot \bar{\beta}_{i,2} = \begin{pmatrix} \bar{\bar{\beta}}_{i,2,1} \\ 0 \end{pmatrix} \\ \bar{\bar{\alpha}}_{i+1} &= \bar{\alpha}_{i+1} \Rightarrow \bar{\bar{\alpha}}_{i+1,1} = \bar{\alpha}_{i+1,1} \end{aligned}$$

The "irrelevant columns" correspond to the last elements of the modified base of V_{2i+1} so that the zero columns of $\bar{\alpha}_{i+1}$ vanish. The "irrelevant rows" correspond to the last elements of the modified base of V_{2i} so that the zero rows of $L(\bar{\beta}_{i,2}) \cdot \bar{\beta}_{i,2}$ vanish.

The minor after the elimination of the irrelevant rows and columns becomes

$$\boxed{\begin{pmatrix} \alpha'_i & \beta'_i \\ 0 & \alpha'_{i+1} \end{pmatrix} := \begin{pmatrix} \bar{\bar{\alpha}}_{i,2} & \bar{\bar{\beta}}_{i,1,2} \\ 0 & \bar{\bar{\alpha}}_{i+1,1} \end{pmatrix}}.$$

Transformation $T_3(\rho, i) = T_3(B_{2i-1}) = T_3 \begin{pmatrix} \alpha_i & -\beta_i \\ 0 & \alpha_{i+1} \end{pmatrix}$:

First take

$$\begin{aligned} \bar{\alpha}_i &= L(\alpha_i) \cdot \alpha_i = \begin{pmatrix} \bar{\alpha}_{i,1} \\ 0 \end{pmatrix} \\ \bar{\beta}_i &= L(\alpha_i) \cdot \beta_i = \begin{pmatrix} \bar{\beta}_{i,1} \\ \bar{\beta}_{i,2} \end{pmatrix} \\ \bar{\alpha}_{i+1} &= \alpha_{i+1} \end{aligned}$$

and then take

$$\begin{aligned} \bar{\bar{\alpha}}_i &= \bar{\alpha}_i \Rightarrow \bar{\bar{\alpha}}_{i,1} = \bar{\alpha}_{i,1} \\ \bar{\bar{\beta}}_{i,1} &= \bar{\beta}_{i,1} \cdot R(\bar{\beta}_{i,2}) = (\bar{\bar{\beta}}_{i,1,1},\ \bar{\bar{\beta}}_{i,1,2}) \\ \bar{\bar{\beta}}_{i,2} &= \bar{\beta}_{i,2} \cdot R(\bar{\beta}_{i,2}) = (\bar{\bar{\beta}}_{i,2,1},\ 0) \\ \bar{\bar{\alpha}}_{i+1} &= \bar{\alpha}_{i+1} \cdot R(\bar{\beta}_{i,2}) = (\bar{\bar{\alpha}}_{i+1,1},\ \bar{\bar{\alpha}}_{i+1,2}) \end{aligned}$$

The minor after the elimination of the irrelevant rows and columns becomes

$$\boxed{\begin{pmatrix} \alpha'_i & \beta'_i \\ 0 & \alpha'_{i+1} \end{pmatrix} := \begin{pmatrix} \bar{\bar{\alpha}}_{i,1} & \bar{\bar{\beta}}_{i,1,2} \\ 0 & \bar{\bar{\alpha}}_{i+1,2} \end{pmatrix}}$$

Transformation $T_4(\rho, i+1) = T_4(B_{2i}) = T_4 \begin{pmatrix} -\beta_i & 0 \\ \alpha_{i+1} & -\beta_{i+1} \end{pmatrix}$:

First take

$$\begin{aligned} \bar{\beta}_i =& \beta_i \\ \bar{\alpha}_{i+1} =& L(\beta_{i+1}) \cdot \alpha_{i+1} = \begin{pmatrix} \bar{\alpha}_{i+1,1} \\ \bar{\alpha}_{i+1,2} \end{pmatrix} \\ \bar{\beta}_{i+1} =& L(\beta_{i+1}) \cdot \beta_{i+1} = \begin{pmatrix} \bar{\beta}_{i+1,1} \\ 0 \end{pmatrix} \end{aligned}$$

and then take

$$\begin{aligned} \bar{\bar{\beta}}_i =& \bar{\beta}_i \cdot R(\bar{\alpha}_{i+1,2}) = (\bar{\bar{\beta}}_{i,1} \ \bar{\bar{\beta}}_{i,2}) \\ \bar{\bar{\alpha}}_{i+1,1} =& \bar{\alpha}_{i+1,1} \cdot R(\bar{\alpha}_{i+1,2}) = (\bar{\bar{\alpha}}_{i+1,1,1} \ \bar{\bar{\alpha}}_{i+1,1,2}) \\ \bar{\bar{\alpha}}_{i+1,2} =& \bar{\alpha}_{i+1,2} \cdot R(\bar{\alpha}_{i+1,2}) = (\bar{\bar{\alpha}}_{i+1,2,1} \ 0) \\ \bar{\bar{\beta}}_{i+1} =& \bar{\beta}_{i+1} \Rightarrow \bar{\bar{\beta}}_{i+1,1} = \bar{\beta}_{i+1,1} \end{aligned}$$

The minor after the elimination of the irrelevant rows and columns is

$$\boxed{\begin{pmatrix} \beta'_i & 0 \\ \alpha'_{i+1} & \beta'_{i+1} \end{pmatrix} := \begin{pmatrix} \bar{\bar{\beta}}_{i,1} & 0 \\ \bar{\bar{\alpha}}_{i+1,1,2} & \bar{\bar{\beta}}_{i+1,2} \end{pmatrix}}.$$

Transformation $T_1(\rho, i+1) = T_1(B_{2i}) = T_1 \begin{pmatrix} -\beta_i & 0 \\ \alpha_{i+1} & -\beta_{i+1} \end{pmatrix}$:

First take

$$\begin{aligned} \bar{\beta}_i =& \beta_i \cdot R(\beta_i) = (\bar{\beta}_{i,1}, \ 0) \\ \bar{\alpha}_{i+1} =& \alpha_{i+1} \cdot R(\beta_i) = (\bar{\alpha}_{i+1,1}, \ \bar{\alpha}_{i+1,2}) \\ \bar{\beta}_{i+1} =& \beta_{i+1} \end{aligned}$$

and then take

$$\begin{aligned} \bar{\bar{\beta}}_i =& \bar{\beta}_i \Rightarrow \bar{\bar{\beta}}_{i,1} = \bar{\beta}_{i,1} \\ \bar{\bar{\alpha}}_{i+1,2} =& L(\bar{\alpha}_{i+1,2}) \cdot \bar{\alpha}_{i+1,2} = \begin{pmatrix} \bar{\bar{\alpha}}_{i+1,2,1} \\ 0 \end{pmatrix} \\ \bar{\bar{\alpha}}_{i+1,1} =& L(\bar{\alpha}_{i+1,2}) \cdot \bar{\alpha}_{i+1,1} = \begin{pmatrix} \bar{\bar{\alpha}}_{i+1,1,1} \\ \bar{\bar{\alpha}}_{i+1,1,2} \end{pmatrix} \\ \bar{\bar{\beta}}_{i+1} =& L(\bar{\alpha}_{i+1,2}) \cdot \bar{\beta}_{i+1} = \begin{pmatrix} \bar{\bar{\beta}}_{i+1,1} \\ \bar{\bar{\beta}}_{i+1,2} \end{pmatrix} \end{aligned}$$

The minor after the elimination of the irrelevant rows and columns becomes

$$\boxed{\begin{pmatrix} \beta'_i & 0 \\ \alpha'_{i+1} & \beta'_{i+1} \end{pmatrix} := \begin{pmatrix} \bar{\bar{\beta}}_{i,1} & 0 \\ \bar{\bar{\alpha}}_{i+1,1,2} & \bar{\bar{\beta}}_{i+1,2} \end{pmatrix}}.$$

7 Conclusions

We have analyzed circle valued maps from the perspective of topological persistence. We show that the notion of persistence for such maps incorporate an invariant that is not encountered in persistence studied erstwhile. Our results also shed lights on computing homology vector spaces and other topological invariants from bar codes and Jordan cells (Theorems 3.1 and 3.2). We have given an algorithm to compute the bar codes and the Jordan cells; the algorithms can be also adapted to compute zigzag persistence. In subsequent work Burghelea and Haller have derived more subtle topological invariants like Novikov homology, monodromy [3], Reidemeister torsion, and others from bar codes and Jordan cells confirming their mathematical relevance. We have not treated in this paper the stability of the invariants. The issue will be addressed in a forthcoming paper, see [3] for partial answer.

The standard persistence is related to Morse theory. In a similar vein, the persistence for circle valued map is related to Morse Novikov theory [16]. The work of Burghelea and Haller applies Morse Novikov theory to instantons and closed trajectories for vector field with Lyapunov closed one form [2]. The results in this paper will very likely provide additional insight on the dynamics of these vector fields [2] and have implications in computational topology in particular and algebraic topology in general.

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Appendix

After applying Step1 of the algorithm to the example described in section 3 one obtains the representation ρ_0, ρ_1 of the graph G_{14}. The representation ρ_0 has all vector spaces one dimensional and all $\alpha_i = \beta_i$ the identity, hence lead only to a Jordan cell $(1; 1)$ in dimension zero. The representation ρ_1, as visible in Fig 2 is given by : is given by

$$\begin{gathered}
\alpha_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \alpha_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}; \alpha_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \alpha_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\alpha_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \alpha_6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \alpha_7 = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 2 & 0 \\ 2 & 3 & 1 \end{pmatrix} \\
\beta_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}; \beta_2 = I \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \beta_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}; \beta_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
\beta_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \beta_6 =; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \beta_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{gathered} \tag{7}$$

Here is what the Step 2 of the algorithm does:

1. inspect B_1 - no change; inspect B_2- no change;
2. inspect B_3, - apply $T_3(B_3)$ and obtain $\rho(1) = T_3(\rho_1, 2)$;
3. inspect B_4, - apply $T_4(B_4)$ and obtain $\rho(2) = T_4(\rho(1), 3)$;
4. inspect B_5 - no change; inspect B_6- no change;
5. inspect B_7, - apply $T_2(B_7)$, and obtain $\rho(3) = T_2(\rho(2), 4)$;
6. inspect B_8 - no change; inspect B_9- no change; inspect B_{10}- no change; inspect B_{11}- no change;
7. inspect B_{12}- apply $T_1(B_{12})$ and obtain $\rho(4) = T_1(\rho(3), 7)$;
8. inspect B_{13}-no change;
9. inspect B_{14}- apply $T_4(B_{14})$, and obtain $\rho(5) = T_4(\rho(4), 8 = 1)$which has all α_i's and β_i's but α_7 the identity and $\alpha_7 = \begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}$. Hence we have one Jordan cell $(3; 2)$.

Recall that:

1. $T_3(B_3) = T_3(\rho, 2)$ eliminates all bar codes of the form $[2, 2]$ and $[2, 3)$, shrinks each bar code of the form $[2, k)$ and $[2, k]$, $k \geq 3$, into bar codes $[3, k)$ and $[3, k]$ respectively,
2. $T_4(B_4) = T_4(\rho, 3)$ eliminates all bar codes of the form $[3, 3]$ and $(2, 3]$, shrinks each bar code of the forms $(l, 3]$ and $[l, 3]$, $l \leq 2$, into the bar codes $(l, 2]$ and $[l, 2]$ respectively,
3. $T_2(B_7) = T_2(\rho, 4)$ eliminates all bar codes of the form $(4, 5)$ and $[4, 5)$, shrinks each bar code of the form $[l, 5)$ and $(l, 5)$, $l \leq 3$, into bar codes $[l, 4)$ and $(l, 4)$ respectively,

4. $T_1(B_{12}) = T_1(\rho, 7)$ eliminates all bar codes of the form $(6, 7)$ and $(6, 7]$, shrinks each bar code of the form $(6, k]$ and $(6, k)$, $k \geq 8$, into bar codes $(7, k]$ and $(7, k)$ respectively,

5. $T_4(B_{14}) = T_4(\rho, 8)$ eliminates all bar codes of the form $[1, 1]$ and $(7, 8]$, shrinks each bar code of the forms $(l, 8]$ and $[l, 8]$, $l \leq 7$, into the bar codes $(l, 7]$ and $[l, 7]$ respectively,

In view of Proposition (6.1) which provides the multiplicity of the bar codes which were eliminated and in view of the shrinking described above we conclude:

1. $\rho(5)$ has no bar codes,

2. $\rho(4)$ has one bar code $(7, 8]$,

3. $\rho(3)$ has one bar code $(6, 8]$,

4. $\rho(2)$ has two bar codes $(6, 8]$ and $(4, 5)$,

5. $\rho(1)$ has three bar codes $[3, 3],\ (6, 8]$ and $(4, 5)$,

6. ρ_1 has three bar codes $[2, 3],\ (6, 8]$ and $(4, 5)$,

which converted in critical angles give exactly the barcodes described in the table associated with the example.