arXiv:1904.02626v2 [math.AT] 13 Apr 2019

Comparison between ”Parametrized homology via zigzag persistence” and ”Refined homology in the presence of a real-valued continuous function”

Dan Burghelea *

Abstract

One shows that the persistence diagrams $Dgm^{\cdots}(H\mathbb{X})$ and the measures $\mu^{\cdots}_{H\mathbb{X}}$ defined in [5] can be derived as restrictions of the maps δ^f, γ^f and of the measures $\dim \mathbb{F}(\cdots)$, $\dim \mathbb{T}(\cdots)$ considered in [2] and [1] to the points and the rectangles above and below diagonal.

Contents

1 Introduction

In a recent paper [5] published in this journal the authors have proposed for an $\mathbb{R}-$ space $\mathbb{X}$ (= a continuous map $f : X \to \mathbb{R}$), under reasonable topological hypotheses, four persistence diagrams

$$Dgm^{\wedge}(H\mathbb{X}), Dgm^{\vee}(H\mathbb{X}), Dgm^{\backslash\backslash}(H_r\mathbb{X}), Dgm^{//}(H\mathbb{X})$$

regarded as the relevant invariants for parametrized homology of $\mathbb{X}$. These persistence diagrams collect the four types of barcodes which can be associated to f via zigzag persistence, are maps with discrete support from $\overline{R}^2_+ = \{-\infty \leq a < b \leq +\infty\}$ to $\mathbb{Z}_{\geq 0}$ and can be interpreted as densities $d\mu$ of four integer valued measures $\mu = \mu^{\wedge}_{H_r(\mathbb{X})}, \mu^{\vee}_{H_r(\mathbb{X})}, \mu^{\backslash\backslash}_{H_r(\mathbb{X})}, \mu^{//}_{H_r(\mathbb{X})}$ for squares $R = [a, b] \times [c.d]$, $-\infty \leq a < b < c, d \leq \infty$.

In cite [2] sections 6 and 7 (cf. also [3]published in this journal) and with more details in the book [1] sections 5 and 6, under essentially the same hypotheses, two maps with discrete support, $\rho^f_r : \mathbb{R}^2 \to \mathbb{Z}_{\geq 0}, \gamma^f_r : \mathbb{R}^2 \setminus \Delta \to \mathbb{Z}_{\geq 0}$ with $\Delta = \{(x, x) \in \mathbb{R}^2\}$, have been defined and studied. [1]. In the case X is compact the support of the maps δ^f_r and γ^f_r is finite, hence these maps are configurations of points with multiplicity, cf. [2], [1] and [3].

---

*Department of Mathematics, The Ohio State University, Columbus, OH 43210,USA. Email: `burghele@math.ohio-state.edu`

[1] In the above references the discussion refers mostly to X compact but the conclusions remain the same without the need of any additional hypotheses when X is locally compact and $f : X \to \mathbb{R}$ is a proper map. In this case the support of the maps δ^f_r and γ^f_r is discrete; this is already used in the case of angle-valued maps

These maps are derived from the vector space-valued maps $\hat{\rho}_r^f$ and $\hat{\gamma}_r^f$ with $\rho^f = \dim \hat{\gamma}_r^f$ and $\gamma_r^f = \dim \hat{\gamma}_r^f$ and are viewed as relevant topological invariants for a real-valued map f. The support of ρ_r consists of points in $\mathbb{R}^2$ which correspond to the closed $r-$bar codes and open $(r-1)-$ barcodes while the support of γ_r of points which correspond to the closed-open and open-closed $r-$barcodes. In both [2] and [1] these two maps [2] are defined in two ways, one being as "densities" of the measures $\dim \mathbb{F}_r$ and $\dim \mathbb{T}_r$ on the sigma algebras associated to the same type of squares. A measure theoretic formulation of both δ_r^f, γ_r^f, and even more general of $\hat{\delta}_r^f$ and $\hat{\gamma}_r^f$, can be explicitly found in [1] subsection 9.2 and is implicit in [2] and [1] subsections 5.1 and 6.1.

The purpose of this note is to show that the persistence diagrams $Dmg^{\cdots}_{\cdots}$ and of the measures $\mu^{\cdots}_{\cdots}$ are the restrictions to the points and the rectangles above the diagonal resp. below diagonal after composing with the map $T(x,y) = (y,x)$ of the maps $\rho^f_{\dots}$ and the measures $\dim \mathbb{F}..., \dim \mathbb{T}...$ facts which might pass unnoticed in view of notational differences.

Precisely, one has:

Proposition 1.1

- *(a):* $Dmg^{\wedge}(H_r(\mathbb{X}))$ *equals* ρ_r^f *restricted to* $\mathbb{R}^2_+$,
- *(b):* $Dmg^{\vee}(H_r(\mathbb{X}))$ *equals* $\rho_{r-1}^f \cdot T$ *restricted to* $\mathbb{R}^2_+$,
- *(c):* $Dmg^{\backslash\backslash}(H_r(\mathbb{X}))$ *equals* ρ_r^f *restricted to* $\mathbb{R}^2_+$,
- *(d):* $Dmg^{//}(H_r(\mathbb{X}))$ *equals* $\rho_{r-1}^f \cdot T$ *restricted to* $\mathbb{R}^2_+$.

$(R^2_+ = \{a, b) \in \mathbb{R}^2, a < b\})$ and

Proposition 1.2

- *(A):* $\mu^{\wedge}_{H_r}([a,b] \times [c,d)) = \dim \mathbb{F}_r((a,b] \times [c,d))$,
- *(B):* $\mu^{\backslash\backslash}_{H_r}([a,b] \times [c,d]) = \dim \mathbb{T}_r((a,b] \times (c,d])$,
- *(C):* $\mu^{//}_{H_{r-1}}([a,b] \times [c,d]) = \dim \mathbb{F}_r((c,d] \times [a,b))$,
- *(D):* $\mu^{\vee}_{H_r}([a,b] \times [c,d]) = \dim \mathbb{F}_r((c,d] \times [a,b))$.

Note that the stability results as stated in [5] follows in a straightforward manner from the stability results of [2] or [1] and the Alexander duality in [5] can be derived without effort from the Poincaré-duality in [2] or [1] in the same way the Alexander-duality can be derived from the Poincaré duality.

(NOTE: After the posting of the first version of this note it was brought to my attention that the result on Alexander-duality as well as most of the arguments in [5] were contained in the thesis of one of the author, Sara Kalisnik, cf. http://www.matknjiz.si/doktorati/2013/Kalisnik-14521-4.pdf, and posted on arXiv cf. Sara Kalisnik , Alexander Duality for Parametrized Homology, arXiv:1303.1591.)

---

[2] They become four when one treats separately the sigma algebras generated by the above diagonal squares and the below diagonal squares

2 Definitions

2.1 CSKM-definitions

For simplicity in writing one shortens the notations in [5] by replacing the notation : $\wedge$, $\vee$, $\backslash\backslash$, $//$ by c, o, co, oc abreviatioins of *closed, open, closed-open, open-closed* and $\mu^{\cdots}_{H_r\mathbb{X}}$ by $\mu^{\cdots}_r$.
Consider
$\mathcal{B}^c_r =$ the multi-set of closed $r-$bar codes,
$\mathcal{B}^o_r =$ the multi-set of open $r-$ bar codes,
$\mathcal{B}^{c,o}_r =$ the multi-set of closed-open $r-$bar codes,
$\mathcal{B}^{o,c}_r =$ the multi-set of open-closed $r-$ bar codes.
The definitions of barcodes $\mathcal{B}^{\cdots}_r$ in [5] (called in [5] "intervals" and / or "decorated pairs") are based on the initial presentation of *zigzag persistence* introduced by Carlsson, de-Silva, Morozov in 2009 . A reformulation of these definitions in terms of "death" and of "observability" is provided in [1] subsection 9.1.1.

If for a barcode I one denotes by $l(I)$ resp. $r(I)$ the left end resp. the right end, then a careful reading of the definitions in [5] shows that for a box $R = [a,b] \times [c,d]$ with $-\infty \leq a < b < c < d \leq \infty$ one has:

$$\begin{aligned}
\mu^c_{H_r}(R) =& \sharp\{I \in \mathcal{B}^c_r \ \mid a < l(I) \leq b,\ c \leq r(I) < d\} \\
\mu^{c,o}_{H_r}(R) =& \sharp\{I \in \mathcal{B}^{c,o}_r \mid a < l(I) \leq b,\ c \leq r(I) < d\} \\
\mu^o_{H_r}(R) =& \sharp\{I \in \mathcal{B}^o_r \ \mid a < l(I) < b,\ c < r(I) < d\} \\
\mu^{o,c}_{H_r}(R) =& \sharp\{I \in \mathcal{B}^{o,c}_r \mid a < l(I) \leq b,\ c \leq r(I) < d\}
\end{aligned} \tag{1}$$

and then

$$\begin{aligned}
Dmg^c_r(a,b) =& \sharp\{I \in \mathcal{B}^c_r \mid a = l(I)\ r(I) = b\} \\
Dmg^{c,o}_r(a,b) =& \sharp\{I \in \mathcal{B}^{c,o}_r \mid a = l(I)\ r(I) = b\} \\
Dmg^o_r(a,b) =& \sharp\{I \in \mathcal{B}^o_r \mid a = l(I)\ r(I) = b\} \\
Dmg^{o,c}_r(a,b) =& \sharp\{I \in \mathcal{B}^{o,c}_r \mid a = l(I)\ r(I) = b\}
\end{aligned} \tag{2}$$

2.2 BH-definitions

Denote by :
$\mathbb{I}_a(r) = img(H_r(f^{-1}((-\infty,a])) \to H_r(X))$,
$\mathbb{I}^a(r) = img(H_r(f^{-1}([a,\infty))) \to H_r(X))$,
$\mathbb{F}_r(a,b) = \mathbb{I}_a(r) \cap \mathbb{I}^b(r)$, $F_r(a,b) := \dim \mathbb{F}_r(a,b)$,
$\mathbb{T}_r(a,b) := \ker(H_r(f^{-1}((-\infty,a]) \to H_r(f^{-1}((-\infty,b]))$ when $a < b$,
$\mathbb{T}_r(a,b) := \ker(H_r(f^{-1}([a,\infty)) \to f^{-1}([b,\infty))$ when $a > b$,
$T_r(a,b) = \dim \mathbb{T}_r(a,b)$.

For a box $B = (a,b] \times [c,d),\ a < b, c < d$ one defines

$$\mathbb{F}_r(B) := \mathbb{F}_r(b,c)/(\mathbb{F}_r(a,c) + \mathbb{F}_r(b,d));$$

and one observes

$$\dim \mathbb{F}_r(B) = F_r(b,c) + F_r(a,d) - F_r(a,c) - F_r(b,d). \tag{3}$$

For a box above diagonal $B' = (a,b] \times (c,d], a < b \leq c < d$ one defines

$$\mathbb{T}_r(B') := \mathbb{T}_r(b,d)/j'\mathbb{T}_r(a,d) + \mathbb{T}_r(b,c),$$

with $j' : \mathbb{T}_r(a,d) \to \mathbb{T}_r(b,d)$ the obviously induced linear map, and one observes

$$\dim \mathbb{T}_r(B') = T_r(b,d) + T_r(a,c) - T_r(a,d) - T_r(b,c). \tag{4}$$

For a box below diagonal $B'' = [c,d) \times [a,b), a < b \leq c < d$ one defines

$$\mathbb{T}_r(B'') := \mathbb{T}_r(c,a)/j''\mathbb{T}_r(d,a) + \mathbb{T}_r(c,b)$$

with $j'' : \mathbb{T}_r(d,a) \to \mathbb{T}_r(c,a)$ the obviously induced linear map, and one observes

$$\dim \mathbb{T}_r(B'') = T_r(c,a) + T_r(d,b) - T_r(c,b) - T_r(d,a). \tag{5}$$

Recall from [4] Theorem 3.2 and Proposition 5.3 or from [2] Proposition 4.1 [3] or from [1] Proposition 4.3 the following equalities:

- $$\dim H_r(f^{-1}(-\infty,a]) = \begin{cases} \sharp\{I \in \mathcal{B}^c_r \mid l(I) \leq a\} \\ \sharp\{I \in \mathcal{B}^o_{r-1} \mid I \subset (-\infty,a)\} \\ \sharp\{I \in \mathcal{B}^{co}_r \mid l(I) \leq a < r(I)\} \end{cases}$$

  and then

  $$\dim \mathbb{I}_a(r) = \begin{cases} \sharp\{I \in \mathcal{B}^c_r \mid l(I) \leq a\} \\ \sharp\{I \in \mathcal{B}^o_{r-1} \mid I \subset (-\infty,a)\} \end{cases},$$

- $$\dim H_r(f^{-1}([a,\infty)) = \begin{cases} \sharp\{I \in \mathcal{B}^c_r \mid r(I) \geq a\} \\ \sharp\{I \in \mathcal{B}^o_{r-1} \mid I \subset (a,\infty)\} \\ \sharp\{I \in \mathcal{B}^{oc}_r \mid l(I) < a \leq r(I)\} \end{cases}$$

  and then

  $$\dim \mathbb{I}^a(r) = \begin{cases} \sharp\{I \in \mathcal{B}^c_r \mid r(I) \geq a\} \\ \sharp\{I \in \mathcal{B}^o_{r-1} \mid I \subset (a,\infty)\} \end{cases}.$$

As a consequence we have

1. for $a > b$

   $$\dim \mathbb{F}_r(a,b) = \begin{cases} \sharp\{I \in \mathcal{B}^c_r \mid l(I) \leq a, r(I) \geq b\} \\ \sharp\{I \in \mathcal{B}^o_{r-1} \mid b < l(I) < r(I) < a\} \end{cases} \tag{6}$$

   $$\delta^f_r(a,b) = \sharp\{I = [a,b] \in \mathcal{B}^o_{r-1} \mid l(I) = b, r(I) = a\} \tag{7}$$

2. for $a \leq b$

   $$\dim \mathbb{F}_r(a,b) = \sharp\{I \in \mathcal{B}^c_r \mid l(I) \leq a \leq b \leq r(I)\} \tag{8}$$

   $$\delta^f_r(a,b) = \sharp\{I = [a,b] \in \mathcal{B}^c_r \mid l(I) = a, r(I) = b\} \tag{9}$$

3. for $a < b$

   $$\dim \mathbb{T}_{a,b}(r) = \sharp\{I \in \mathcal{B}^{co}_r \mid l(I) \leq a < r(I) \leq b\} \tag{10}$$

   $$\gamma^f_r(a,b) = \sharp\{I = [a,b) \in \mathcal{B}^o_{r-1} \mid l(I) = a, r(I) = b\} \tag{11}$$

4. for $a > b$

   $$\dim \mathbb{T}^{a,b}(r) = \sharp\{I \in \mathcal{B}^{oc}_r \mid b \leq l(I) < a \leq r(I)\} \tag{12}$$

   $$\gamma^f_r(a,b) = \sharp\{I = [a,b] \in \mathcal{B}^o_{r-1} \mid l(I) = b, r(I) = a\} \tag{13}$$

---

[3] a real valued map can be regarded as an angle valued map

3 Equalities

For $R = [a, b] \times [c, d]$ with $a < b < c < d$

(1)+ (8) +(3) imply Proposition1.2 (A) and (2)+ (9) imply Proposition1.1(a),
(1)+ (6) +(3) imply Proposition 1.2(B) and (2)+ (7) imply Proposition1.1(b),
(1)+ (10) +(4) imply Proposition1.2(C) and (2)+ (11) imply Proposition1.1(c),
(1)+ (12) +(4) imply Proposition1.2(D) and (2)+ (13) imply Proposition1.1(d).

References

[1] Dan Burghelea, *New topological invariants for real- and angle-valued maps; an alternative to Morse-Novikov theory* Word scientific Publishing Co. Pte. Ltd, 2017

[2] Dan Burghelea, Stefan Haller, *Topology of angle valued maps, bar codes and Jordan blocks,* J. Appl. and Comput. Topology, vol 1, pp 121-197, 2017, arXiv:1303.4328; Max Plank preprints

[3] D.Burghelea, A refinement of Betti numbers and homology in the presence of a continuous function I, Algebr. Geom. Topol. 17. pp 2051-2080, 2017, arXiv 1501.02486

[4] D. Burghelea and T. K. Dey, *Persistence for circle valued maps. Discrete Comput. Geom..* **50** 2013 pp69-98 ; arXiv:1104.5646

[5] Gunnar Carlsson, Vin de Silva, Sara Kalisnik, Dmitry Morozov. Parametrized homology via zigzag persistence, Algebr. Geom. Topol. 19 657-700, 2019