DYNAMICS, SPECTRAL GEOMETRY AND TOPOLOGY

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ABSTRACT. The paper is an informal report on joint work with Stefan Haller on Dynamics in relation with Topology and Spectral Geometry. By dynamics one means a smooth vector field on a closed smooth manifold; the elements of dynamics of concern are the rest points, instantons and closed trajectories. One discusses their counting in the case of a generic vector field which has some additional properties satisfied by a still very large class of vector fields.

CONTENTS

1. INTRODUCTION

This paper provides an informal presentation of results obtained in collaboration with Stefan Haller; some of them have been already published in [3] and [5]. We follow here an unpublished version [4] of the paper [3] which provides a slightly different formulation of the results. There is an additional gain in generality of the results presented in this paper. The rest points of the vector fields under consideration rather than of "Morse type" are "hyperbolic".

By dynamics we mean a smooth vector field X on a closed smooth manifold M. The elements of dynamics are the rest points, instantons and closed trajectories. In the case of a generic vector field the rest points are hyperbolic and because M is closed they are finitely many, but the instantons and closed trajectories are at most countable and possibly infinitely many. However, under additional hypotheses, in each homotopy class of continuous paths between two rest points and in each homotopy class of closed curves there are finitely many instantons respectively finitely many closed trajectories. Therefore the instantons and closed trajectories can be counted by "counting functions" defined on the set of corresponding homotopy classes. When M is connected, in the first case this set is in bijective correspondence with the set of elements of the fundamental group and in the second case with the set of conjugacy classes of elements of the fundamental group. For a still large class of vector fields the counting functions referred above have Laplace transforms with respect to some cohomology classes $\xi \in H^1(M\ \mathbb{R})$ and these Laplace

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transforms are holomorphic functions in some regions of the complex plane. These holomorphic functions can be described using differential geometry and topology.

In this paper we express these holomorphic functions in terms of geometric data independent of closed trajectories and instantons (Theorems 3.2 and 3.3 below). We also provide a precise description of the class of vector fields for which our results hold and show that this class is large enough to insure that the results are relevant (Theorem 2.4 below). These results are not true for all vector fields but provide some patterns which might be recognized in a much larger class of vector fields than the one we describe and they deserve additional investigations.

Section 2 of this paper provides the definition of the above mentioned elements of dynamics and of the properties the vector fields are supposed to have, in order to make our theory work, and this section is referred to as Basics in Dynamics. Section 3 discusses the spectral geometry needed to formulate the results stated in Theorems 3.2 and 3.3 which provide directly or indirectly interpretation of the Laplace transform of the counting functions of instantons and closed trajectories. Section 4 treats some relations with topology. This section is very sketchy because more substantial results will need additional lengthy definitions. We review only Novikov results about the rest points and Huchings-Lee and Pajitnov results about closed trajectories. To find more of what we know at this time the interested reader is invited to consult [5].

2. Basics in Dynamics

Let M be a smooth manifold, $TM \xrightarrow{p} M$ the tangent bundle and X a smooth vector field. Recall that a smooth vector field can be regarded as a smooth map $X : M \to TM$ s.t. $p \cdot X = id$.

Any vector field X has a flow $\Psi_t : M \to M$, a smooth one parameter group of diffeomorphisms. We denote by $\theta_m : \mathbb{R} \to M$, the unique trajectory which passes through $m \in M$ at $t = 0$, which is exactly $\theta_m(t) := \Psi_t(m)$.

Rest points: The set of rest points is the set $\mathcal{X} = \{x \in M | X(x) = 0\}$ where the vector field X vanishes.

At any such point the differential of the the map $X : M \to TM$ defines the linear map

$$D_x(X) : T_xM \to T_xM$$

with $T_x(M) = p^{-1}(x)$, as follows. Choose an open neighborhood U of x in M, and a trivialization of the tangent bundle above U, $\theta : TU \to U \times T_x(M)$, with $\theta|_{T_x(M)} = \mathrm{id}$. Consider $X^\theta := \mathrm{pr}_2 \circ \theta \circ X : U \to T_x(M)$ with pr_2 the projection on the second component. Clearly $X^\theta(x) = 0$. Observe that $D_x(X^\theta)$ is independent of θ, which justifies the notation $D_x(X) := D_x(X^\theta)$.

Definition 2.1. A rest point $x \in \mathcal{X}$ is called **nondegenerate = hyperbolic** if the real part $\Re\lambda \neq 0$ of each eigenvalue λ of $D_x(X)$ is different from zero. For x hyperbolic rest point the number of the eigenvalues (counted with multiplicity) whose real part is larger than 0 is called the **Morse index** of x.

Stable/Unstable sets: For a rest point x the set $W_x^\pm := \{y \in M | \lim_{t\to\pm\infty} \theta_x(t) = x\}$ is called stable /unstable set. If x is nondegenerate and has Morse index q then, by Perron-Hadamard theorem [10], W_x^+ resp. W_x^- is the image of a one to one immersion

$$i_x^+ : \mathbb{R}^{(n-q)} \to M,$$

resp.

$$i_x^- : \mathbb{R}^q \to M,$$

and is called the **stable manifold** resp. **unstable manifold** of x.

Trajectories: A smooth map $\theta : \mathbb{R} \to M$ is a **parametrized trajectory** if

$$d\theta(t)/dt = X(\theta(t)).$$

Two trajectories θ_1 and θ_2 are regarded as equivalent ($\theta_1 \equiv \theta_2$) if $\theta_1(t) = \theta_2(t+a)$ for some real number a.

A **nonparametrized trajectory** is an equivalence class $[\theta]$ of parametrized trajectories. If $m' = \Psi_t(m)$ the differential $D_m(\Psi_t) : T_m(M) \to T_{m'}(M)$ induces $\hat{D}_m(\Psi_t) : T_m(M)/T_m(\Gamma) \to T_{m'}(M)/T_{m'}(\Gamma)$ where Γ denotes the trajectory containing m and m' and is referred to as **Poincaré map** along Γ.

An **instanton** between the rest points x and y is an isolated [1] nonparametrized trajectory $[\theta]$ with $\lim_{t\to -\infty} \theta(t) = x$ and $\lim_{t\to\infty} \theta(t) = y$. The set of instantons from x to y is denoted by $\mathcal{I}_{x,y}$.

A **closed trajectory** is a pair $[\hat{\theta}] = ([\theta], T)$ s.t. $\theta(t+T) = \theta(t)$. The number T is called the **time period** of $[\hat{\theta}]$ as opposed to the **period** $p([\hat{\theta}])$ of $[\hat{\theta}]$ introduced below. The set of closed trajectories is denoted by $\mathcal{C}$.

Nondegeneracy:

Definition 2.2. :

(a) A rest point $x \in \mathcal{X}$ is called nondegenerate if is hyperbolic, in which case it has a Morse index $\operatorname{ind}(x) \in \mathbb{Z}_{\geq 0}$. The set of rest points of index k is denoted by $\mathcal{X}_k$.

(b) An instanton $[\theta]$ from x to y, with x, y nondegenerate rest points is nondegenerate if the maps i_x^- and i_y^+ are transversal at any point of $[\theta] \subset M$. Equivalently $W_x^- \pitchfork W_y^+$ along $[\theta]$, in which case $\operatorname{ind}(x) - \operatorname{ind}(y) = 1$. Moreover, orientations o_x of W_x^- and o_y of W_y^- induce an orientation of $[\theta]$ and implicitly a sign $\epsilon^{o_x,o_y}([\theta])$ with

$$\epsilon^{o_x,o_y}([\theta]) = +1 \text{ or } -1$$

if the induced orientation is consistent or not with the orientation from x to y.

(c) A closet trajectory $[\hat{\theta}] = ([\theta], T)$ is nondegenerate if the Poincaré map

$$\hat{D}_m(\Psi_T) : T_m(M)/T_m(\Gamma) \to T_m(M)/T_m(\Gamma)$$

induced from the linear map $D_m(\Psi_T) : T_m(M) \to T_m(M)$ for some m (and then for any other $m \in [\theta]$) satisfies $\det(\hat{D}_m(\Psi_T) - \lambda Id) \neq 0,\ |\lambda| = 1$. In this case denote by

$$\epsilon([\hat{\theta}]) = \mathrm{sign}\det(\hat{\mathrm{D}}_\mathrm{m}(\Psi_\mathrm{T}) - \mathrm{Id}).$$

As the closed trajectory $[\hat{\theta}]$ defines a map $\underline{[\hat{\theta}]} : S^1 \to M$ one denotes by $K([\hat{\theta}])$ the set of integers so that $\underline{[\hat{\theta}]}$ factors by a self map of S^1 of degree k. Define the **period** of $[\hat{\theta}]$ by

$$p([\hat{\theta}]) := \sup K(\hat{\theta}).$$

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[1] Isolated here means that there exists an open neighborhood U of the underlying set of $[\theta]$ which does not contain any other trajectory between these rest points.

Definition 2.3. :

A closed one form $\omega \in \Omega^1(M)$, ($d(\omega) = 0$) is called **Lyapunov** for X if:

(a) $\omega(X) \leq 0$ and

(b) $\omega(X)(x) = 0$ iff $X(x) = 0$.

A Lyapunov form ω which is Morse (i.e. locally is the differential of smooth function with non degenerate critical points) is called **Morse–Lyapunov**.

The cohomology class $\xi \in H^1(M; \mathbb{R})$ is called Lyapunov for X if it contains Lyapunov forms for X.

It is straightforward to check that given a Lyapunov form ω for X, a neighborhood U of $\mathcal{X}$ and $r \geq 2$, one can find a smooth function f arbitrary small in C^r topology which vanishes on $\mathcal{X}$, has support in U, and with $\omega' = \omega + df$ a Morse–Lyapunov form for X. So if a vector field which satisfies **H** (see below the definition of **H**) has a Lyapunov form ω it also has a Morse–Lyapunov form ω' arbitrary close to ω.

Properties of a smooth vector field:

- **G** (Genericity)= **H**(Hyperbolicity) + **MS**(Morse Smale) +**NCT**(Nondegenerate closed trajectories)

  Property **H** requires all rest points are nondegenerate;

  Property **MS** requires that i^+_x and i^+_y are transversal for any pair of rest points;

  Property **NCT** requires that all closed trajectories are nondegenerate.
- **L**(Lyapunov)

  Property **L** is satisfied if the set of Lyapunov cohomology classes in $H^1(M; \mathbb{R})$ is nonempty, equivalently if X admits a Lyapunov form.

Let X be a vector field which satisfies property **H** and g a Riemannian metric on M. For any $x \in \mathcal{X}$ denote by $\mathrm{Vol}(\mathrm{B_x(r)})$ the volume of the ball centered at 0 of radius r in $\mathbb{R}^{\mathrm{ind}\, x}$ w.r. to the pull back of the Riemannian metric g by i^-_x.

- **EG**(Exponential growth)

  The vector field X has exponential growth property **EG** at the rest point x if for some (and therefore every) Riemannian metric g on M there exists $C > 0$ so that $\mathrm{Vol}(\mathrm{B_x(r)}) \leq \mathrm{e^{Cr}}$ for all $r \geq 0$. It has exponential growth property **EG** if it has property **EG** at all rest points.

These properties are satisfied by a large class of vector fields, as the following theorem indicates.

Theorem 2.4. *1. (Kupka–Smale) For any r the set of vector fields which satisfy* **G** *is residual in the C^r topology.*

2. (Smale) Suppose X is a smooth vector field which satisfies **L**. *Then in any C^r neighborhood of X there exists vector fields which coincide with X in a neighborhood of X and satisfy* **L** *and* $\mathbf{G} = \mathbf{H} + \mathbf{MS} + \mathbf{NCT}$.

3. Suppose X is a smooth vector field which satisfies **L**. *Then in any C^0 neighborhood of X there exists vector fields which coincide with X in a neighborhood of X and satisfy* $\mathbf{L}, \mathbf{G}, \text{and } \mathbf{EG}$.

One expects the following Conjecture to be true.

Conjecture 2.5. *Statement (3) in the theorem above remains true for an arbitrary r.*

This is indeed the case for $n = 2$. In fact for the rest points of Morse index $0,\ 1$ and n **EG** holds; in the case of index 0 and n one has nothing to verify.

In [3] we have also considered a stronger version of **EG** referred there as **SEG**. Theorem 2.4 remains true for **SEG** replacing **EG**. Property **SEG** is of the same nature but takes a little longer to describe.

A very readable proof of Theorem 2.4 (1) is contained in [16]. An inspection of the proof of (1) leads easily to (2) which can also be derived from a slightly stronger version of (1). The proof of (3) is considerably more elaborated. A complete proof is contained in [3] and uses the work of Pajitnov [14].

In view of the compacity of M Property **L** insures that there are only finitely many rest points, Property **MS** insures that there are at most countable number of instantons and Property **NCT** insures that there are at most countable number of closed trajectories.[2]

The following proposition is of crucial importance.

Proposition 2.6. *Suppose X satisfies* $\mathbf{G}$ *and ω is Lyapunov for X representing the cohomology class ξ. Then for any real number R one has:*

1. The set of instantons $[\theta]$ so that $\int_{[\theta]} \omega < R$ is finite.

2 . The set of closed trajectories $[\hat\theta]$ so that $\xi([\hat\theta]) < R$ is finite.

Statement (1) is due to Novikov [12]. Statement (2) is due to Fried and Hutchings-Lee [8].

Proposition 2.6 indicates that despite their infiniteness, the instantons and the closed trajectories can be counted with the help of counting functions. To explain this we need additional definitions.

For M a closed connected smooth manifold and $x, y \in M$ define:

- $\mathcal{P}_{x,y}$:=homotopy classes of continuous paths from x, y. This set can be put in bijective correspondence to the fundamental group of M.
- $[S^1, M]$ the set of homotopy classes of continuous maps from S^1 to M. This set is in bijective correspondence with the conjugacy classes of elements of the fundamental group of M.

Let $\xi \in H^1(M; \mathbb{R})$ and ω a closed one form representing ξ. Define

(a) $\omega(\alpha) = \int_\alpha \omega,$ for $\alpha \in \mathcal{P}_{x,y}$,
(b) $\xi(\gamma) := \int_\gamma \omega,$ for $\gamma \in [S^1, M]$.

Let X be a smooth vector field on M which satisfies Property **G**. As noticed the set $\mathcal{X}_k$ of rest points of Morse index k is finite. Denote their number by n_k.

Suppose in addition that X has Property **L**.

In view of Proposition 2.6 for $x \in \mathcal{X}_{k+1}, y \in \mathcal{X}_k$, o_x orientation of W^-_x and o_y orientation of W^-_y define the counting functions $\mathcal{I}^{o_x,o_y}_{x,y} : \mathcal{P}_{x,y} \to \mathbb{Z}$ and $\mathcal{Z} : [S^1, M] \to \mathbb{Q}$ by

$$\mathcal{I}^{o_x,o_y}_{x,y}(\alpha) := \sum_{[\theta]\in\alpha} \epsilon^{o_x,o_y}([\theta])$$

with $[\theta]$ instanton in the homotopy class $\alpha \in \mathcal{P}_{x,y}$ and

$$\mathcal{Z}(\gamma) := \sum_{[\hat\theta]\in\gamma} \epsilon([\hat\theta])/p([\hat\theta])$$

with $[\hat\theta]$ closed trajectory in the homotopy class $\gamma \in [S^1, M]$.

[2]In fact much more is true: For any positive real number T the set of closed trajectories with time period smaller that T is finite and similarly the set of instantons whose needed time to go from $i^-_x(S^-_x)$ to $i^+_y(S^+_y)$ is smaller than T is finite. Here S^-_x resp. S^+_y denote the unit sphere in $R^{\operatorname{ind} x}$ resp $R^{n-\operatorname{ind} y}$.

Laplace transform:

If ω is a closed one form representing a cohomology class ξ one consider the following formal expressions:

$$I^{o_x,o_y,\omega}_{x,y}(z) =: \sum_{\alpha\in\mathcal{P}_{x,y}} \mathcal{I}^{o_x,o_y}_{x,y}(\alpha)e^{-z\omega(\alpha)}$$

and

$$Z^{\xi}(z) =: \sum_{\gamma\in[S^1,M]} \mathcal{Z}(\gamma)e^{-z\xi(\gamma)}$$

and one can ask when they define holomorphic functions in some parts of the complex plane.

Note that if $\omega' = \omega + dh$ then $I^{\mathcal{O},\omega}_{x,y}(z) = e^{z(h(y)-h(x)}I^{\mathcal{O},\omega'}_{x,y}(z)$, and a change of the orientations $\mathcal{O}$ might change the sign of the function $I^{\mathcal{O},\omega}_{x,y}(z)$. So an affirmative or negative answer to the question above for the second formal expression depends only on the cohomology class of ω.

Theorem 2.7. *Suppose X satisfies* **G** *and* **L** *and $\mathcal{O} = \{o_x, x \in \mathcal{X}\}$ is a collection of orientations of the unstable manifolds W^-_x.*

1. If $I^{\mathcal{O},\omega}_{x,y}(z)$ is absolutely convergent for $\Re z > \rho$ then $Z^{\xi}(z)$ is absolutely convergent for $Rez > \rho$.

2. If X satisfies **G**, **L** *and* **EG** *then there exists $\rho \in \mathbb{R}$ so that $I^{\mathcal{O},\omega}_{x,y}(z)$ and $Z^{\xi}(z)$ are absolutely convergent for $\Re z > \rho$. Moreover, for any $u \in \mathcal{X}_{k+1}$ and $w \in \mathcal{X}_{k-1}$*

$$\sum_{v\in\mathcal{X}_k} I^{\mathcal{O},\omega}_{u,v}(z) \cdot I^{\mathcal{O},\omega}_{v,w}(z) = 0.$$

Note that the formal sums $I^{\mathcal{O},\omega}_{x,y}(z)$ and $Z^{\xi}(z)$ are the Dirichlet series as defined in [17]. The first is associated with the discrete sequence of real numbers $\omega(\alpha), \alpha \in \mathcal{P}_{x,y}$ and the corresponding numbers $\mathcal{I}^{o_x,o_y}_{x,y}(\alpha) \in \mathbb{Z} \subset \mathbb{C}$. The second is associated with the discrete sequence $\xi(\gamma), \gamma \in [S^1, M]$ and the corresponding numbers $\mathcal{Z}(\gamma) \in \mathbb{Q} \subset \mathbb{C}$. Proposition 2.6 insures that the formal sums $I^{\mathcal{O},\omega}_{x,y}(z)$ and $Z^{\xi}(z)$ are Dirichlet series and consequently have an abscissa of convergence $\rho \leq \infty$. They define holomorphic functions provided the abscissa of convergence is $\neq \infty$.

Theorem 2.7 is the general result of our work . Ultimately to prove it boils down to show that the abscissa of convergence for the Dirichlet series $I^{\mathcal{O},\omega}_{x,y}(z)$ and $Z^{\xi}(z)$ are finite. For this purpose it suffices to consider these series for z a real parameter. This will bring us to Witten deformation described in the next section.

Suppose that X satisfies **G, L** and **EG** and $\mathcal{O} = \{o_x, x \in \mathcal{X}\}$ is a collection of orientations for the unstable manifolds $W^-_x, x \in \mathcal{X}$.

In view of Theorem 2.7 (2) one can use the functions $I^{\mathcal{O},\omega}_{x,y}(z)$ to define for any $z,\ \Re z > \rho$ a holomorphic family of cochain complexes with base $\mathbb{C}^*(M, X, \omega, \mathcal{O})(z) := (C^*(M, X), \delta^*_{\mathcal{O},\omega}(z))$ with

- $C^k((M, X) := \mathrm{Maps}(\mathcal{X}_\mathrm{k}, \mathcal{C})$. The base is determined by characteristic functions associated to the rest points.
- $\delta^*_{\mathcal{O},\omega}(z); C^*(M; X) \to C^{*+1}(M, X)$, given by

$$\delta^k_{\mathcal{O},\omega}(z)(f)(u) := \sum_{v\in\mathcal{X}_k} I^{\omega}_{u,v}(z)f(v),\ \ u \in \mathcal{X}_{k+1}$$

If ω_1, ω_2 are two Lyapunov forms representing the same cohomology class ξ and $\mathcal{O}_1, \mathcal{O}_2$ are two sets of orientations then there exists a canonical isomorphism between the cochain complexes $(C^*(M,X), \delta^*_{\mathcal{O}_1,\omega_1}(z))$ and $(C^*(M,X), \delta^*_{\mathcal{O}_2,\omega_2}(z))$.

The isomorphism send the base element corresponding to the rest point u into $\pm e^{zh(u)}$ with $\omega_2 = \omega_1 + df$ with $\pm$ if $o_{1,u}$ is the same or not with $o_{2,u}$. We can therefore denote this holomorphic family of cochain complexes, well defined up to an holomorphic isomorphism, by $\mathbb{C}^*(M,X,\xi)(z)$.

3. Spectral Geometry

In this section we will describe a few results in geometric analysis and use them to express the holomorphic functions $I^{\mathcal{O},\omega}_{x,y}(z)$ and $Z^\xi(z)$ in terms of more geometric invariants.

Witten deformation:

Let ω be a real valued closed one form, $\Omega^*(M)$ the real or complex valued differential forms and $d^*_\omega(t) : \Omega^*(M) \to \Omega^{*+1}(M)$ the perturbed exterior differential defined by

$$d^*_\omega(t) := d + t\omega\wedge$$

with $t \in \mathbb{R}$. The family of cochain complexes of deRham type, $(\Omega^*(M), d^*_\omega(t))$ is referred to as the "Witten deformation" (of the deRham complex $(\Omega^*(M), d^*)$).

Suppose that M is endowed with a Riemannian metric g. Then a differential operator from $\Omega^*(M)$ to $\Omega^*(M)$, in particular for $d^*_\omega(t) : \Omega^*(M) \to \Omega^{*+1}(M)$, has a formal adjoint $(d_\omega(t)^\sharp)^* : \Omega^*(M) \to \Omega^{*-1}(M)$. Following Witten [18] consider $\Delta^\omega_*(t) = (d_\omega(t)^\sharp)^{*+1} \cdot d^*_\omega(t) + d_\omega(t)^{*-1} \cdot (d_\omega(t)^\sharp)^*$ which is equal to

$$\Delta_* + t(L_X + L^\sharp_X) + t^2||X||^2$$

with $X = -\operatorname{grad}_g(\omega)$, L_X the Lie derivative w.r. to X, $L^\sharp_X$ the formal adjoint of L_X and $||X||^2$ the operator of multiplication by the square of the fiber-wise norm of X.

The operators $\Delta^\omega_*(t)$ are a zero order perturbation of Δ_q, the Laplace–Beltrami operators associated to the Riemannian metric g.

Recall that a closed one form is called Morse if locally is the differential of a function with all critical points are nondegenerate. As already noticed if X satisfies **H** and **L** then it admits Morse Lyapunov form.

We will consider the Witten deformation for ω a closed Morse one form and a Riemannian metric flat near the rest points [3] of $-\operatorname{grad}_g \omega$. Let n_k be the number of rest points of Morse index k.

Proposition 3.1. *There exists positive constants C_1, C_2, C_3, T so that for $t \geq T$ exactly n_k eigenvalues of $\Delta^{\omega,sm}_k(t)$ counted with their multiplicity are smaller that $C_1e^{-C_2t}$ and all others are larger than C_3t.*

This is a known observation first made by Witten [18] for a Morse exact one form but provable by the same arguments for a Morse closed one form.

As a consequence, for t large enough say $t > \rho$, there is a canonical orthogonal decomposition

$$(\Omega^*(M); d^\omega(t)) := (\Omega^*(M)_{\text{sm}}(M)(t), d^\omega(t)) \oplus (\Omega^*(M)_{\text{la}}(M)(t), d^\omega(t))$$

[3]We believe that the flatness requirement is not necessary but considerably more effort is needed to finalize the arguments without this hypothesis.

which diagonalizes $\Delta^\omega_*(t)$ i.e., $\Delta^\omega_*(t) = \Delta^{\omega,\text{sm}}_*(t) \oplus \Delta^{\omega,\text{la}}_*(t)$ for $t > \rho$. The small resp. large complex is generated by the eigenforms corresponding to the small resp. large eigenvalues i.e. the eigenvalues which for $t > \rho$ are bounded from above resp. below by a fix number, say 1. The small complex $(\Omega^*(M)_{\text{sm}}(M)(t), d^\omega(t))$ is finite dimensional with $\dim \Omega^k(M)_{\text{sm}}(M)(t) = n_k$ while the large complex is acyclic. The number ρ can be any number which insures that $t > \rho$ implies $C_1 e^{-C_2 t} < 1 < C_3 t$.

Theorem 3.2. *Suppose X satisfies* **G** *and* **EG***, ω is Morse Lyapunov for X and $\mathcal{O}$ is a collection of orientations of the unstable manifolds $W^-_x, x \in \mathcal{X}$. Choose a Riemannian metric on M which is flat near the rest points of X. For any $x \in \mathcal{X}$ let $h_x : \mathbb{R}^{\text{ind}\, x} \to \mathbb{R}$ be the only smooth function which satisfies $dh_x = (i^-_x)^*(\omega)$ and $h_x(0) = 0$.*

There exists $\rho' > 0$ so that :

(a) *for any t with $t > \rho'$, $x \in \mathcal{X}$ and $a \in \Omega^{\text{ind}\, x}(M)$ the integral*

$$\int_{\mathbb{R}^{\text{ind}\, x}} e^{-th_x}(i^-_x)^*(a) \in \mathbb{C}$$

is absolutely convergent,

(b) *the map $a \rightsquigarrow \int_{\mathbb{R}^{\text{ind}\, x}} e^{-th_x}(i^-_x)^*(a) \in \mathbb{C}$ defines the linear maps $Int^k : \Omega^k(M) \to C^k(M;X)$ which, when restricted to $(\Omega^k(M)_{sm}$, provide an isomorphism from $(\Omega^*(M)_{sm}(M)(t), d^\omega(t))$ to $\mathbb{C}^*(M, X, \omega, \mathcal{O})(t)$.*

Theorem 3.2 identifies the cochain complex $\mathbb{C}^*(M, X, \omega, \mathcal{O})(t)$, an object determined by dynamics, to a subcomplex of $(\Omega^*(M), d^*_\omega(t))$, precisely to $(\Omega^*(M)_{\text{sm}}(M)(t), d^\omega(t))$. Moreover $(\Omega^*(M)_{\text{sm}}(M)(t), d^\omega(t))$ gets a canonical base, the image by $(Int^*)^{-1}$ of the base of $C^*(M;X)$ determined by the rest points. Note that this base depends only on the integration on unstable manifolds, the metric g and the closed one form ω (which ultimately determines $\Omega^*_{sm}(M)(t)$). However, if one regards $d^*_\omega(t)$ with respect to this base as a matrix, its entries are exactly the functions $I^{\mathcal{O},\omega}_{x,y}(t)$ which count the instantons.

Theorem 3.2 is first proved for $X = -\operatorname{grad}_g \omega$. To obtain the result as stated one needs additional arguments. There is a qualitative difference between these two vector fields. At rest points the linearization of the first vector field has all eigenvalues real numbers $\neq 0$ while for the second vector field the eigenvalues can be complex numbers only with real part $\neq 0$.

The invariant $\mathcal{R}$:

Consider (M, g) a Riemannian manifold of dimension n and ω a closed one form. Let $\Psi(g) \in \Omega^{n-1}(TM \setminus M; \pi^*\mathcal{O}_M)$ be the global angular form associated to the Riemannian manifold (M, g) introduced by Mathai-Quillen [11]. Here $\pi^*\mathcal{O}_M$ denotes the pull back of the orientation bundle $\mathcal{O}_M$ by tangent bundle map on $TM \setminus M$.

Suppose $\omega \in \Omega^1(M)$ is a real valued closed one form and $X : M \to TM$ a smooth vector field with no rest points. In [3] and [6] the following quantity

$$\mathcal{R}(X, g, \omega) := \int_M \omega \wedge X^*(\Psi(g))$$

was introduced as a numerical invariant of the triple (X, g, ω). It was noticed that this invariant can be extended to vector fields with isolated rest points (in particular hyperbolic) using a "geometric regularization" of the possibly divergent integral $\int_{M\setminus\mathcal{X}} \omega \wedge X^*(\Psi(g))$. The invariant has a number of remarkable properties which have been presented in [6].

The function $\log Vol(t)$:

Regard $Int^k(t) : \Omega^k(M)_{\text{sm}}(t) \to C^k(M, X)$ as an isomorphism between two finite dimensional vector spaces equipped with scalar products[4]. The first with the scalar product induced from the Riemannian metric g, the second with the only scalar product which makes the base provided by the rest points orthonormal. Recall that for $\alpha : V \to W$ an isomorphism between two vector spaces with scalar products

$$\log \mathrm{Vol}(\alpha) = 1/2 \log \det(\alpha^\sharp \cdot \alpha)$$

with $\alpha^\sharp$ the adjoint of α. Write $\log \mathrm{Vol_k(t)} := \log \mathrm{Vol(Int^k(t))}$ and define

$$\log \mathrm{Vol(t)} := \sum (-1)^{\mathrm{k}} \log \mathrm{Vol_k(t)}. \tag{1}$$

The Ray- Singer large torsion $\log T_{la}(t)$:

Recall that for a positive self adjoint elliptic pseudo differential operator Δ, the spectrum is a countable collection of positive real numbers $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} \cdots$. Ray and Singer have defined the regularized determinant $\det \Delta$ by the formula

$$\log \det \Delta = -d/ds_{s=0} \zeta^{\Delta}(s) \tag{2}$$

where $\zeta^{\Delta}(s)$, the zeta function of Δ, is the analytic continuation of $\sum_i (\lambda_i)^{-s}$ well defined for $\Re s > 0$. It is known that $\zeta^{\Delta}(s)$ is a meromorphic function in the entire complex plane and has 0 as a regular value. The definition can be applied to the operator $\Delta_k^{\omega,\text{la}}(t)$. Define

$$\log T_{\text{la}}(t) := 1/2 \sum_k k(-1)^{k+1} \log \det \Delta_k^{\omega,\text{la}}(t) \tag{3}$$

which is a real analytic function[5] in t for $t > \rho$

Theorem 3.3. *Suppose ω is Morse Lyapunov for X, and X satisfies* **G** *and* **EG** *and g is a Riemannian metric flat near the rest points of X. Then there exists $\rho > 0$ so that for $t > \rho$*

$$\log T_{la}(M, g, \omega)(t) - \log \mathrm{Vol(t)} + \mathrm{t}\mathcal{R}(\mathrm{g, X}, \omega) = \mathrm{Z}^{\mathrm{X}}_{[\omega]}(\mathrm{t}). \tag{4}$$

The right side of the above equality is a dynamical quantity while the left side in a spectral geometry quantity.

Next observe that the family of operators

$$\Delta_q^{\omega}(z) = \Delta_q + z(L_X + L_X^*) + z^2 || \operatorname{grad}_g \omega||^2$$

is a selfadjoint holomorphic family of type A in the sense of Kato [9], and therefore there exists a family of functions $\lambda_n(z)$ and a family of differential forms $a_n(z) \in \Omega(M)^q$ both holomorphic in z in a neighborhood of $[0, \infty)$ in the complex plane so that

(a) $\lambda_n(z)'$s exhaust the eigenvalues of $\Delta_q^{\omega}(z)$,
(b) $a_n(z)$ is an eigenform corresponding to the eigenvalue $\lambda_n(z)$.

As a consequence the left side of equality (4) has an analytic continuation to a neighborhood of $[0, \infty)$ and since the right side is a well defined holomorphic function for $\Re z > \rho$ so is the left side. It also follows from Kato's theory that:

Theorem 3.4. *The two complexes $(\Omega^*(M)_{sm}(M)(t), d^{\omega}(t))$ and $(\Omega^*(M)_{la}(M)(t), d^{\omega}(t))$ have analytic continuation to a neighborhood of $[0, \infty)$.*

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[4] If the vector space is over $\mathbb{C}$ then "scalar product" means "Hermitian scalar product".

[5] Actually $e^{2 \log T_{\text{la}}(t)}$ can be extended to a holomorphic function in z for $\Re z > \rho$.

One expects this to be the case for $\Re z > 0$.

The proofs of Theorems 3.2 and 3.3 are contained in [4] and can be also derived from [3] while Theorem 3.4 is a consequence of Kato's theory.

Theorems 3.2, 3.3 and 3.4 can be generalized. Both the deRham complex $(\Omega^*(M), d^*)$ and the Witten deformation $\Omega^*(M), d^*_\omega(t) = d^* + t\omega\wedge$ can be twisted by a closed complex valued one-form η. Precisely $(\Omega^*(M), d^*)$ can be replaced by $(\Omega^*(M), d^*_\eta = d^* + \eta\wedge)$ and $(\Omega^*(M), d^*_\omega(t) = d^* + t\omega\wedge)$ by $(\Omega^*(M), d^*_{\eta;\omega} = d^* + (\eta + t\omega)\wedge)$.

Theorems 3.2, 3.3 and 3.4 remain true with proper modification. For example the conclusion of Theorem 3.3 remains the same by replacing $t\omega$ by $\eta + t\omega$ and $Z^{[\omega]}(z)$ by $Z^{[\eta];[\omega]}(z)$ with

$$Z^{\xi_1 \cdot \xi_2}(z) =: \sum_{\gamma \in [S^1, M]} \mathcal{Z}(\gamma) e^{-(\xi_1 + z\xi_2)(\gamma)}$$

for $\xi_1 \in H^1(M; \mathbb{C}), \xi_2 \in H^1(M; \mathbb{R})$. An equivalent form of this stronger result is proved in [3].

An interesting particular case:

If X has no rest points then **G** and **EG** reduce to **NCT** i.e. all closed trajectories are nondegenerate. Moreover $(\Omega^*(M), d^*_\omega(t)) = (\Omega^*_{\mathrm{la}}(t), d^*_\omega(t))$ and for any $t > 0$ the operator $\Delta^\omega_k(t)$ is invertible and therefore positive for any k.

The following statement is a minor improvement of a result of J. Marksick [4].

Corollary 3.5. *Suppose X is a smooth vector field with no rest points and $\xi \in H^1(M; \mathbb{R})$ and ω a Lyapunov form representing ξ and g a Riemannian metric on M. Suppose that all closed trajectories are nondegenerate. Denote by*

$$\log T_{an}(t) := 1/2 \sum_q (-1)^q \log \det \Delta^\omega_q(t).$$

Then

$$\log T_{an}(t) + t \int \omega \wedge X^*(\Psi_g) = Z^\xi(t).$$

Corollary 3.5 is a particular version of Theorem 3.3.

4. Topology

In this section we review results relating elements of dynamics with topological invariants.

Twisted cohomology $H^*(M; \xi)$:

Recall that a cohomology class $\xi \in H^1(M; \mathbb{C})$ permits the definition of twisted cohomology $H^*(M; \xi)$. This can be defined in any setting, simplicial, singular, Cech, deRham. All settings lead to isomorphic vector spaces at least for a smooth manifold M. The deRham version of $H^*(M; \xi)$ is defined as the cohomology of the complex of smooth differential forms equipped with the differential $d^*_\eta : \Omega^* \to \Omega^{*+1}$ with $d_\eta = d + \eta \wedge$. Here η is a closed one form representing ξ (in de Rham cohomology). The twisted cohomology for $t[\omega] \in H^1(M : \mathbb{R}) \subset H^1(M; \mathbb{C})$ is therefore the cohomology of the complex $(\Omega^*, d^*_\omega(t))$ in Witten deformation. It is not hard to see that given ξ the dimension of $H^*(M; z\xi)$ changes for only finitely many z so we write $\underline{\beta}^\xi_q$ for $\dim H^q(M; t\xi)$ for t large. The following well known result of Novikov provides strong restrictions for the numbers of rest points of a vector field which satisfies **H, MS** and **L**.

Theorem 4.1. *(Novikov)*
If X satisfies **H, MS** *and* **L** *with ξ a Lyapunov class for X then:*

$$\begin{aligned} n_k &\geq \underline{\beta}_k^{\xi} \\ \sum_{0\leq k\leq r} (-1)^k n_k &\geq \sum_{0\leq k\leq r} \underline{\beta}_k^{\xi},\ r \text{ even} \\ \sum_{0\leq k\leq r} (-1)^k n_k &\geq \sum_{0\leq k\leq r} \underline{\beta}_k^{\xi},\ r \text{ odd.} \end{aligned}$$

Torsion:
The cochain complex $(C^*(M;X), \delta^*_{\mathcal{O},\omega}, (z))$ is equipped with a base and has well defined *torsion* not explained here in the full generality. In case that the $z[\omega]$-twisted cohomology is trivial the torsion is a complex number defined up to a sign and its square depends holomorphically on z. The domain of this function consists of the complex numbers z with $H^*(M; z[\omega]) = 0$ which is the complement of a finite set provided that is nonempty.

The cohomology class $z\xi$, $\xi \in H^1(M;\mathbb{C})$ can be interpreted as a rank one complex representation of the fundamental group. The manifold M equipped with this representation and with an *Euler structure* has a *Milnor–Turaev torsion* not described in this paper. In case that the $z\xi$-twisted cohomology is trivial the Milnor–Turaev torsion is a complex number defined up to sign and its square depends holomorphically on z. The domain of this function consists of the complex numbers z with $H^*(M; z\xi) = 0$. The vector field X and some minor additional data determine an Euler structure.

It is possible to show that if X satisfies **G, L** and **EG** with $\xi = [\omega]$ a Lyapunov cohomology class for X then the torsion of $(C^*(M;X), \delta^*_{\mathcal{O},\omega}(z))$ multiplied by the function $e^{Z^{\xi}(z)}$ is essentially the Milnor–Turaev torsion of $M, z\xi$ and the Euler class defined by X. For more details consult [5]. This relates the functions $I^{\mathcal{O},\omega}_{x,y}(z)$, $Z^{\xi}(z)$ and the topology. The statement above is a reformulation and a minor extension of results of Hutchings–Lee and Pajitnov [8] and [15] which will not be explained here in details. However, we will explain below this statement in case t is a real number.

Suppose $H^*(M; t\omega) = 0$ for $t > \rho$. For such t, $\Delta^{\omega}_*(t)$ and then $\Delta^{\omega,\mathrm{sm}}_*$, are invertible and therefore have non vanishing determinants. The determinant of Δ^{ω}_k refers to the $\zeta-$ regularized determinant as explained in the previous section and defined by formula (1). In particular, we have $\log T_{\mathrm{an}}(t)$ and $\log T_{\mathrm{sm}}(t)$ defined by the formula (2) in the previous section. Clearly

$$\log T_{\mathrm{an}}(t) = \log T_{\mathrm{sm}}(t) + \log T_{\mathrm{la}}(t). \tag{5}$$

The cochain complex $(C^*(M;X), \delta^*_{\mathcal{O},\omega}, (z))$ has a base and therefore we can regard its components as equipped with the scalar product which makes the base orthonormal and implicitly define the corresponding Laplacians $\Delta_X(t)$.

Note also that by Theorem 2.7 the cohomology of the finite dimensional cochain complex $(C^*(M;X), \delta^*_{\mathcal{O},\omega}, (z))$ for t large ($t > \sup\{\rho, \rho'\}$) is trivial and then by formula (3) defines $\log T_X(t)$ for t large. As $\log \mathrm{Vol(t)} = \log \mathrm{T_{sm}(t)} - \log \mathrm{T^{\omega}_X(t)}$ combining (4) and (5) one concludes that :

$$\log T_{\rm an}(t) - t\mathcal{R}(M,\omega,g) = \log T_X(t) + Z^{[\omega]}(t)$$

which, in the case X satisfies in addition to **G** and **L** the property **EG**, is a generalization of Marsick's result to the case when X has rest points. This statement is equivalent via the work of Bismut–Zhang [1] to an analytic version of Hutchings- Lee Pajitnov theorem.

REFERENCES

[1] J.M. Bismut and W. Zhang, *An extension of a theorem by Cheeger and Müller,* Astérisque 205, Société Mathématique de France, 1992.

[2] M. Braverman and M. Farber, *Novikov type inequalities for differential forms with nonisolated zeros,* Math. Proc. Cambridge Philos. Soc. 122(1997), 357Ð375.

[3] D. Burghelea and S. Haller, *Dynamics, Laplace transform and spectral geometry,* Journal of Topology, Vol 1, 2008, 115-151.

[4] D. Burghelea and S. Haller, *Laplace transform, dynamics and spectral geometry (version 1),* arXiv: math/0405037v2.

[5] D. Burghelea and S. Haller, *Torsion a a function on the space of representations ,* C*-algebra and Elliptic Theory II, Trends in Mathematics , 41-66, 2007-66.

[6] D. Burghelea and S. Haller, *A Riemannian invariant, Euler structures and some topological applications,* C*-algebra and Elliptic Theory , Trends in Mathematics , 37-60.

[7] D. Burghelea and S. Haller, *Complex valued ray-Singer torsion* Journal of Functional Analysis 248 (2007) 27-78.

[8] M. Hutchings, Y.J. Lee, *Circle–valued Morse Theory, Reidemeister torsion and Seiberg–Witten invariants of* 3 *manifolds*, Topology **38**(1999), 861–888.

[9] T. Kato, *Perturbation theory for linear operators.* Second edition. Grundlehren der Mathematischen Wissenschaften **132**. Springer-Verlag, Berlin-New York, 1976.

[10] A. Katok, B. Hasselblatt, *Introduction to the modern theory of Dynamical systems.* Cambridge University press, 1995.

[11] V. Mathai and D. Quillen, Superconnections, Thom Classes, and Equivariant differential forms, *Topology* 25 (1986) 85–110.

[12] S.P. Novikov, *Quasiperiodic structures in topology*, in Topological methods in modern mathematics, Proceedings of the symposium in honor of John Milnor's sixtieth birthday held at the State University of New York, Stony Brook, New York, 1991, eds L. R. Goldberg and A. V. Phillips, Publish or Perish, Houston, TX, 1993, 223–233.

[13] V.A. Pajitnov, *The incidence coefficients in the Novikov complex are generically rational functions,* St. Petersburg Math. J. **9**(1998), 969–1006. Translation from Algebra i Analiz **9**(1997), 92–139.

[14] V.A. Pajitnov, C^0*–generic properties of boundary operators in the Novikov complex.* Pseudoperiodic topology, 29–115, Amer. Math. Soc. Transl. **197**, Amer. Math. Soc., Providence, RI, 1999.

[15] V.A. Pajitnov *On closed orbits of gradient flows of circle-valued mappings,* St. Petersburg Math. J. **14**(2003), 499–534. Translation from Algebra i Analiz **14**(2002), 186–240.

[16] M.M. Peixoto, *On an Approximation Theorem of Kupka and Smale*, J. Differential Equations **3**(1967), 214–227.

[17] J.P. Serre, *A* course in arithmetic. Translated from the French. Graduate Texts in Mathematics 7, Springer Verlag, New YorkÐHeidelberg, 1973.

[18] E. Witten, *Supersymmetry and Morse theory*, J. Differential Geom. 17(1982), 661Ð692.

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