New topological (computable) invariants for real-valued maps and multivalued maps; application to dynamics

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Plan of the lecture

1. **Background**
   - Maps, TC1-forms, critical values and critical points
   - Configurations
   - Dynamics
2. **The invariants and the results**
3. **The description of the invariants δ and γ**

Background

MAPS, MULTIVALUED MAPS, TC1-forms

- **(tame)map** $f : M \to \mathbb{R}$
  (i). M ANR-space, f continuous,
  (ii). ($f^{-1}(t)$ ANR)
- **(tame) multivalued map.** $\{U_i, f_i : U_i \to \mathbb{R}\}$ or 1-Cech cocycle on M.
  (i) U_i open sets and $\bigcup U_i = M$,
  (ii) $f_i - f_j|_{U_i \cap U_j}$ constant
- **equivalent multivalued maps** $\{U_i, f_i\} \sim \{V_j, g_j\}$ iff $\{U_i.V_j, f_i, g_j\}$ is a multivalued map
- A **(tame) TC 1-form** ω is an equivalency class of (tame) multivalued maps
- ($\mathcal{Z}^1_t(M)$) $\mathcal{Z}^1(M)$ the set of all (tame)TC1-forms is a $\mathbb{R}-$ vector space equipped with the uniform convergence topology.

what a TC1-form ω does ?

- ω defines $\xi = \xi_\omega \in H^1(M;\mathbb{R})$, equivalently the homomorphism $\underline{\omega} : H_1(M;\mathbb{Z}) \to \mathbb{R}$,
- when M is compact $\underline{\omega}$ gives:
    1. the group $\Gamma \subset \mathbb{R}$, $\Gamma = img(\underline{\omega}) \subset \mathbb{R}$, $\Gamma \simeq \mathbb{Z}^n$
    2. the $\Gamma-$ principal covering $\pi : \tilde{M} \to M$,
- ω provides the real-valued $\Gamma-$equivariant maps $f : \tilde{M} \to \mathbb{R}$ with $\pi^*(\omega) = df$ called lifts of ω unique up to additive constant .

Critical values and critical points

For a tame map $f : M \to \mathbb{R}$

- $t_0 \in \mathbb{R}$ is a *critical value* iff the topology of $f^{-1}(t)$ changes for in t the neighborhooud of t_0.
- $x \in M$ is a *critical point* if $f(x)$ is a critical value for the restriction of f to any neighbordood U if x.
- *Critical points* make sense for a TC1-form ω

CONFIGURATIONS

For Y topological space

- $\mathcal{C}onf(Y) := \{\delta : Y \to \mathbb{Z}_{\geq 0} \mid \sharp(\mathrm{s}upp\ \delta) < \infty\}$

  $\delta = \{{y_1, y_2, \cdots, y_k \atop n_1, n_2, \cdots, n_k}\}$
- $\mathcal{C}onf_n(Y) := \{\delta \in Conf(Y) \mid \sum_{y \in M} \delta(y) = n\}$
  - $Conf_n(Y) = Y^n/\Sigma_n$
  - $Conf(Y) = \bigsqcup_n Conf_n(Y)$

A closed subset $K \subset Y$ induces a topology on $\mathcal{C}onf(Y \setminus K)$ the *bottleneck topology*. When $K = \emptyset$ this topology on $\mathcal{C}onf(Y)$ is referred to as the *collision topology*

Bottleneck topology

A *fundamental neigborhood* of $\delta \in Conf(Y \setminus K)$,

$\mathcal{C}onf_k(Y) \ni \delta = \{{}^{y_1, y_2, \cdots y_k}_{n_1, n_2, \cdots n_k}\}$ is indexed by a collection of disjoint open subsets $\{U_1, \cdots U_k, V\}$ with $U_i \ni y_i$ and $V \supset K$

$$\mathcal{U}(\delta; U_1, \cdots U_k, V) := \left\{ \delta' \in Conf(Y \setminus K) \;\middle|\; \begin{array}{l} \mathrm{s}upp\delta' \subset \bigcup U_i \cup V \\ \sum_{x \in U_i} \delta'(x) = \delta(y_i) = n_i \end{array} \right.$$

DYNAMICS

Flows on M, M compact ANR= one parameter group of homeomorphisms $\boxed{\mu : \mathbb{R} \times M \to M}$, $\mu(t, \mu(s, x)) = \mu(t+s, x)$ and $\mu(0, x) = x$.

Trajectory through $x \in M$: the set $\tau \subset M$ of the from $\tau = \mu(\mathbb{R} \times x)$

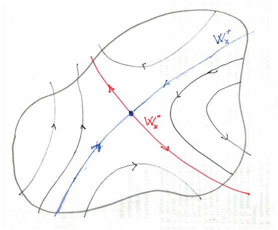
EXAMPLE: *A smooth vector field* X, *on a smooth closed manifold M* generates a smooth flow, a one parameter group of diffeomorphisms $\mu^X : \mathbb{R} \times M \to M$.

ELEMENTS OF DYNAMICS

1. **Rest points** $\mathcal{R}(X) := \{x \in M \mid X(x) = 0\}$,
2. **Instantons** $\mathcal{I}(x, y) =$ isolated trajectories, from x to y, $x, y \in \mathcal{R}(X)$,
3. **Closed trajectories** (isolated).

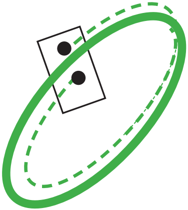
Rest - points For $x \in \mathcal{R}(X)$

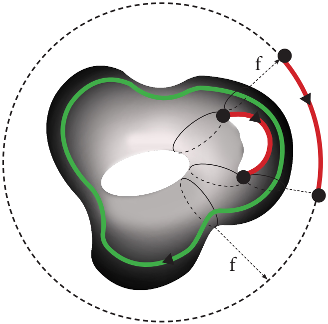
- W_x^- unstable set, $i(x) := \dim W_x^-$ **Morse index** of x
- W_x^+ stable set



Closed trajectories

A closed trajectory τ has a Poincaré return map, (conjugacy class of) linear isomorphism $T_\tau; V_\tau \to V_\tau$. The closed trajectory τ is *isolated* iff T_τ has no eigenvalue = 1.





an example

Morse-Smale vector fields satisfiy:

1. All rest points are hyperbolic, (i.e. $W_x^{\mp}$ are smooth sub manifolds diffeomorphic to $\mathbb{R}^{i(x)}, \mathbb{R}^{n-i(x)}$).
2. For any two rest points $x, y,\quad W_x^-$ and W_y^+ are transversal.
3. Any closed trajectory is isolated.

The set of Morse-Smale vector fields is generic (in C^1- topology).

As a consequence:

- The set $\mathcal{R}(X)$ is finite,
- $\mathcal{I}(x, y) \neq \emptyset$ then $i(x) - i(y) = 1$ and each homotopy class of paths from x to y, $x, y \in \mathcal{R}(X)$, contains a finite number of elements in $\mathcal{I}(x, y)$,
- If an orientation on each W_x^- is given then each $\tau \in I(x, y)$ has a sign, $\boxed{\epsilon(\tau) = \pm 1.}$
- Each homotopy class of closed curves contains a finite number (possibly zero) of closed trajectories.

LYAPUNOV FUNCTION / Lyapunov TC1-FORM

Definition

1. A tame map $f : M \to \mathbb{R}$ is Lyapunov for X iff it is strictly decreasing on nonconstant trajectories (for f smooth iff $df(X)(x) < 0$ iff $x \in M \setminus \mathcal{R}(X)$).
2. A TC1-from $\omega \in \mathcal{Z}^1(M)$ is Lyapunov for X if for one (and then for any) representative U_i, f_i of ω the maps f_i are strictly decreasing on each non constant trajectory (for ω smooth iff $\omega(X)(x) < 0$ iff $x \in M \setminus \mathcal{R}(X)$)

The new invariants

For a compact ANR M, a field κ, an integer $r \geq 0$, based on homology with coefficients in κ, one produces:

- for a tame map $f : M \to \mathbb{R}$ the configurations

$$\boxed{\delta^f_r \in Conf(\mathbb{C}) \ , \ \gamma^f_r \in Conf(\mathbb{C} \setminus \Delta)}$$

$\Delta = \{z \in \mathbb{C} \mid z = i\overline{z}\}$, the first diagonal in $\mathbb{C}$,

- for a tame TC1-form $\omega \in \mathcal{Z}^1_t(M)$ the configurations

$$\boxed{\delta^\omega_r \in Conf(\mathbb{R}) \ , \ \gamma^\omega_r \in Conf(\mathbb{R} \setminus 0)}$$

- $Conf(C) \ni \delta = \{{z_1, z_2, \cdots z_k \atop n_1, n_2, \cdots n_k}\} \Leftrightarrow P^\delta(z) = \prod(z - z_i)^{n_i}$

Results; The case of a real-valued map

Topological results

Let $f : M \to \mathbb{R}$ be a tame map, M a compact ANR. Then

1. $\sum_{z \in \mathbb{C}} \delta^f_r(z) = degP^f_r(z) = \dim H_r(M; \kappa)$,

2. $\dim H_r(f^{-1}([a,b])$ can be expressed in terms of $\delta^f_r(z)$, and $\gamma^f_r(z)$ (homology calculations),

3. If M is a closed $\kappa-$orientable n-manifold then

$$\delta^f_r(z) = \delta^f_{n-r}(i\overline{z}) \text{ and } \gamma^f_r(z) = \gamma^f_{n-1-r}(i\overline{z})$$

(Poincaré duality).

Dynamical results

1. If f is a Morse function on a closed smooth manifold, $Cr_r(f)$ denotes the set of critical points of index r and $C^+ = \{z = (a + ib) \mid a < b\}$ then

$$\sharp Cr_r(f) = \sum_{z \in \mathbb{C}} \delta_r^f(z) + \sum_{z \in \mathbb{C}^+} \gamma_r^f(z) + \sum_{z \in \mathbb{C}^+} \gamma_{r-1}^f(z).$$

2. If X is a vector field with f Lyapunov function and $\mathcal{I}_r$ denotes the set of instantons between rest points of index r and $r-1$ then

$$\sum_{z \in C^+} \gamma_r^f(z) \neq 0 \textit{ implies } \sharp\mathcal{I}_r \neq \emptyset.$$

Robustness results

Let M be a compact ANR, $C(M)$ denote real-valued continuous functions and $C_t(M) \subset C(M)$ tame functions equipped with the uniform convergence topology. Let $\beta_r := \dim H_r(M; \kappa)$.

1. The assignment $C_t(M) \ni f \rightsquigarrow \delta^f_r, \in Conf_{\beta_r(M)}(\mathbb{C})$ extends to a continuous map on $C(M)$ (Burghelea-Haller).

2. The assignment $C_t(M) \ni f \rightsquigarrow \gamma^f_r \in Conf(\mathbb{C} \setminus \Delta)$ is continuous (Edelsbruner-Harer).

A more precise statements involving metrics on both spaces permit to calculate the configurations with arbitrary accuracy. Note that the computer accepts only rational numbers.

Topological results

1. Let $\omega \in \mathcal{Z}_t^1(M)$, M a compact ANR and $NH_r(M;\xi_\omega)$ be the Novikov homology associated with the pair $(M, \xi_\omega \in H^1(M,R))$. Then

$$\sum_{t\in\mathbb{R}} \delta_r^\omega(t) = \dim NH_r(M;\xi_\omega).$$

2. If M is a closed $\kappa-$orientable n-manifold then

$$\delta_r^\omega(t) = \delta_{n-r}^\omega(-t)) \text{ and } \gamma_r^\omega(t) = \gamma_{n-1-r}^\omega(-t).$$

Dynamical results

Let $\omega \in \mathcal{Z}^1(M)$ be a TC1-form Lyapunov for a Morse-Smale vector field X on a closed smooth manifold M with $Cr_r(\omega)$ the set of critical points of index r, and $\mathcal{I}_r$ the set of instantons between rest points of index r and $r-1$. Then

1. $\sharp Cr_r(\omega) = \sum_{t\in\mathbb{R}} \delta^\omega_r(t) + \sum_{t\in\mathbb{R}_{>0}} \gamma^\omega_r(t) + \sum_{t\in\mathbb{R}_{>0}} \gamma^\omega_{r-1}(t)$,
2. $\sum_{t\in R_{>0}} \gamma^\omega_r(t) \neq 0$ implies $\sharp\mathcal{I}_r \neq \emptyset$,
3. $\sum_{t\in\mathbb{R}} \delta^\omega_r(t) \neq \dim H_r(M;\kappa)$ implies that the set of closed trajectories in not empty.

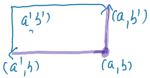
Robustness results

For M a compact ANR let $\mathcal{Z}(M)$ denote the set of TC1-forms, $\mathcal{Z}_t(M) \subset \mathcal{Z}(M)$ tame TC1-forms equipped with the uniform convergence topology.

1. The assignment $\mathcal{Z}_t(M) \ni f \rightsquigarrow \delta^\omega_r \in Conf_{\eta^N_r}(\mathbb{R})$ with $\beta^N_r = \dim NH_r(M;\omega)$ extents to a continuous map defined on $\mathcal{Z}(M)$.
2. The assignment $\mathcal{Z}_t(M) \ni f \rightsquigarrow \delta^\omega_r \in Conf_{\beta^N_r}(\mathbb{R} \setminus 0)$ is continuous.

A more precise statements involving metrics insures the possibility to calculate these configurations with arbitrary accuracy. Note that the computer accept only rational numbers.

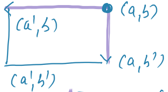
Boxes $B=(a',a]\times[b,b')$



$a'<a$, $b<b'$

(a,b) the first vertex
(a',b') the last vertex

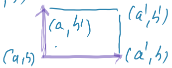
Boxes above diagonal $B=(a',a]\times(b',b]$



$a'<a\leq b'<b$

Boxes below diagonal $B=(a',a]\times[b,b')$

$a'>a\geq b'>b$



- For a tame map $f : M \to \mathbb{R}$ denote

$$\mathbb{I}_a^f(r) := img(H_r(f^{-1}((-\infty, a]) \to H_r(M)),$$
$$\mathbb{I}_f^a(r) := img(H_r(f^{-1}([a, \infty)) \to H_r(M)).$$

and for $(a, b) \in \mathbb{R}^2$

$$\boxed{F_r^f(a, b) := \mathbb{I}_a^f(r) \cap \mathbb{I}_f^b(r)}$$

For $a' \leq a, b' \geq b$ one has $F_r(a', b') \subseteq F_r(a, b)$.

- Consider sets = **boxes** $B = (a', a] \times [b, b') \subset \mathbb{R}^2,\ a' < a,\ b' > b$ and define

$$F_r(B) := F_r(a,b)/F_r(a',b) + F_r(a,b')$$

$$\begin{array}{ccc} \mathbb{F}_r(a',b') & \longrightarrow & \mathbb{F}_r(a,b') \\ \downarrow & & \downarrow \subseteq \\ \mathbb{F}_r(a',b) & \xrightarrow{\subseteq} & \mathbb{F}_r(a,b) \end{array}$$

An inclusion $B' \subseteq B$ of boxes with the same first vertex induces the surjective linear map $\pi^B_{B'} : F_r(B) \to F(B')$ and an inclusion of boxes with the same last vertex induces the injective linear map $\iota^{B'}_B : F_r(B') \to F(B)$.

- Denote $B(a,b;\epsilon) := (a-\epsilon, a] \times [b, b+\epsilon)$, $\epsilon > 0$. For $\epsilon' > \epsilon''$ the linear map $F_r^f(B(a,b;\epsilon')) \to F_r^f(B(a,b;\epsilon''))$ is surjective. Define

$$\boxed{\hat{\delta}_r^f(a,b) := \lim_{\epsilon \to 0} F_r^f(B(a,b;\epsilon), \quad \delta_r^f(a,b) = \dim \hat{\delta}_r^f(a,b)}.$$

Since f is tame $\delta_r^f(a,b) \neq 0 \Rightarrow a, b \in CR(f)$.

Proposition

- *If f is proper or if f is a lift of a TC1-form on a compact ANR then $\delta_r^f(a,b)$ is finite .*
- *If M is compact then the map δ_r^f is a configuration and the assignement $B \rightsquigarrow \dim F_r(B)$ defines a $\mathbb{Z}-$valued measure on the sigma-algebra generated by boxes, whose density is δ_r^f.*

Measure theoretic approach; the map γ_r^f

- For $a < b$ define

$$T_r^f(a,b) := \ker(H_r(f^{-1}((-\infty a]) \to H_r(f^{-1}((-\infty, b])$$

$a' \leq a,\ b' \leq b,\ a' < a,\ a < b$ induce

$$\iota_{a',b'}^{a,b} : T_r(a',b') \to T(a,b);$$

when $a' = a$ this linear map is injective.

- For $a > b$ define

$$T_r^f(a,b) := \ker(H_r(f^{-1}([a,\infty)]) \to H_r(f^{-1}([b,\infty)).$$

$a' \geq a,\ b' \geq b,\ a' > b',\ a > b$ induce

$$\iota_{a',b'}^{a,b} : T_r(a',b') \to T(a,b);$$

when $a' = a$ this linear map is injective.

- For sets = **boxes above diagonal**, $B = (a' a] \times (b', b]$ i.e. $a' < a \leq b' < b$ with (a, b) the first vertex and (a', b') the last vertex define

$$T^f_r(B) = T^f_r(a, b)/\iota T^f_r(a', b) + T^f_r(a, b').$$

- For set = **boxes below diagonal**, $B = [a, a'') \times [b, b'')$, i.e. $b < b" \leq a < a''$ with (a, b) the first vertex and (a'', b'') the last vertex define

$$T^f_r(B) = T^f_r(a, b)/\iota T^f_r(a'', b) + T^f_r(a, b'').$$

- For $a < b$, $\epsilon < (b-a)$ let $B(a,b;\epsilon) = (a-\epsilon, a] \times (b-\epsilon, b]$,

  For $a > b$ $\epsilon < (a-b))$ let $B(a,b;\epsilon) = [a, a+\epsilon) \times [b, b+\epsilon)$,

  $\epsilon' > \epsilon'' \Rightarrow T_r^f(B(a,b;\epsilon')) \geq T_r^f(B(a,b;\epsilon''))$.
- Define

$$\hat{\gamma}_r^f(a,b) := \lim_{\epsilon\to 0} T_r^f(B(a,b;\epsilon)) \quad \gamma_r^f(a,b) := \dim \hat{\gamma}_r^f(a,b)$$

Since f is tame $\gamma_r^f(a,b) \neq 0 \Rightarrow a, b \in CR(f)$.

Proposition

- *If f is proper or if f is a lift of a TC1-form then* $\gamma_r^f(a,b)$ *is finite .*
- *If M is compact then the map* γ_r^f *is a configuration and the assignment* $B \rightsquigarrow \dim T_r(B)$ *defines a* $\mathbb{Z}-$*valued measure on the sigma-algebra generated by boxes above diagonal and below diagonal whose density is* γ_r^f.

Proposition

If $f : \tilde{M} \to \mathbb{R}$ is the lift of a tame TC1-form then $S^\delta := \mathrm{supp}\delta^f_r : \mathbb{R}^2 \to \mathbb{Z}_{\geq 0}$ and $S^\gamma := \mathrm{supp}\gamma^f_r : \mathbb{R}^2 \to \mathbb{Z}_{\geq 0}$ satisfy the following:

1. $S^\delta \subset CR(f) \times CR(f)$ $\;S^\gamma \subset CR(f) \times CR(f) \setminus \Delta$
2. $S^{\cdots}$ *is* $\Gamma-$ *invariant, i.e. if* $(a, b) \in S^{\cdots}$ *then* $(a + g, b + g) \in S^{\cdots}$ *for any* $g \in \Gamma$
3. S^δ *resp* S^γ *is located on a finite collection of lines* $y = x + a^\delta_i,\ i = 1, 2, \cdots N^\delta_r$ *resp.* $y = x + a^\gamma_i,\ i = 1, 2, \cdots N^\gamma_r$ *of finite multiplicities.* $n^\delta_1, n^\delta_2, \cdots n^\delta_{N^\delta_r}$ *resp* $n^\gamma_1, n^\gamma_2, \cdots n^\gamma_{N^\gamma_r}$

Then define

$$\delta^\omega_r(t) := \sum_{t=a^\delta_i} n^\delta_i$$

$$\gamma^\omega_r(t) := \sum_{t=a^\gamma_i} n^\gamma_i$$

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