LAPLACE TRANSFORM, DYNAMICS AND SPECTRAL GEOMETRY (VERSION 1)

DAN BURGHELEA AND STEFAN HALLER

Abstract. We consider vector fields $X$ on a closed manifold $M$ with rest points of Morse type. For such vector fields we define the property of exponential growth. A cohomology class $\xi \in H^1(M; \mathbb{R})$ which is Lyapunov for $X$ defines counting functions for isolated instantons and closed trajectories. If $X$ has exponential growth property we show, under a mild hypothesis generically satisfied, that these counting functions can be recovered from the spectral geometry associated to $(M, g, \omega)$ where $g$ is a Riemannian metric and $\omega$ is a closed one form representing $\xi$, via inverse Laplace transform. The analytic tool for this is are the Dirichlet series and their Laplace transform.

Contents

1. Introduction 1
2. Exponential growth property and the invariant $\rho$ 9
3. Topology of the space of trajectories and unstable sets 17
4. Integration map 21
5. The regularization $R(X, \omega, g)$ 25
6. Proof of Theorems 2, 3, 4 and WHS theory revisited 29
7. Appendix 33
References 34

1. Introduction

In this paper we consider vector fields $X$ on a closed manifolds $M^n$ with all rest points of Morse type and with Lyapunov cohomology class, see the Definition 1 below. The elements of dynamics of such vector fields are:

(i) the rest points, which are finitely many, hence can be counted,
(ii) the instantons,
(iii) the closed trajectories.

For a $C^1$ generic set of such vector fields the instantons and the closed trajectories are discrete, but in general, infinitely many. Despite their infiniteness the existence of Lyapunov cohomology class makes them countable with the help of counting...
functions. A cohomology class $\xi \in H^1(M; \mathbb{R})$ which is Lyapunov for $X$ provides such "counting functions".

We show in this paper that these counting functions can be interpreted as Dirichlet series. This is not very new but an interpretation of previous work of Novikov, Hutchings and others. What it is new is that, when the vector field has exponential growth property, cf. Definition 4, these Dirichlet series have finite abscissa of convergence and therefore have Laplace transform. We also provide explicit formulas for such Laplace transforms in terms of the spectral geometry of $(M, g, \omega)$ where $g$ is a Riemannian metric and $\omega$ a closed form representing the Lyapunov cohomology class $\xi$. We conjecture that the exponential growth property holds for a $C^0$ generic set of vector fields (with Lyapunov cohomology class). We are able to prove that this is the case for a $C^0$ generic set, cf Theorem 1. To make the definitions and the results precise we need some background.

1.1. Vector fields with zeros of Morse type and Lyapunov cohomology class. Let $X$ be a smooth vector field on a smooth manifold $M$. A point $x \in M$ is called a rest point or a zero of $X$ if $X(x) = 0$. The collection of these points will be denoted by $X := \{x \in M | X(x) = 0\}$.

Recall that:

(i) A parameterized trajectory is a map $\theta : \mathbb{R} \to M$ so that $\theta'(t) = X(\theta(t))$. A trajectory is an equivalence class of parameterized trajectories with $\theta_1 \equiv \theta_2$ if $\theta_1(t + a) = \theta_2(t)$ for some real number $a$. Any representative $\theta$ of a trajectory is called a parameterization.

(ii) An instanton from the rest point $x$ to the rest point $y$ is an isolated trajectory with the property that for one and then any parameterization $\theta$, $\lim_{t \to -\infty} \theta(t) = x$, $\lim_{t \to +\infty} \theta(t) = y$.

(iii) A parameterized closed trajectory is a pair $[\theta, T]$, with $\theta$ a parameterized trajectory and $T$ a positive real number so that $\theta(t + T) = \theta(t)$.

A parameterized closed trajectory gives rise to a smooth map $\tilde{\theta} : S^1 := \mathbb{R}/T\mathbb{Z} \to M$. A closed trajectory is an equivalence class $[\tilde{\theta}]$ of parameterized closed trajectories with $[\theta_1, T_1] \equiv [\theta_2, T_2]$ iff $\theta_1 \equiv \theta_2$ and $T_1 = T_2$.

Recall that a rest point $x \in X$ is said to be of Morse type if there exist coordinates $(t_1, \ldots, t_n)$ around $x$ so that

$$X = \sum_{i=1}^{q} t_i \frac{\partial}{\partial t_i} - \sum_{i=q+1}^{n} t_i \frac{\partial}{\partial t_i}.$$  \hspace{1cm} (1)

The integer $q$ is called the Morse index of $x$ and denoted by $\text{ind}(x)$. A rest point of Morse type is non-degenerate and its Hopf index is $(-1)^{n-q}$. The Morse index is independent of the chosen coordinates $(t_1, \ldots, t_n)$. Denote by $X_q$ the set of rest points of Morse index $q$ and let $X = \bigsqcup_q X_q$.

For any rest point of Morse type $x$, the stable resp. unstable set is defined by:

$$D^\pm(x) := \{y | \lim_{t \to \pm\infty} \Psi_t(y) = x\}$$

where $\Psi_t : M \to M$ denotes the flow of $X$. The stable and unstable sets are images of injective smooth immersions $i^\pm : W^\pm \to M$. The manifold $W^+_x$ resp. $W^-_x$ is diffeomorphic to $\mathbb{R}^{\text{ind}(x)}$ resp. $\mathbb{R}^{n-\text{ind}(x)}$. 

**Convention.** Unless explicitly mentioned all vector fields in this paper are assumed to have all rest points of Morse type,\(^1\), hence isolated.

**Definition 1.** A closed one form \(\omega\) which satisfies \(\omega(X) < 0\) on \(M \setminus \mathcal{X}\) is called Lyapunov form for \(X\).

A cohomology class \(\xi \in H^1(M; \mathbb{R})\) which contains Lyapunov forms is called Lyapunov cohomology class for \(X\).

The vector field \(X\) is said to satisfy the Lyapunov property, L for short, if there exists \(\xi\) Lyapunov cohomology class for \(X\).

We have

**Proposition 1.** 1. If \(\xi\) is a Lyapunov cohomology class for \(X\) one can choose \(\omega \in \xi\) a Lyapunov form for \(X\) such that in the neighborhood of any rest point \(x \in X\) there exists coordinates \((t_1, \ldots, t_n)\) with \(\omega\) given by \(1/2(-\sum_{i=1}^q t_i^2 + \sum_{i=q+1}^n t_i^2)\) and \(X\) in the canonical form (1).

2. If \(\omega\) is a Lyapunov form for \(X\), \(g_0\) a Riemannian metric, \(K\) a compact subset of \(M\) so that \(X = -\operatorname{grad}_{g_0} X\) on a neighborhood of \(K\) and \(\omega(X)(x) < 0\) for \(x \in M \setminus K\), then there exists a Riemannian metric \(g\) which agrees with \(g_0\) on a neighborhood of \(K\) and \(X = -\operatorname{grad}_g \omega\).

**Proposition 2.** If \(0 \in H^1(M; \mathbb{R})\) is a Lyapunov cohomology class for \(X\) then any other cohomology class \(\xi \in H^1(M; \mathbb{R})\) is Lyapunov for \(X\).

The proof of these two propositions will be given in the Appendix.

**Definition 2.** The vector field \(X\) is said to satisfy the Morse–Smale property, MS for short, if for any \(x, y \in X\) the maps \(i^-\) and \(i^+\) are transversal.

Let \(\Psi_t\) denote the flow of \(X\). The closed trajectory \([\theta]\) is called non-degenerate if for some (and then any) \(t_0 \in \mathbb{R}\) and representative \(\theta = (\theta, T)\) the differential \(D_{\theta(t_0)} \Psi_T : T_{\theta(t_0)} M \to T_{\theta(t_0)} M\) is invertible with the eigenvalue 1 of multiplicity one.

**Definition 3.** The vector field \(X\) is said to satisfies the non-degenerate closed trajectories property, NCT for short, if all closed trajectories of \(X\) are non-degenerate.

As long as properties MS and NCT are concerned we have the following genericity result.

**Proposition 3.** Let \(\xi \in H^1(M; \mathbb{R})\) be Lyapunov cohomology class for \(X\). Then arbitrary close in the \(C^r\) topology for any \(r \geq 1\) one can find \(Y\) such that:

(i) \(Y\) satisfies MS and NCT
(ii) \(Y\) agrees with \(X\) on a neighborhood of \(\mathcal{X}\)
(iii) \(Y\) has \(\xi\) as Lyapunov cohomology class.

---

\(^1\)The concept of Morse type rest point as defined is not very fortunate (if \(X\) has Morse type rest points as defined \(2X\) does not) however it is convenient to work with. The theory can be probably developed with the same results by replacing *Morse type* by *Elementary type*, cf [1] page.61. This means that the differential of \(X\) at a rest point \(x\) has all eigenvalues complex numbers with nontrivial real part, in which case Morse index is actually the number of eigenvalues with negative real part.
For a proof consult [7] and the references in [10].

If the vector field $X$ satisfies MS then the set $\mathcal{M}(x, y) = W_x^+ \cap W_y^+$, $x, y \in X$ is the image by an injective immersion of a smooth manifold of dimension $\text{ind}(x) - \text{ind}(y)$ on which $\mathbb{R}$ acts freely. The quotient is a smooth manifold $T(x, y)$ of dimension $\text{ind}(x) - \text{ind}(y) - 1$ called the manifold of trajectories from $x$ to $y$. If $T(x, y)$ is zero dimensional its elements are isolated trajectories called instantons. If $\mathcal{O}_x$ and $\mathcal{O}_y$ are orientations of $W_x^-$ and $W_y^-$ then any instanton $[\theta]$ from $x \in X_q$ to $y \in X_{q-1}$ has a sign $\epsilon_{\mathcal{O}_x, \mathcal{O}_y}([\theta]) = \pm 1$ defined as follows: The orientations $\mathcal{O}_x$ and $\mathcal{O}_y$ induce an orientation on $[\theta]$. Take $\epsilon_{\mathcal{O}_x, \mathcal{O}_y}([\theta]) = +1$ if this orientation is compatible with the orientation from $x$ to $y$ and $\epsilon_{\mathcal{O}_x, \mathcal{O}_y}([\theta]) = -1$ otherwise.

If the vector field satisfies NCT then any non-degenerate closed trajectory $[\bar{\theta}]$ has a period $p([\bar{\theta}]) \in \mathbb{N}$ and a sign $\epsilon([\bar{\theta}]) := \pm 1$ defined as follows:

(i) $p([\bar{\theta}])$ is the largest positive integer $p$ such that $\theta : S^1 \to M$ factors through a self map of $S^1$ of degree $p$.

(ii) $\epsilon([\bar{\theta}]) := \text{sign det}(\text{Id} - A_{\theta(t_0)})$ for some (and hence any) $t_0 \in \mathbb{R}$, and parameterization $\bar{\theta}$. Here $A_{\theta(t_0)}$ denotes the linear isomorphism induced by $D_{\theta(t_0)}\Psi_T$ in the normal space of the trajectory.

A cohomology class $\xi \in H^1(M; \mathbb{R})$ induces the homomorphism $\xi : H_1(M; \mathbb{Z}) \to \mathbb{R}$ and then the injective group homomorphism

$$\xi : \Gamma_\xi \to \mathbb{R}, \text{ with } \Gamma_\xi := H_1(M; \mathbb{Z})/\ker \xi.$$ 

For any two points $x, y \in M$ denote by $\mathcal{P}_{x,y}$ the space of continuous paths from $x$ to $y$. We say that $\alpha \in \mathcal{P}_{x,y}$ is $\xi$–equivalent to $\beta \in \mathcal{P}_{x,y}$ iff the closed path $\beta^{-1} \ast \alpha$ represents an element in $\ker \xi$. (Here $\ast$ denotes the juxtaposition of paths. Precisely if $\alpha, \beta : [0, 1] \to M$ and $\beta(0) = \alpha(1)$, then $\beta \ast \alpha : [0, 1] \to M$ is given by $\alpha(2t)$ for $0 \leq t \leq 1/2$ and $\beta(1 - 2t)$ for $1/2 \leq t \leq 1$.) We denote by $\mathcal{P}_{x,y}^\xi$ the set of $\xi$–equivalence classes of elements in $\mathcal{P}_{x,y}$. We also denote by $\tilde{\mathcal{P}}_{x,y}$ the set of homotopy classes of elements in $\mathcal{P}_{x,y}$. Note that $\Gamma_\xi$ acts freely and transitively, both from the left and from the right, on $\mathcal{P}_{x,y}^\xi$. The action $\ast$ is defined by juxtaposing at $x$ resp. $y$ a closed curve representing an element $\gamma \in \Gamma_\xi$ to a path representing the element $\tilde{\alpha} \in \mathcal{P}_{x,y}^\xi$.

Any closed one form $\omega$ representing $\xi$ defines a map, $\varpi : \mathcal{P}_{x,y} \to \mathbb{R}$, by

$$\varpi(\alpha) := \int_{[0,1]} \alpha^* \omega$$

which in turn induces the map $\varpi : \mathcal{P}_{x,y}^\xi \to \mathbb{R}$ resp. $\varpi : \tilde{\mathcal{P}}_{x,y} \to \mathbb{R}$. We have:

$$\varpi(\gamma \ast \tilde{\alpha}) = \xi(\gamma) + \varpi(\tilde{\alpha})$$

$$\varpi(\tilde{\alpha} \ast \gamma) = \varpi(\tilde{\alpha}) + \xi(\gamma)$$

Note that for $\omega' = \omega + dh$ we have $\varpi' = \varpi + h(y) - h(x)$.

**Proposition 4.** Let $X$ be a vector field and $\xi \in H^1(M; \mathbb{R})$ a Lyapunov cohomology class for the vector field $X$.

(i) (Novikov) If $X$ satisfies MS, $x \in X_q$ and $y \in X_{q-1}$ then the set of instantons from $x$ to $y$ in each class $\tilde{\alpha} \in \mathcal{P}_{x,y}^\xi$ is finite and therefore the same is true in each class $\tilde{\alpha} \in \tilde{\mathcal{P}}_{x,y}$.
(ii) (Hutchings) If $X$ satisfies both MS and NCT then for any $\gamma \in \Gamma_\xi$ the set of closed trajectories representing the class $\gamma$ is finite and therefore the same is true in each class $\gamma \in [S^1, M]$.

The proof is a straightforward consequence of the compacity of the space of trajectories of bounded energy, cf. [7] and [10].

Suppose $X$ is a vector field which satisfies MS and NCT and suppose $\xi$ is a Lyapunov class for $X$. In view of Proposition 4 we can define the counting function of closed trajectories by

$$Z_X^\xi : \Gamma_\xi \rightarrow \mathbb{Q}, \quad Z_X^\xi(\gamma) := \sum_{[\tilde{\theta}] \in \gamma} \frac{\epsilon([\tilde{\theta}])}{\rho([\tilde{\theta}])} \in \mathbb{Q}.$$

Denote by $O(x)$ the set of the two possible orientations of $W_x^-$ and define the counting function of the instantons from $x$ to $y$ by

$$I_{X, \xi}^{\mathcal{O}_x, \mathcal{O}_y} : \hat{\mathcal{P}}_{x,y} \times O(x) \times O(y) \rightarrow \mathbb{Z}, \quad I_{X, \xi}^{\mathcal{O}_x, \mathcal{O}_y}(\hat{\alpha}, \epsilon_{\mathcal{O}_x, \mathcal{O}_y}([\theta])) := \sum_{[\theta] \in \hat{\alpha}} \epsilon_{\mathcal{O}_x, \mathcal{O}_y}([\theta]).$$

Note that

$$I_{X, \xi}^{\mathcal{O}_x, \mathcal{O}_y}(\hat{\alpha}, \epsilon'_{\mathcal{O}_x, \mathcal{O}_y}) = \epsilon' \epsilon'' I_{X, \xi}^{\mathcal{O}_x, \mathcal{O}_y}(\hat{\alpha}, \epsilon_{\mathcal{O}_x, \mathcal{O}_y}), \quad \epsilon', \epsilon'' = \pm 1.$$

While both the instantons and the closed trajectories depend only on $X$ their counting functions change in general with $\xi$. A key observation in this work is the fact that the counting functions $I_{X, \xi}^{\mathcal{O}_x, \mathcal{O}_y}$ and $Z_X^\xi$ can be interpreted as Dirichlet series.

1.2. Dirichlet series and their Laplace transform. Recall that a Dirichlet series $f$ is given by a pair of finite or infinite sequences:

$$\left( \lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda_{k+1} < \cdots \right) \quad \left( a_1 \quad a_2 \quad \cdots \quad a_k \quad a_{k+1} \quad \cdots \right)$$

The first sequence is a sequence of real numbers with the property that $\lambda_k \to \infty$ if the sequences are infinite. The second sequence is a sequence of non-zero complex numbers. The associated series

$$L(f)(z) := \sum_{i} e^{-z\lambda_i} a_i$$

has an abscissa of convergence $\rho(f) \leq \infty$, characterized by the following properties, cf. [21] and [22]:

(i) If $\Re z > \rho(f)$ then $f(z)$ is convergent and defines a holomorphic function.

(ii) If $\Re z < \rho(f)$ then $f(z)$ is divergent.

A Dirichlet series can be regarded as a complex valued measure with support on the discrete set $\{\lambda_1, \lambda_2, \ldots\} \subseteq \mathbb{R}$ where the measure of $\lambda_i$ is equal to $a_i$. Then the above series is the Laplace transform of this measure, cf. [22]. The restriction of $L(f)(z)$ to any interval $(\rho', \infty)$, $\rho' > \rho(f)$ determines the Dirichlet series.

The following observation is a reformulation of Proposition 4.

Observation 1. (i) Suppose $X$ is a vector field on a closed manifold $M$ which satisfies MS and $\xi$ as a Lyapunov cohomology class for $X$. Suppose $\omega$ is a

---

The multiplicative group $\pm 1$ acts freely and transitively on $O(x)$
closed one form representing $\xi$. Then for any $x \in \mathcal{X}_q$ and $y \in \mathcal{X}_{q-1}$ the collection of pairs of numbers

$$X_{x,y}^{\omega} \mathcal{O}_x \mathcal{O}_y := \left\{ (-\mathfrak{w}(\hat{a}), \mathbb{I}^{X,Y}_{\mathfrak{w}}(\hat{a}, \mathcal{O}_x \mathcal{O}_y)) \mid \mathbb{I}^{X,Y}_{\mathfrak{w}}(\hat{a}, \mathcal{O}_x \mathcal{O}_y) \neq 0, \hat{a} \in \hat{\mathcal{P}}_{x,y} \right\}$$

defines a Dirichlet series. The sequence of $\lambda$’s consists of the numbers $-\mathfrak{w}(\hat{a})$ with $\mathbb{I}^{X,Y}_{\mathfrak{w}}(\hat{a}, \mathcal{O}_x \mathcal{O}_y) \neq 0$, and the sequence of $\alpha$’s consists of the numbers $\mathbb{I}^{X,Y}_{\mathfrak{w}}(\hat{a}, \mathcal{O}_x \mathcal{O}_y) \in \mathbb{Z}$. If $\omega$ changes in $\xi$ all numbers $-\mathfrak{w}(\hat{a})$ change by addition of the same constant $C_{x,y}$ hence the abscissa of convergence does not change and when smaller than $\infty$ the Laplace transform changes up to multiplication by $e^{-c_{x,y}z}$.

(ii) If in addition $X$ satisfies NCT then the collection of pairs of numbers

$$Z_X^\xi := \left\{ (-\xi(\gamma), Z_X(\gamma)) \mid Z_X(\gamma) \neq 0, \gamma \in \Gamma_\xi \right\}$$

defines a Dirichlet series. The sequence of $\lambda$’s consists of the real numbers $-\xi(\gamma)$ when $Z_X^\xi(\gamma)$ is non-zero and the sequence of $\alpha$’s consists of the numbers $Z_X^\xi(\gamma) \in \mathbb{Q}$.

**Definition 4.** A vector field $X$ is said to have the exponential growth property at a rest point $x$ if for some (and then any) Riemannian metric $g$ there exists a positive constant $C$ so that $\text{Vol}(D_r(x)) \leq e^{Cr}$, for all $r > 0$. Here $D_r(x) \subseteq W^-_x$ denotes the disk of radius $r$ with respect to the induced Riemannian metric $(i_x^*)^*g$ on $W^-_x$ centered at $x \in W^-_x$. A vector field $X$ satisfies the exponential growth property, EG for short, if it has the exponential growth property at all rest points.

A vector field which has $0 \in H^1(M; \mathbb{R})$ as Lyapunov cohomology class satisfies EG.

The interest of EG property stems from the fact that it assures the finite abscissa of convergence for the Dirichlet series associated with the counting functions of instantons and of closed trajectories.

For the sake of Theorems 3 and 4 we introduce in section 3, cf. Definition 9, the strong exponential growth property, SEG for short. If our Conjecture (cf section 2) is true both concepts, exponential growth and strong exponential growth, are however superfluous.

Also in this paper, for any vector field $X$ and cohomology class $\xi \in H^1(M; \mathbb{R})$ we define an invariant $\rho(\xi, X) \in \mathbb{R} \cup \{\pm \infty\}$ and show that if $\xi$ is Lyapunov for $X$ then exponential growth property is equivalent to $\rho(\xi, X) < \infty$. The finiteness of the abscissa of convergence comes from the finiteness of the invariant $\rho$. We show in this paper.

**Theorem 1.** Suppose $X$ is a smooth vector field which has $\xi$ as Lyapunov cohomology class. In any open $C^0$ neighborhood of $X$ one can find a vector field $Y$ which satisfies $L$, $MS$, $NCT$, $EG$ (SEG) properties. One can choose $Y$ to agree with $X$ in a neighborhood of the rest points.

The vector field $Y$ was constructed by Pajitnov, we only checked the EG (SEG) property.

Choose a collection of orientations $\mathcal{O} = \{\mathcal{O}_x\}$ for the unstable sets $W^-_x$, $x \in \mathcal{X}$. For $a \in \Omega^1(M)$, $x \in \mathcal{X}_q$ consider the integral

$$\text{Int}_{\mathcal{X}_q}(a) := \int_{W^-_x} e^{iH_x(i_x^*)a}$$

(2)
where \( h_x : W_x^- \to \mathbb{R} \) is the unique function with \( h_x(x) = 0 \) and \( dh_x = (i_x^-)^* (\omega) \).

For any \( x \in \mathcal{X}_q \) and \( y \in \mathcal{X}_{q+1} \) consider the sum

\[
(\delta^q_{X,\omega,\mathcal{O}}(t))_{x,y} := \sum_{\tilde{a} \in \mathcal{P}^q_{X,\omega}} \| X,\omega,\xi \|_{x,y} (\tilde{a}) e^{\|X,\omega,\xi\|} (\tilde{a})
\]  

(3)

Using the results about the completion of the unstable sets of \( X \) and the compactification of the spaces of trajectories, cf Theorems 5 or 6, we prove that:

**Theorem 2.** If \( X \) has exponential growth, \( \xi \) is a Lyapunov cohomology class for \( X \), and \( \omega \) is a closed one form in \( \xi \), then there exists \( T \in \mathbb{R} \) so that for \( t > T \)

(i) the integrals (2) are absolutely convergent and define the linear maps

\[
\text{Int}^q_{X,\omega,\mathcal{O}}(t) : \Omega^q(M) \to C^q(X,\mathcal{O}) := \text{Maps}(\mathcal{X}_q,\mathbb{R}),
\]

(ii) the sums (3) are absolutely convergent and define the linear maps

\[
\delta^q_{X,\omega,\mathcal{O}}(t) : C^q(X,\mathcal{O}) \to C^{q+1}(X,\mathcal{O}),
\]

(iii) \( \mathbb{C}^*(X,\omega,\mathcal{O})(t) := \{ C^q(X,\mathcal{O}), \delta^q_{X,\omega,\mathcal{O}}(t) \} \) is a cochain complex and \( \text{Int}^*_{X,\omega,\mathcal{O}}(t) \) is a surjective morphism of cochain complexes,

(iv) the morphism of cochain complexes \( \text{Int}^*_{X,\omega,\mathcal{O}}(t) \) is an isomorphism for \( t \) large enough.

As a consequence we will prove the following.

**Theorem 3.** Suppose \( X \) has \( \xi \) as Lyapunov cohomology class.

1. If \( X \) satisfies MS and EG then the Dirichlet series defined by the counting functions \( I^q_{X,\omega,\mathcal{O}} \) have finite abscissa of convergence.

2. If \( X \) satisfies MS, NCT and SEG \(^3\) then the Dirichlet series defined by the counting function \( Z^q_X \) has finite abscissa of convergence.

In particular the Laplace transform of the above Dirichlet series is defined and can be explicitly calculated with the help of spectral geometry. See formulas 28 and 32 in section 6.5.

Suppose \( g \) is a Riemannian metric on \( M \) and \( \omega \) a closed one form representing \( \xi \). The metric induces a scalar product on the space of differential forms, and then the operators \( \Delta^q(t) \) nonnegative self-adjoint elliptic differential operators referred to as the Witten Laplacians, cf section 6.4. They are zero order perturbations of the Laplace Beltrami operators \( \Delta^q \) associated to \((M,g)\) cf section 6, for definition. Then one can consider the Ray Singer torsion \( T_{an}(t) := T_{an}(\omega,g)(t) \) of the elliptic complex \( \mathcal{O}^*(M), d^e_{an}(t) \) with the metric \( g \) defined by

\[
\log T_{an}(t) := 1/2 \sum (-1)^{q+1} q \log \det' \Delta^q_{an}(t).
\]  

(4)

Here \( \det' \Delta^q_{an}(t) \) denotes the zeta regularized determinant calculated by ignoring the eigenvalue 0. Similarly, for \( t > T \) one can consider the torsion \( T_X(t) \) of the finite dimensional cochain complex \( \mathbb{C}^*(X,\omega,\mathcal{O})(t) \) equipped with the scalar products which makes the canonical base (provided by the characteristic functions of the rest points) orthonormal. \( T_X(t) \) is defined by a similar formula where \( \Delta^q_{an}(t) \) are replaced by the Laplacians in the finite dimensional cochain complex \( \mathbb{C}^*(X,\omega,\mathcal{O})(t) \). The

\(^3\)As this conclusion can be also derived from Theorem 4 below in the case \( H^*(M;\omega) = 0 \) for \( t \) large enough SEG can be replaced by EG.
Witten-Helffer-Sjöstrand theory extended from Morse functions to Morse closed one forms as discussed in [7] permits to prove

**Observation 2.** There exists $T \in \mathbb{R}$ so that for $t > T$, $Int_{X,\omega,O}(t)$ induces an isomorphism in cohomology.

In fact much more is true see section 6.4 and in particular Theorems 8 and 9.

As a consequence, if we equip the cohomology of $(\Omega^*(M), d_n(t))$, with the scalar product induced from $g$ via Hodge theory and the cohomology of $C^*(X,\omega,O)(t)$ with the scalar product induced from the canonical base $\{E_x\}$ we obtain for $t > T$ the function

$$\log V(t) := \log \text{Vol}_{X,g,\omega}(t) = \log \text{Vol}(Int_{X,\omega,O}(t)).$$

The Bismut Zhang main theorem in [2] permits to compare the functions $\log T_{an}(t)$, $\log T_X(t)$ and $\log V(t)$ when $X$ is a generalized triangulation (i.e. admits $0 \in H^n(M;\mathbb{R})$ as Lyapunov cohomology class). We extends this theorem to $X$ which satisfies MS, NCT, L and SEG. Since such $X$ can have closed trajectories the formula will involve in addition to the terms discussed in [2], the Laplace transform of the counting function for closed trajectories. To formulate precisely this result one needs one more geometric invariant.

1.3. **A geometric invariant associated to $(X,\omega,g)$**. For any Riemannian manifold $(M,g)$ of dimension $n$ there is a differential form $\Psi_g \in \Omega^{n-1}(TM \setminus M;\mathcal{O}_M)$ called in [3] the angular form. Here $\mathcal{O}_M$ denotes the orientation bundle of $M$ pulled back to $TM$. For any closed one form $\omega$ on $M$ we consider the form $\omega \wedge X^*\Psi_g \in \Omega^n(M \setminus X;\mathcal{O}_M)$, cf [2]. The integral $\int_{M \setminus X} \omega \wedge X^*\Psi_g$ is in general divergent. However it does have a regularization defined by the formula

$$R(X,\omega,g) := \int_{M} \omega_0 \wedge X^*\Psi_g = \int_{M} f E_g + \sum_{x \in X} (-1)^{\text{ind}(x)} f(x) \quad (5)$$

where

(i) $f$ is a smooth function whose differential $df$ is equal to $\omega$ in a small neighborhood of $X$ and therefore $\omega_0 := \omega - df$ vanishes in a small neighborhood of $X$ and

(ii) $E_g \in \Omega^n(M;\mathcal{O}_M)$ is the Euler form associated with $g$.

It will be shown in section 5 below that the definition is independent of the choice of $f$, see also [8]. Finally we have

**Theorem 4.** Suppose $X$ is a vector field which has $\xi$ as a Lyapunov cohomology class, $\omega \in \xi g$ a Riemannian metric and suppose $X$ satisfies MS, NCT, L and SEG. Then we have for $t$ large enough

$$\log T_{an}(\omega,g,t) - \log T_X(\omega)(t) - tR(X,\omega,g) - \log \text{Vol}(\omega,g,t) = \pm L(\xi_X)(t).$$

In case $H^*(M;\xi) = 0$ then SEG can be replaced by EG.

As a consequence we have

**Corollary 1** (J. Marcsik cf. [13] or [8]). Suppose $X$ is a vector field with no rest points, $\xi \in H^1(M;\mathbb{R})$ a Lyapunov class for $X$, $\omega$ a closed one form representing $\xi$ and $g$ a Riemannian metric on $M$. Suppose all closed trajectories of $X$ are non-degenerate and denote by

$$\log T_{an}(t) := 1/2 \sum (-1)^{q+1} q \log \det(\Delta^q(\xi,t)).$$
Then
\[ \log T_{an}(t) + t \int_{M} \omega \wedge X^* \Psi_g \]
is the Laplace transform of the Dirichlet series \( Z_X \), which counts the set of closed trajectories of \( X \) with the help of \( \xi \).

Here \( \Delta_q^{\xi}(t) \) denote the Witten laplacians introduced in section 6.4.

Remark 1. In case \( M = N_\phi \) is the mapping torus of a diffeomorphism \( \phi : N \to N \), whose periodic points are all non-degenerate, the Laplace transform of the Dirichlet series \( Z_X \) is the Lefschetz zeta function of \( \phi \) in the variable \( e^t \).

We end this paper with a number of additional comments/results on Witten-Helffer-Sj"{o}strand theory for a closed Morse form \( \omega \) Lyapunov for a vector field \( X \) which satisfies MS, NCT and SEG.

Here is the contents of the remaining part of this paper.

In section 2 we define the invariant \( \rho \), discuss the relationship with the exponential growth property, formulate the main conjecture (cf. section 2.2) and finally derive Theorem 1 from the work of Pajitnov [19].

In section 3 we discuss the completion of the unstable sets and the compactification of the space of unparameterized trajectories, cf. Theorem 5. This theorem was also proved in [7]. In this paper we provide a significant short cut in the proof and a slightly more general formulation.

In section 5 we discuss the invariant \( R(X, \omega, g) \).

In section 4 we discuss the integration map from the complex \( (\Omega^*(M), d_\omega(t)) \) to the complex \( \mathcal{C}^*(X, \omega, O)(t) \) which is the essential ingredient used in this paper.

In section 6 we prove Theorems 2, 3 and Theorem 4 and we discuss the additional features of the Witten-Helffer-Sj"{o}strand theory for closed Morse form which are Lyapunov for a vector field which satisfies L, MS, NCT, SEG properties. The proofs use results of Hutchins-Lee and Bismut-Zhang which will be reviewed.

We close this introduction with the following remarks:

1. Theorems 2, 3, 4 and Corollary 1 can be routinely extended to the case of a compact manifolds with boundary.

2. If \( \omega \) is a closed Morse form which is Lyapunov for \( X \) we have the decomposition of the deRham complex into the direct sum of small and large cf section 6.4. As Theorem 3 and 4 show the small complex is responsible for the counting of instantons while the large is largely (but not entirely) responsible for the counting of closed trajectories.

2. Exponential growth property and the invariant \( \rho \)

In this section we introduce for a pair \( (X, \xi) \) consisting of a vector field \( X \) and a cohomology class \( \xi \in H^1(M; \mathbb{R}) \) an invariant \( \rho(\xi, X) \in \mathbb{R} \cup \{\pm \infty\} \). We are interested in the case this invariant is smaller than \( \infty \) since it implies that the abscissas of convergence of the Dirichlet series considered in this paper are finite. One expects that this is always the case if \( \xi \) is Lyapunov for \( X \), or at least that this happens for a class of vector fields which is \( C^1 \)-generic. cf Conjecture section 2.2. If \( X \) has \( \xi \) as a Lyapunov cohomology class we prove that the exponential growth and \( \rho < \infty \) are equivalent. This section ends up with a criteria which recognizes exponential growth property. This criterion can be applied to the class of vector fields considered by Pajitnov in [19] and proved to be \( C^0 \)-generic.
2.1. The invariant \( \rho \). For a critical point \( x \in X \), i.e. a zero of \( X \), we let \( i_x^- : W_x^- \to M \) denote the smooth immersion of the unstable manifold into \( M \). If \( M \) is equipped with a Riemannian metric we get an induced Riemannian metric \( g_x := (i_x^-)^* g \) on \( W_x^- \) thus a volume form \( \mu(g_x) \) on \( W_x^- \) and hence the spaces \( L^p(W_x^-) \), \( p \geq 1 \). Though the \( L^p \)-norm depends on the metric \( g \) the space \( L^p(W_x^-) \) and its topology does not. Indeed for another Riemannian metric \( g' \) on \( M \) we find a constant \( C > 0 \) so that \( \frac{\mu(g')}{\mu(g)} \leq C \) for all tangent vectors \( X \) and \( Y \) which implies \( 1/C' \leq \frac{\mu(g')}{{\mu(g)}} \leq C' \) for some constant \( C' > 0 \).

Given a closed 1–form \( \omega \) on \( M \) we let \( h^\omega_x \) denote the unique smooth function on \( W_x^- \) which satisfies \( dh^\omega_x = (i_x^-)^* \omega \) and \( h^\omega_x(x) = 0 \). We are interested in the space of closed 1–forms for which \( e^{h^\omega_x} \in L^1(W_x^-) \). This condition actually only depends on the cohomology class of \( \omega \). Indeed we have \( h^{\omega+\delta \phi} = h^\omega_x + (i_x^-)^* f - f(x) \) and so \( |h^\omega_x+h^\delta| \leq C' \) and \( e^{-C'} \leq e^{h^\omega_x+h^\delta} \leq e^{C'} \) for some constant \( C' > 0 \).

We introduce the sets \( R_x(X), R(X) \subset H^1(M) \) defined by

\[
R_x(X) := \left\{ [\xi] \in H^1(M) \mid e^{h^\omega_x} \in L^1(W_x^-) \right\}
\]

and define the numbers \( \rho_x(\xi, X), \rho(\xi, X) \in \mathbb{R} \) by

\[
\rho_x(\xi, X) := \inf \{ t \in \mathbb{R} \mid t \xi \in R_x(X) \} \in \mathbb{R} \cup \{ \pm \infty \}
\]

and

\[
\rho(\xi, X) = \max_{x \in \mathcal{C}(X)} \rho_x(\xi, X).
\]

Observe that if \( t > 0 \) then \( \rho(t \xi, X) = 1/t \cdot \rho(\xi, \rho) \).

We define

\[
L(X) := \{ \xi \in H^1(M) \mid \xi \text{ is Lyapunov class for } X \}
\]

The main result of this of this subsection is the following Proposition.

**Proposition 5.**

1. The set \( L(X) \) is open and a convex cone, i.e. if \( t > 0 \) and \( \xi \in L(X) \) then \( t \xi \in L(X) \).

2. If \( \xi', \xi'' \in L(X) \) then \( \rho(\xi', X) < \infty \) implies \( \rho(\xi'', X) < \infty \).

In order to check \( \rho(\xi, X) < \infty \) by (2) it suffices to show this is actually the case for one cohomology class in \( L(X) \) and by (1) that this class can be chosen to be integral hence represented by a map \( M \to S^1 \).

**Proof of Proposition 5 (1).** To check the openness of \( L(X) \) observe that one can change \( \omega \) by adding a form whose support is disjoint form \( \mathcal{X} \) representing any sufficiently small cohomology but not affecting the Lyapunov property (\( \omega(X) < 0 \) away from \( \mathcal{X} \)). The convex cone property is straightforward.

**Proof of Proposition 5 (2).** First observe that

O1: The definition of Lyapunov form and implies that \( R(X) + L(X) \subseteq R(X) \).

Next observe that

O2: In view of Proposition 5 (1.) any ray in \( L(X) \), i.e.a half line starting at the origin which is contained in \( L(X) \), intersects \( \xi + L(X) \) for any \( \xi \).

If \( \rho(\xi', X) < \infty \) then there exists \( t_1 \) so that \( t_1 \xi' \in R(X) \). If \( \xi'' \in L(X) \) there exists \( t_2 \) so that by O2 we have \( t_2 \xi'' \in t_1 \xi' + L(X) \subset R(X) + L(X) \). Then by O1 \( t_2 \xi'' \in R(X) \), hence \( \rho(\xi'', X) < \infty \).
2.2. Exponential growth versus $\rho$. Let $x \in X$ be a zero of $X$, $W_x^-$ the unstable manifold. Let $g$ be a Riemannian metric on $M$ and let $r := \text{dist}(x, \cdot) : W_x^- \to [0, \infty)$ denote the distance to $x$ with respect to the induced metric $g_x = (i_x^-)^* g$ on $W_x^-$. Clearly $r(x) = 0$. Moreover let $B_s(x) := \{ y \in W_x^- : |r(y)| \leq s \}$ denote the ball of radius $s$ centered at $x$.

Recall from Definition 4 that $X$ has the exponential growth property at a zero $x$ if there exists a constant $C \geq 0$ such that $\text{Vol}(B_s(x)) \leq e^{C s}$ for all $s \geq 0$. This does not depend on the Riemannian metric $g$ on $M$ even though the constant $C$ will depend on $g$.

Let $g$ be a Riemannian metric on $M$, $\omega$ a closed one form representing $\xi$, and suppose $X = -\text{grad}_g \omega$.

Let $x \in X$. Recall that we have a smooth function $h_x^\omega : W_x^- \to (-\infty, 0]$ defined by $(i_x^-)^* \omega = dh_x^\omega$ and $h_x^\omega(x) = 0$. Observe that to show that $\rho(\xi, X) < \infty$ we have to check that there exists $C > 0$ so that $e^{C \omega_x} \in L^1(W_x^-)$. We have the following

Proposition 6. Suppose $\xi$ is Lyapunov for $X$ and let $x$ be a zero of $X$. Then the following are equivalent.

(i) $X$ has the exponential growth property at $x$ with respect to one (and hence every) Riemannian metric on $M$.

(ii) For one (and hence every) Riemannian metric on $M$ there exists a constant $C \geq 0$ such that $e^{-C r} \in L^1(W_x^-)$.

(iii) $\rho(\xi, X) < \infty$.

We begin with four lemmas.

Lemma 1. Suppose we have $C \geq 0$ such that $\text{Vol}(B_s(x)) \leq e^{C s}$ for all $s \geq 0$. Then $e^{-(C+\epsilon)r} \in L^1(W_x^-)$ for every $\epsilon > 0$.

Proof. We have

$$\int_{W_x^-} e^{-(C+\epsilon)r} = \sum_{n=0}^{\infty} \int_{B_{n+1}(x) \setminus B_n(x)} e^{-(C+\epsilon)r} \leq \sum_{n=0}^{\infty} \int_{B_{n+1}(x) \setminus B_n(x)} \text{Vol}(B_{n+1}(x)) e^{-(C+\epsilon)n} \leq e^{C(n+1)} e^{-(C+\epsilon)n} = e^C e^{-\epsilon n}$$

On $B_{n+1}(x) \setminus B_n(x)$ we have $e^{-(C+\epsilon)r} \leq e^{-(C+\epsilon)n}$ and thus

$$\int_{B_{n+1}(x) \setminus B_n(x)} e^{-(C+\epsilon)r} \leq \text{Vol}(B_{n+1}(x)) e^{-(C+\epsilon)n} \leq e^{C(n+1)} e^{-(C+\epsilon)n} = e^C e^{-\epsilon n}$$

So (6) implies

$$\int_{W_x^-} e^{-(C+\epsilon)r} \leq e^C \sum_{n=0}^{\infty} e^{-\epsilon n} = e^C (1 - e^{-\epsilon})^{-1} < \infty$$

and thus $e^{-(C+\epsilon)r} \in L^1(W_x^-)$. \hfill $\square$

Lemma 2. Suppose we have $C \geq 0$ such that $e^{-Cr} \in L^1(W_x^-)$. Then there exists a constant $C_0$ such that $\text{Vol}(B_s(x)) \leq C_0 e^{Cs}$ for all $s \geq 0$. 

Proof. We start with the following estimate for $N \in \mathbb{N}$:
\[
\Vol(B_{N+1}(x)) e^{-CN} = \sum_{n=0}^{N} \Vol(B_{n+1}(x)) e^{-C(n+1)} = \sum_{n=0}^{\infty} (\Vol(B_{n+1}(x)) - \Vol(B_{n}(x))) e^{-C(n+1)} = \sum_{n=0}^{\infty} \Vol(B_{n+1}(x) \setminus B_{n}(x)) e^{-C(n+1)} \leq \sum_{n=0}^{\infty} \int_{B_{n+1}(x) \setminus B_{n}(x)} e^{-Cr} = \int_{W_{x}^-} e^{-Cr}
\]
Given $s \geq 0$ we choose an integer $N$ with $N \leq s \leq N + 1$. Then $\Vol(B_{s}(x)) e^{-Cs} \leq \Vol(B_{N+1}(x)) e^{-CN} = e^{C} \Vol(B_{N+1}(x)) e^{-C(N+1)}$. So the computation above shows
\[
\Vol(B_{s}(x)) e^{-Cs} \leq e^{C} \int_{W_{x}^-} e^{-Cr} =: C_{0} < \infty
\]
and thus $\Vol(B_{s}(x)) \leq C_{0} e^{Cs}$ for all $s \geq 0$. \hfill \Box

Lemma 3. There exists a constant $C_{\omega,g} \geq 0$ such that $r \leq 1 - C_{\omega,g} h_{x}^{\omega}$ on $W_{x}^{-}$.

Proof. The proof is exactly the same as the one in [7, Lemma 3(2)]. Note that the MS property is not used there. \hfill \Box

Lemma 4. There exists a constant $C_{\omega,g}^{'} \geq 0$ such that $-h_{x}^{\omega} \leq C_{\omega,g}^{'} r$.

Proof. Let $\gamma : [0, 1] \to W_{x}^{-}$ be any path starting at $\gamma(0) = x$. For simplicity set $h := h_{x}^{\omega}$. Since $h(x) = 0$ we find
\[
|h(\gamma(1))| = \left| \int_{0}^{1} (dh)(\gamma'(t))dt \right| \leq ||\omega||_{\infty} \int_{0}^{1} |\gamma'(t)|dt = ||\omega||_{\infty} \text{length}(\gamma)
\]
with $||\omega||_{\infty}$ the supremum norm of $\omega$. We conclude
\[
||\omega||_{\infty} r(\gamma(1)) = ||\omega||_{\infty} \text{dist}(x, \gamma(1)) \geq |h(\gamma(1))| \geq -h(\gamma(1))
\]
and thus $-h \leq C_{\omega,g}^{'} r$ with $C_{\omega,g}^{'} := ||\omega||_{\infty}$. \hfill \Box

Proof. Lemma 1 shows (i) implies (ii), Lemma 2 that (ii) implies (i), Lemma 3 that (ii) implies (iii) and Lemma 4 that (iii) implies (ii). \hfill \Box

We end this subsection with the following conjecture.

Conjecture (Exponential growth). Let $\omega$ be a Morse closed one form and $X$ a smooth vector field with $\omega$ Lyapunov for $X$.

(the strong form: ) If $X$ has Morse Smale property then $X$ has exponential growth property.

(the weak form:) Arbitrary close to $X$ in the $C^{1}$- topology there exists a smooth vector field $Y$ which has $\omega$ as Lyapunov form and has exponential growth property.
2.3. A criterion for exponential growth. In this subsection we describe a criterion which when satisfied guarantees that the exponential growth property, and hence $\rho < \infty$, holds. This criterion is satisfied for a class of vector fields referred here as cellular gradients introduced by Pajitnov, who also proved that they are $C^0$ open and dense in the family of vector fields which have Lyapunov cohomology class. We begin by introducing a concept of virtual interaction.

For $x, y \in \mathcal{X}$ the virtual interaction of $y$ with $x$ is a compact set

$$K_x(y) \subseteq \text{Gr}_{\text{ind}(x)}(T_y(W_y^{-})).$$

Here $\text{Gr}_q(V)$ denotes the Grassmannian of $k$-dimensional subspaces in the vector space $V$. The virtual interaction $K_x(y)$ is empty if $\text{ind}_y > \text{ind}(x)$, although the interaction which when $X$ satisfies MS is the compact manifold $\hat{T}(x, y)$ of trajectories from $x$ to $y$ might not be empty. If $\text{ind}_y \geq \text{ind}_x$ and $X$ satisfies MS then there is no interaction of $y$ and $x$ but we may have nontrivial virtual interaction.

Definition of virtual interactions: Let $\text{Gr}_q(M) := \text{Gr}_q(T(M))$ the $q$ Grassmannian-tangent bundle of $M$ which is compact manifold. If $n = q$ then $\text{Gr}_q(M) = M$. If $N^0 \subseteq M$ is a smooth submanifold not necessary closed let $\iota(N) := \text{Gr}_q(N) \subseteq \text{Gr}_q(M)$ the image of $N$ by the obvious embedding $\iota : N \to \text{Gr}_q(M)$ with $\iota(y) = T_y(N) \subseteq \text{Gr}_q(T_y(M)) \subseteq \text{Gr}_q(M)$. For any $y \in M$ define the compact set $K^N_q(y) := \iota(N) \cap \text{Gr}_q(T_yM) \subset \text{Gr}_q(T_yM)$ if $y \neq N$ and $K^N_q(y) := \iota(N \setminus B_y) \cap \text{Gr}_q(T_yM) \subset \text{Gr}_q(T_yM)$ where $B_y$ is any compact ball in $N$ centered in $y$. Note that $K^N_q(y)$ is independent of the chosen ball.

Remark 2. Even though we removed a ball from $N$ the set $K^N_q(y)$ need not be empty. However if we did not remove $B$ the set $K^N_q(x)$ would never be vacuous for trivial reasons.

We define the virtual interaction by

$$K_x(y) := K_{\text{ind}_x}^{W^-}.$$

The main result of this subsection is the following Proposition.

Proposition 7. Let $\xi$ be Lyapunov for $X$ and suppose that the virtual interactions $K_x(y) = \emptyset$ for all $y \in \mathcal{X}$. Then $\rho_x(\xi, X) < \infty$.

To prove Proposition 7 we will need the following Lemma.

Lemma 5. Let $(V, g)$ be an Euclidean vector space and $V = V^+ \oplus V^-$ an orthogonal decomposition. For $\kappa \geq 0$ consider the endomorphism $A_\kappa := \kappa \text{id} \oplus -\text{id} \in \text{End}(V)$ and the function

$$\delta^A_\kappa : \text{Gr}_q(V) \to \mathbb{R}, \quad \delta^A_\kappa(W) := \text{tr}_{|W}(p_W^\perp \circ A_\kappa \circ i_W),$$

where $i_W : W \to V$ denotes the inclusion and $p_W^\perp : V \to W$ the orthogonal projection. Suppose we have a compact subset $K \subseteq \text{Gr}_q(V)$ for which $\text{Gr}_q(V^+) \cap K = \emptyset$. Then there exists $\kappa > 0$ and $\epsilon > 0$ with $\delta^A_\kappa \leq -\epsilon$ on $K$.

---

4The definition of $\hat{T}(x, y)$ is given in section 3.1 however this is an informal statement not used in this section.
Proof. Consider the case $\kappa = 0$. Let $W \in \text{Gr}_q(V)$ and choose a $g|_W$ orthonormal base $e_i = (e_i^+, e_i^-) \in V^+ \oplus V^-$, $1 \leq i \leq q$, of $W$. Then

$$\delta^{A_0}(W) = \sum_{i=1}^{q} g(e_i, A_0 e_i) = -\sum_{i=1}^{q} g(e_i^-, e_i^-).$$

So we see that $\delta^{A_0} \leq 0$ and $\delta^{A_0}(W) = 0$ iff $W \in \text{Gr}_q(V^+)$. Thus $\delta^{A_0}|_K < 0$. Since $\delta^{A_\kappa}$ depends continuously on $\kappa$ and since $K$ is compact we certainly find $\kappa > 0$ and $\epsilon > 0$ so that $\delta^{A_\kappa}|_K \leq -\epsilon$. \qed

Proof of Proposition 7. Let $S \subseteq W_x^-$ denote a small sphere centered at $x$. Let $\tilde{X} := (i_x^-)^*X$ denote the restriction of $X$ to $W_x^-$ and let $\Phi_t$ denote the flow of $\tilde{X}$ at time $t$. Then

$$\varphi : S \times [0, \infty) \rightarrow W^+_x, \quad \varphi(x, t) = \varphi_t(x)$$

parameterizes $W_x^-$ with a small neighborhood of $x$ removed.

Let $\kappa > 0$. For every $y \in X$ choose a chart $u_y : U_y \rightarrow \mathbb{R}^n$ centered at $y$ so that

$$X|_{U_y} = \kappa \sum_{i \in \text{ind}(y)} u_{iy}^\kappa \frac{\partial}{\partial u_{iy}} - \sum_{i > \text{ind}(y)} u_{iy}^\kappa \frac{\partial}{\partial u_{iy}}.$$ 

Let $g$ be a Riemannian metric on $M$ which restricts to $\sum_i du_i^g \otimes du_i^g$ on $U_y$ and set $g_x := (i_x^-)^* g$. Then

$$\nabla X|_{U_y} = \kappa \sum_{i \in \text{ind}(y)} du_i^g \otimes \frac{\partial}{\partial u_{iy}} - \sum_{i > \text{ind}(y)} du_i^g \otimes \frac{\partial}{\partial u_{iy}}.$$ 

In view of our assumption $K_x(y) = 0$ for all $y \in X$ Lemma 5 permits us to choose $\kappa > 0$ and $\epsilon > 0$ so that after possibly shrinking $U_y$ we have

$$\text{div}_{g_x}(\tilde{X}) = \text{tr}_{g_x}(\nabla \tilde{X}) \leq -\epsilon < 0 \quad \text{on} \quad \varphi(S \times [0, \infty)) \cap (i_x^-)^{-1}(\bigcup_{y \in X} U_y). \quad (7)$$

Next choose a closed 1–form $\omega$ so that $[\omega] = \xi$ and $\omega(X) < 0$ on $M \setminus X$. Choose $\tau > 0$ so that

$$\tau \omega(X) + \text{ind}(x)||\nabla X||_g \leq -\epsilon < 0 \quad \text{on} \quad M \setminus \bigcup_{y \in X} U_y. \quad (8)$$

Using $\tau \tilde{X} : h_x^\omega \leq 0$ and

$$\text{div}_{g_x}(\tilde{X}) = \text{tr}_{g_x}(\nabla \tilde{X}) \leq \text{ind}(x)||\nabla \tilde{X}||_{g_x} \leq \text{ind}(x)||\nabla X||_g$$

(7) and (8) yield

$$\tau \tilde{X} : h_x^\omega + \text{div}_{g_x}(\tilde{X}) \leq -\epsilon < 0 \quad \text{on} \quad \varphi(S \times [0, \infty)). \quad (9)$$

Choose an orientation of $W_x^-$ and let $\mu$ denote the volume form on $W_x^-$ induced by $g_x$. Consider the function

$$\psi : [0, \infty) \rightarrow \mathbb{R}, \quad \psi(t) := \int_{\varphi(S \times [0, t])} e^{\tau h_x^\omega} \mu \geq 0.$$ 

For its first derivative we find

$$\psi'(t) = \int_{\varphi(S)} e^{\tau h_x^\omega} i_{\tilde{X}} \mu > 0.$$
and for the second derivative, using (9),
\[
\psi''(t) = \int_{\varphi_t(S)} (\tau \tilde{X} \cdot h_x^\omega + \operatorname{div}_{\theta}(\tilde{X})) e^{\tau h_x^\omega} i_{\tilde{X}}\mu \\
\leq -\epsilon \int_{\varphi_t(S)} e^{\tau h_x^\omega} i_{\tilde{X}}\mu = -c\psi'(t).
\]
So \((\ln \psi')'(t) \leq -\epsilon\) hence \(\psi'(t) \leq \psi'(0)e^{-\epsilon t}\) and integrating again we find
\[
\psi(t) \leq \psi(0) + \psi'(0)(1 - e^{-\epsilon t})/\epsilon \leq \psi'(0)/\epsilon.
\]
So we have \(e^{\tau h_x^\omega} \in L^1(\varphi(S \times [0, \infty))\) and hence \(e^{\tau h_x^\omega} \in L^1(W_x^-)\) too. We conclude \(\rho_x(\xi, X) \leq \tau < \infty\).

\(\square\)

Remark 3.

(i) Proposition 7 implies \(\rho_x(\xi, X) < \infty\) whenever \(\xi\) is Lyapunov for \(X\) and \(\text{ind}(x) = \dim(M)\). However there is a much easier argument for this special case. Indeed, in this case \(W_x^-\) is an open subset of \(M\) and therefore its volume has to be finite. Since \(\xi\) is Lyapunov for \(X\) we immediately even get \(\rho_x(\xi, X) \leq 0\).

(ii) In the case \(\text{ind}(x) = 1\) we certainly have \(\rho_x(\xi, X) \leq 0\).

Note that throughout the whole subsections 2.1-2.3 we did not make use of a Morse–Morse condition.

2.4. The proof of Theorem 1. First observe that \(L(X)\) open and convex cone implies:

Observation 3. If \(X\) satisfies \(L\) then there exists \(\bar{x} \in \text{Im}(H^1(M; \mathbb{Z}) \to H^1(M; \mathbb{R}))\) Lyapunov for \(X\). In particular there exists a smooth map \(\theta : M \to S^1\) so that the closed one form \(\omega = \theta^*(dt)\) is Lyapunov for \(X\) and a Morse form.

Indeed because \(L(X)\) is open and convex cone one can choose \(\xi \in \text{Im}(H^1(M; \mathbb{Z}) \to H^1(M; \mathbb{R}))\), cf Proposition 5 (1), Lyapunov for \(X\), and by Proposition 1 (1) one can represent this \(\xi\) by a Morse closed one form \(\omega\) Lyapunov for \(X\). As \(\xi\) is integral \(\omega = \theta^*(dt)\).

Next we observe that \(X\) can be supposed to satisfy MS otherwise we can use Proposition 3 to change by an arbitrary small \(C^\infty\) perturbation the vector field \(X\) into a vector field \(Y\) which agrees with \(X\) in a small neighborhood of \(\mathcal{X}\) and satisfies MS. By choosing the perturbation small enough \(Y\) will continue to have \(\theta^*(dt)\) as Lyapunov form.

\(\text{Proof of Theorem 1.}\) We can start with \(X\) a vector field which satisfies MS and with a smooth function \(\theta : M \to S^1\) so that \(\omega = \theta^*(dt)\) is Lyapunov for \(X\).

Choose \(s_0 \in S^1\) a regular value and \(\epsilon\) small enough to have any point in the \(\epsilon\)-neighborhood of \(s_0\) a regular value. Denote by \(V := \theta^{-1}(s_0)\) and by \(W\) the bordism obtained by cutting \(M\) along \(V\) \((\partial W = V)\). We continue to denote by \(X\) the vector field on \(W\) induced from \(X\) and by \(\bar{\theta} : W \to [0, 1]\) the map induced from \(\theta\) by the "cutting off" at \(\theta^{-1}(s_0)\) (after the identification of \(S^1 \setminus s_0\) to the interval \((0, 1)\)). We are exactly in the situation of Pajitnov [19], with \(X\) a \(\theta\)-gradient like vector field on a bordism. By a arbitrary small \(C^0\) modification of \(X\) into the smooth vector field \(Y\) which equals \(X\) on \(\theta^{-1}(1 - \epsilon/2, 1] \cup [0, \epsilon/2)\) one obtains, as proved in [19], a \(\theta\)-gradient like vector field \(Y\) which satisfies the Pajitnov \(\delta\)-cellular property.
or Condition $\mathfrak{G}$, cf [19]. Since $X$ and $Y$ agree in a neighborhood of $\theta^{-1}(\{0,1\})$, $Y$ defines a vector field (still denoted by) $Y$ on $M$ which has $\theta^\ast(dt)$ as a Lyapunov form and satisfies on the nose the criterion $K_\ast(y) = \emptyset$, hence $Y$ has exponential growth property. As the set of $\delta-$cellular vector fields is also open in the $C^0$ topology one can make sure that $Y$ continues to satisfy MS.

For the reader convenience we review below Pajitnov’s condition $\mathfrak{G}$ or $\delta-$cellular in a notation as close as possible from Pajitnov’s.

First recall that a smooth vector field $-X$ on a closed manifold $M^n$, 5 which satisfies MS and is $f-$gradient like in the sense of [15] for some Morse function $f$, provides a partition of the manifold in cells, the unstable sets of the rest points of $-X$. (Such partition will be referred to as a generalized triangulation $\tau$, cf Definition 6.) From this perspective the dual triangulation $D(\tau)$ is associated to the vector field $X$ which has the same properties with respect to $-f$. Also from this perspective the union of the unstable sets (w.r. to $-X$) of the rest point of Morse index $\leq k$ represents the $k-$skeleton of $\tau$ and will be denoted (following Pajitnov [19]) by

$$D(\text{ind} \leq k,-X).$$

The union of stable sets of rest points of Morse index $\geq (k+1)$ (w.r.to $-X$) represents the $(k+1)-$ skeleton of $D\tau$. Both are compact sets.

Given a Riemannian metric $g$ on $M$ we will also write

$$B_\delta(\text{ind} \leq k,-X) \text{ resp. } D_\delta(\text{ind} \leq k,-X)$$

for the open resp. closed $\delta-$thickening of $D(\text{ind} \leq k,-X)$. They are the sets of points which lie on trajectories (of $-X$) which depart from the open resp. closed ball of radius $\delta$ centered at the rest points of Morse index $\leq k$. It is not hard to see that if $\delta$ is small enough then $B_\delta(\text{ind} \leq k,-X)$ resp. $D_\delta(\text{ind} \leq k,-X)$ provide a fundamental system of open resp. closed neighborhoods of $D(\text{ind} \leq k,-X)$.

We also write

$$C_\delta(\text{ind} \leq k,-X) := M \setminus B_\delta(\text{ind} \geq n-k-1,X).$$

These definitions and notations can be also used in the case of a bordism with gradient like vector field in the sense of Milnor [15]. Suppose we have $-X$ a vector field on the bordism $W$ which is $\theta-$ gradient like for a Morse function $\theta : W \to [0,1]$ as defined in [15], which satisfies the MS conditions. As above, provided that a Riemannian metric $g$ on $W$ is given, we have the sets $B_\delta(\text{ind} \leq k,-X)$ and $D_\delta(\text{ind} \leq k,-X)$.

Denote by $U_\pm \subseteq \partial_\pm W$ the set of points $y \in \partial_\pm W$ so that the trajectory $\Psi_t(y)$ of the vector field $-X$, arrives / departs from $\partial W_\mp$ at some positive/negative time $t$. They are open sets. Following again Pajitnov’s notation we denote by

$$(-Y)^\sim : U_+ \to U_- \text{ resp. } Y^\sim : U_- \to U_+$$

the obvious diffeomorphisms induced by the flow of $X$ which are inverse one to the other. If $A \subseteq \partial_\pm W$ we write for simplicity $(-Y)^\sim(A)$ instead of $(-Y)^\sim(A \cap U_\pm)$.

**Definition 5.** The gradient like vector field $-Y$ is $\delta-$cellular if for some (and then any other metric $g$ on $W$ one can find the generalized triangulations $X_\pm$ of $\partial W_\pm$ and $\delta$ small enough so that:

---

5We use $-X$ instead of $X$ to be consistent with the rest of the paper and with the references.
(B1) \((-Y)^\sim(C_3(\text{ind} x \leq k, X_+)) \cap (D_3(\text{ind} \leq k + 1, Y) \cap \partial_+ W) \subseteq B_3(\text{ind} \leq k, X_-)\)

(B2) \(Y^\sim(C_3(\text{ind} x \leq k, -X_-)) \cap (D_3(\text{ind} \leq k + 1, -Y) \cap \partial_+ W) \subseteq B_3(\text{ind} \leq k, -X_+)\)

If the vector field \(Y\) on \(W\) constructed by the cutting off construction is \(\delta\) cellular then, when regarded on \(M\), in view of B1 and B2, it has the following property. For any rest point \(y\) of index \(k\) there exists an open neighborhood so that any trajectory departing from \(x\) of index smaller or equal to \(k\) stay away from this neighborhood, hence the virtual interaction \(K_x(y)\) is empty. \(\square\)

All the definitions and results stated in this section including Theorem 1 routinely extend to bordisms and vector fields on bordisms which satisfies the conditions formulated in section 3.2.

The strengthening "strong exponential growth" in Theorem 1 which actually means exponential growth for a vector field on a bordism follows simply from the above extension when applied to the bordism \((M \times [-1, 1], M \times \{1\}, M \times \{-1\})\) combined with the Pajitnov result that \(C^{0,\ast}\) small perturbation of a vector field which satisfies condition \(\Phi\) continue to satisfy this condition.

3. Topology of the Space of Trajectories and Unstable Sets

In this section we consider a smooth vector field \(X\) on a closed manifold \(M\) with \(L(X) \neq \emptyset\).\(^6\) We discuss the completion of the unstable manifolds and the compactification of the manifolds of trajectories to manifolds with corners. Since we have to consider homotopies between such vector fields too we also discuss the extension of those results to the case of a vector field on a bordism.

3.1. Trajectories and Unstable Sets. Let \(X\) be vector field on the closed manifold \(M\) and suppose that \(X\) satisfies MS. Let \(\pi : \tilde{M} \to M\) denote the universal covering.

Denote by \(\tilde{X}\) the vector field \(\pi^* X\) and consider \(\pi : \tilde{X} = \pi^{-1}(X)\) and \(\tilde{X}_+ = \pi^{-1}(X_+)\).

Given \(\tilde{x} \in \tilde{X}\) let \(i_\tilde{x}^\pm : W^\pm_{\tilde{x}+\tilde{x}} \to \tilde{M}\) denote the one to one immersions whose images define the stable and unstable sets of \(\tilde{x}\) with respect to the vector field \(\tilde{X}\)

For any \(\tilde{x}\) with \(\pi(\tilde{x}) = x\) one can canonically identify \(W^\pm_{\tilde{x}}\) to \(W^\pm_x\) so that \(\pi \circ i^\pm_{\tilde{x}} = i^\pm_x\). Define \(\mathcal{M}(\tilde{x}, \tilde{y}) := W^-_{\tilde{x}} \cap W^+_{\tilde{y}}\) if \(\tilde{x} \neq \tilde{y}\) and \(\mathcal{M}(\tilde{x}, \tilde{x}) := \emptyset\). As the maps \(i^\pm_{\tilde{x}}\) and \(i^\pm_{\tilde{y}}\) are transversal \(\mathcal{M}(\tilde{x}, \tilde{y})\) is submanifold of \(\tilde{M}\) of dimension \(\text{ind}(\tilde{x}) - \text{ind}(\tilde{y})\). It is equipped with the free action of \(\mathbb{R}\) defined by the flow generated by \(\tilde{X}\). Denote the quotient \(\mathcal{M}(\tilde{x}, \tilde{y})/\mathbb{R}\) by \(\mathcal{T}(\tilde{x}, \tilde{y})\). The quotient \(\mathcal{T}(\tilde{x}, \tilde{y})\) is a smooth manifold of dimension \(\text{ind}(\tilde{x}) - \text{ind}(\tilde{y}) - 1\), possibly empty. If \(\text{ind}(\tilde{x}) \leq \text{ind}(\tilde{y})\), in view the transversality required by the hypothesis MS, the manifolds \(\mathcal{M}(\tilde{x}, \tilde{y})\) and \(\mathcal{T}(\tilde{x}, \tilde{y})\) are empty.

An unparameterized broken trajectory from \(\tilde{x} \in \tilde{X}\) to \(\tilde{y} \in \tilde{X}\), is an element of the set \(\mathcal{T}(\tilde{x}, \tilde{y}) := \bigcup_{k \geq 0} \mathcal{T}(\tilde{x}, \tilde{y})_k\), where

\(B(\tilde{x}, \tilde{y})_k := \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_k, \tilde{y}_{k+1})\) \hspace{1cm} (10)

\(^6\)Recall that \(L(X)\) denotes the set of Lyapunov cohomology classes of \(X\).

\(^7\)The maps \(i^\pm_{\tilde{x}}\) are actually smooth embeddings i.e. the manifold topology on \(W^\pm_{\tilde{x}}\) coincides with the topology induced from \(\tilde{M}\), although we do not need this fact here.
and the union is over all (tuples of) critical points $\tilde{y}_k \in \tilde{X}$ with $\tilde{y}_0 = \tilde{x}$ and $\tilde{y}_{k+1} = \tilde{y}$.

For $\tilde{x} \in \tilde{X}$ introduce the completed unstable set $\tilde{W}_x^- := \bigcup_{k \geq 0} (\tilde{W}_x^-)_k$, where

$$(W_x^-)_k := \bigcup \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_{k-1}, \tilde{y}_k) \times W_{\tilde{y}_k}^-$$

and the union is over all (tuples of) critical points $\tilde{y}_k \in \tilde{X}$ with $\tilde{y}_0 = \tilde{x}$.

Let $\tilde{i}_x^- : \tilde{W}_x^- \to M$ denote the map whose restriction to $\mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_{k-1}, \tilde{y}_k) \times W_{\tilde{y}_k}^-$ is the composition of the projection on $W_{\tilde{y}_k}^-$ with $\tilde{i}_{\tilde{y}_k}$.

Recall that an $n$-dimensional manifold with corners $P$, is a paracompact Hausdorff space equipped with a maximal smooth atlas with charts $(\partial P = P)$ by

Theorem 5. Let $M$ be a closed manifold, and suppose $X$ is a smooth vector field which satisfies $MS$ and $L$.

(i) For any two rest points $\tilde{x}, \tilde{y} \in \tilde{X}$ the smooth manifold $\mathcal{T}(\tilde{x}, \tilde{y})$ has $\mathcal{B}(\tilde{x}, \tilde{y})$ as a compactification. Moreover $\mathcal{B}(\tilde{x}, \tilde{y})$ has a natural structure of a compact smooth manifold with corners, whose $k$-corner is $\mathcal{B}(\tilde{x}, \tilde{y})_k$ from (10).

(ii) For any rest point $\tilde{x} \in \tilde{X}$, the smooth manifold $W_{\tilde{x}}^-$ has $\tilde{W}_x^-$ as a completion. Moreover $\tilde{W}_x^-$ has a natural structure of a smooth manifold with corners, whose $k$-corner coincides with $(\tilde{W}_x^-)_k$ from (11).

(iii) $\tilde{i}_x^- : \tilde{W}_x^- \to M$ is smooth and proper, for all $\tilde{x} \in \tilde{X}$.

(iv) If $\omega$ is a closed one form representing $\xi$ and $h : M \to \mathbb{R}$ a smooth function with $dh = \pi^* \omega$ then the function $h \circ \tilde{i}_x^-$ is smooth and proper, for all $\tilde{x} \in \tilde{X}$.

Proof. This follows from Theorem 1 in [7] by lifting everything to the universal covering. Note that the proof in [7] can be significantly shortened by observing two facts: First, that Theorem 5 is equivalent to a more general statement which claim the same result for any covering $\pi : \tilde{M} \to M$ with $\pi^*(\xi) = 0$, $\xi \in H^1(M; \mathbb{R})$ Lyapunov cohomology class for $X$, and that $L(X) \neq \emptyset$ implies the existence of a Lyapunov cohomology class in $\text{img}(H^1(M; \mathbb{Z}) \to H^1(M; \mathbb{R}))$ cf. Observation 3.

It will be convenient to formulate Theorem 5 without any reference to the the lifts $\tilde{x}$ of the rest points $x$.

As in section 1 denote by $\mathcal{P}_{x,y}$ the set of continuous paths from $x$ to $y$ and by $\tilde{\mathcal{P}}_{\tilde{x},y}$ the set of homotopy classes of paths from $x$ to $y$.

Note that any two lifts $\tilde{x}, \tilde{y} \in \tilde{M}$ determine an element $\tilde{\alpha} \in \tilde{\mathcal{P}}_{\tilde{x},y}^\tilde{M}$ and the set of trajectories from $\tilde{x}$ to $\tilde{y}$ identifies to the set $\mathcal{T}(x, y, \tilde{\alpha})$ of trajectories of $X$ from $x$ to $y$ in the class $\tilde{\alpha}$.

Theorem 5 can be reformulated in the following way:

Theorem 6 (Reformulation of Theorem 5). Let $M$ be a closed manifold and suppose that $X$ is a smooth vector field which satisfies $MS$ and has $L(X) \neq \emptyset$.

(i) For any two rest points $x, y \in X$ and every $\tilde{\alpha} \in \tilde{\mathcal{P}}_{\tilde{x},y}^\tilde{M}$, $T(x, y, \tilde{\alpha})$ is a smooth manifold of dimension $\text{ind}(x) - \text{ind}(y) - 1$ which has a natural
compactification to a compact smooth manifold with corners that \( T(x, y, \hat{\alpha}) \).

Its \( k \)-corner is

\[
\hat{T}(x, y, \hat{\alpha})_k = \bigcup \mathcal{T}(y_0, y_1, \hat{\alpha}_0) \times \cdots \times \mathcal{T}(y_k, y_{k+1}, \hat{\alpha}_k)
\]

where the union is over all (tuples of) critical points \( y_i \in \mathcal{X} \) and \( \hat{\alpha}_i \in \hat{\mathcal{P}}_{y_i, y_{i+1}} \) with \( y_0 = x, y_{k+1} = y \) and \( \hat{\alpha}_0 \cdots \hat{\alpha}_k = \hat{\alpha} \).

(ii) For any rest point \( x \in \mathcal{X} \) the smooth manifold \( W_x^- \) has a natural completion to a smooth manifold with corners \( \hat{W}_x^- \). Its \( k \)-corner is

\[
(W_x^-)_k = \bigcup \mathcal{T}(y_0, y_1, \hat{\alpha}_0) \times \cdots \times \mathcal{T}(y_{k-1}, y_k, \hat{\alpha}_{k-1}) \times W_{y_k}^-
\]

where the union is over all (tuples of) critical points \( y_i \in \mathcal{X} \) and \( \hat{\alpha}_i \in \hat{\mathcal{P}}_{y_i, y_{i+1}} \) with \( y_0 = x \).

(iii) The mapping \( \hat{i}_x^- : \hat{W}_x^- \to M \) which on \( (W_x^-)_k \) is given by the composition of the projection onto \( W_{y_k}^- \) with \( \hat{i}_{y_k}^- : W_{y_k}^- \to M \) is smooth, for all \( x \in \mathcal{X} \).

(iv) If \( \omega \) is a closed one form representing \( \xi \) and \( h : \hat{M} \to \mathbb{R} \) a smooth function with \( dh = \pi^*\omega \) then the function \( h \cdot \hat{i}_x^- \) is smooth and proper, for all \( x \in \hat{\mathcal{X}} \).

Theorem 5 (or 6) discusses only the unstable manifolds. The same results remain true for the stable manifolds of \( X \) which are the same as the unstable manifolds of \(-X\).

**Definition 6.** A vector field \( X \) which satisfies \( MS \) and has \( 0 \in H^1(M : \mathbb{R}) \) as Lyapunov cohomology class is called a generalized triangulation.

In this case the smooth manifolds with corners \( \hat{W}_x^- \) are compact and \( W_x^- \) provide a partition of \( \hat{M} \) in cells (cell complex). Moreover \( \mathcal{X} \) has no closed trajectories.

### 3.2. Bordisms and homotopies

Theorems 5 and 6 above can be extended to the case of bordisms.

Recall that a bordism \( (M, \partial_\pm, \partial M_-) \) is a compact smooth manifold with boundary \( (M, \partial M) \) whose boundary \( \partial M \) is decomposed in two components (not necessary connected) \( \partial M_+ \) and \( \partial M_- \).

A smooth vector field \( X \) on a bordism \( (M, \partial M_+, \partial M_-) \) is assumed to satisfy the following conditions:

(i) there exist collars neighborhoods \( \varphi : \partial \pm M \times [0, \epsilon) \to M \) so that \( \varphi^*(X) = X_\pm + \pm \partial \partial \partial M \) with vectors fields on \( \partial \pm M \).

(ii) all rest points are of Morse type (i.e. in some coordinate system in the neighborhood of each rest point is of the form (1)).

Note that if \( x \in \mathcal{X} \cap \partial M_- \) then \( \text{ind}_{\partial M}(x) = \text{ind}(x) - 1 \). Denote by

(i) \( \mathcal{X}' := \mathcal{X} \cap \partial M_- \), \( \mathcal{X}_\partial := \mathcal{X} \cap \partial M_+ \),

(ii) \( \mathcal{X}' = \mathcal{X} \cap \partial M = \mathcal{X}_\partial \cup \mathcal{X}_\partial \)

(iii) \( \mathcal{X}' := \mathcal{X} \setminus \mathcal{X}_\partial \).

For \( x \in \mathcal{X}' \) denote by \( \hat{i}_x^- : W_x^- \to M \) the unstable manifold with respect to \( X \) and by \( j_x^- : W_{\partial M, x}^- \to \partial M \) the unstable manifold with respect to \( X_{\partial M} \).

**Remark 4.**

(i) If \( x \in \mathcal{X}_\partial \) then the unstable manifold of \( x \) with respect to \( X \) and \( X_{\partial M} \) are the same. More precisely \( \hat{i}_x^- : W_x^- \to M \) identifies to \( j_x^- : W_{\partial M, x}^- \to \partial M \) followed by the inclusion of \( \partial M \subset M \).
(ii) If \( x \in \mathcal{X}''_+ \) then \((W^-_x, W^-_{\partial M, x})\) is a smooth manifold with boundary diffeomorphic to \((\mathbb{R}^k, \mathbb{R}^{k-1})\) with \( k = \text{ind}(x) \), \( i^-_x : W^-_x \to M \) is transversal to the boundary of \( M \) and \( i^-_x|_{W^-_{\partial M, x}} = j_x \)

**Remark 5.** Theorems 5 and 6 remain true as stated with the following specifications. Set \( P_y := W_y \setminus W^-_{\partial M, y} \) for \( y \in \mathcal{X}''_+ \), and \( P^-_y := W^-_y \) for \( y \in \mathcal{X}'' \). For \( x \in \mathcal{X}''_+ \) the \( k \)-corner of \( \hat{W}_x \) is

\[
(\hat{W}_x)_k = (\hat{W}^-_{\partial M, x})_{k-1} \cup \bigcup \mathcal{T}(y_0, y_1, \alpha_0) \times \cdots \times \mathcal{T}(y_{k-1}, y_k, \alpha_{k-1}) \times P^-_{y_k}
\]

where the big union is over all (tuples of) \( y_i \in \mathcal{X} \) and \( \alpha_i \in \hat{P}_{y_i, y_{i+1}} \) with \( y_0 = x \).

Let \( \xi \in H^1(M; \mathbb{R}) \), and \( \pi : \hat{M} \to M \) be a covering so that \( \pi^* \xi = 0 \).

**Definition 7.** A homotopy from the vector field \( X^1 \) to the vector field \( X^2 \) is a smooth family of sections \( \hat{X} := \{X_s\}_{s \in [-1,1]} \) of the tangent bundle so that for some \( \epsilon > 0 \) \( X_s = X^1 \) for \( s < -1 + \epsilon \) and \( X_s = X^2 \) for \( s > 1 - \epsilon \).

To a homotopy \( \hat{X} := \{X_s\}_{s \in [-1,1]} \) one associates the vector field \( Y \) on the compact manifold with boundary (cf appendix to section 3 for definition) \( N := M \times [-1,1] \) defined by

\[
Y(x, s) := X(x, s) + 1/2(s^2 - 1) \frac{\partial}{\partial s}.
\]  

(12)

The vector field \( Y \) is a vector field on the bordism \( (N, \partial N_+, \partial - N) \). Note that \( \mathcal{Y}' \), the set of interior rest points, is empty.

**Definition 8.** The homotopy \( \hat{X} \) satisfies L, MS, EG properties if if so does the vector field \( Y \).

We write \( \rho(\xi, \mathcal{X}) := \rho(p^* \xi, Y) \). Clearly \( \rho(\xi, \mathcal{X}) \geq \rho(\xi, X^i) \) for \( i = 1, 2 \).

We have

**Proposition 8.** 1. If \( \mathcal{X} \) is a homotopy between two vector fields \( X^1 \) and \( X^2 \) which both have \( \xi \) as a Lyapunov cohomology class, then the vector field \( Y \) has \( p^* \xi \) as a Lyapunov cohomology class, where \( p : N = M \times [-1,1] \to M \) is the first factor projection.

2. If \( X^1 \) and \( X^2 \) are two vector fields which satisfy MS and \( \mathcal{X} \) is a homotopy from \( X^1 \) to \( X^2 \), then arbitrarily close to \( \mathcal{X} \) in the \( C^1 \)-topology there exists homotopy \( \mathcal{X}' \) which satisfy MS.

**Proof of 1.** Choose \( \omega^1 \) resp. \( \omega^2 \) Lyapunov forms for \( X^1 \) and \( X^1 \). Choose \( \lambda : [-1,1] \to \mathbb{R} \) a non-negative smooth function satisfying

\[
\lambda(s) = \begin{cases} 
0 & \text{for } s \leq -1 + \epsilon \\
1 & \text{for } s \geq 1 - \epsilon.
\end{cases}
\]

Choose a closed 1-form \( \omega \) on \( N \) which restricts to \( p^* \omega^1 \) on \( M \times [-1,1 + \epsilon] \) and to \( p^* \omega^2 \) on \( M \times [1 - \epsilon, 1] \). This is possible since \( \omega^1 \) and \( \omega^2 \) define the same cohomology class \( \xi \) and can be constructed in the following way.

Choose a function \( h \) on \( M \) with \( \omega_2 - \omega_1 = dh \) and set \( \omega := p^* \omega^1 + d(\lambda p^* h) \).

Choose a function \( u : [-1,1] \to \mathbb{R} \), such that:

(i) \( u(s) = -\frac{1}{2}(s^2 - 1) \) for all \( s \leq -1 + \epsilon \) and all \( s \geq 1 - \epsilon \).

(ii) \( u(s) \geq \frac{1}{4}(\omega(Y)(x,s)) \) for all \( s \in [-1 + \epsilon, 1 - \epsilon] \) and all \( x \in M \).
This is possible since \[ \left\{ \frac{-\omega(Y)(x,s)}{s^2 - 1} \right\} \leq 0 \text{ for } s = -1 + \epsilon \text{ and } s = 1 - \epsilon. \]

The form \( \tilde{\omega} := \omega + u(s) ds \) represents the cohomology class \( p^! \xi \) and \( \tilde{\omega}(Y) < 0 \) on \( N \setminus \mathcal{Y}. \)

Proof of 2. First we modify the vector field \( Y \) into \( Y' \) by a small change in the \( C^1 \)-topology, and only in the neighborhood of \( M \times \{0\} \), in order to have the Morse–Smale condition satisfied for any \( y \in \mathcal{Y}' \) and \( z \in \mathcal{Y}'' \). This can be done using Proposition 3. The vector field \( Y' \) might not have the \( I \)-component equal to \( 1/2(s^2 - 1) \partial/\partial s \), it is nevertheless \( C^1 \)-close to, so by multiplication with a function which is \( C^1 \)-close to 1 and equal to 1 on the complement of a small compact neighborhood of the locus where \( Y \) and \( Y' \) are not the same, one obtains a vector field \( Y'' \) whose \( I \)-component is exactly \( 1/2(s^2 - 1) \partial/\partial s \). The \( M \)-component of \( Y'' \) defines the desired homotopy. By multiplying a vector field with a smooth positive function the stable and unstable sets do not change, and their transversality continues to hold.

Finally we introduce SEG property.

**Definition 9.** The vector field \( X \) satisfies strong exponential growth property, SEG for short if there exists a homotopy \( X \) from \( X = X^1 \) to a generalized triangulation \( X^2 \) which satisfies EG.

Clearly it can be chosen to satisfy also MS.

4. Integration map

Let \( X \) be a vector field with \( \xi \in H^1(M; \mathbb{R}) \) a Lyapunov cohomology class.

Choose \( \mathcal{O} \equiv \{ O_x, x \in X \} \) a collection of orientations of \( W_x \). In Section 1 we have defined the instanton counting function \( \hat{I} \xi : \hat{P} \xi \times O(x) \times O(y) \rightarrow \mathbb{Z} \). Denoted by \( I^{X, \mathcal{O}, \xi} \).

The following observation completes Observation 1 and is a straightforward consequence of the compacity of the 2-dimensional manifold with corners \( \bar{T}(x, z, \hat{\gamma}) \) when \( \text{ind } x = \text{ind } z = 2 \).

**Observation 4.** If \( \omega \) is a closed form representing \( \xi \) then one has.

(i) For any \( x \in \mathcal{X}_q, \ y \in \mathcal{X}_{q-1} \) and every real number \( R \) the set
\[ \{ \hat{\alpha} \in \hat{P} \xi \mid I^{X, \mathcal{O}, \xi} (\hat{\alpha}) \neq 0, -\pi(\hat{\alpha}) \leq R \} \]

is finite.

(ii) For any \( x \in \mathcal{X}_q, \ z \in \mathcal{X}_{q-2} \) and \( \hat{\gamma} \in \hat{P} \xi \) one has
\[ \sum I^{X, \mathcal{O}, \xi} (\hat{\alpha}) \cdot I^{X, \mathcal{O}, \xi} (\hat{\beta}) = 0. \]

where the sum is over all \( y \in \mathcal{X}_{q-1}, \ \hat{\alpha} \in \hat{P} \xi, \) and all \( \hat{\beta} \in \hat{P} \xi \) with \( \hat{\alpha} \ast \hat{\beta} = \hat{\gamma} \).

Formula (13) implicitly states that the left side of the equality contains only finitely many non-zero terms.

Observation 4 above is equivalent to Theorem 2 parts 1 and 2 in [7]. The proof, originally due to Novikov can be also found in [7].

The following proposition will be the main tool in the proof of Theorem 2.
Proposition 9. Suppose \( t \in \mathbb{R}, \omega \) a closed one form representing \( \xi \) and \( t > \rho(\xi, X) \). Then:

(i) For every \( a \in \Omega^n(M) \) and every \( x \in X_q \) the integral

\[
(\text{Int}^q_{X,\omega,\mathcal{O}}(t)(a))(x) := \int_{W^x_a} e^{i\bar{h}s} (i_x^a)^* a
\]

converges absolutely and then defines a linear map \( \text{Int}^q_{X,\omega,\mathcal{O}}(t) : \Omega^n(M) \to \text{Maps}(X_q, \mathbb{R}) \).

(ii) The map \( \text{Int}^q_{X,\omega,\mathcal{O}}(t) : \Omega^n(M) \to \text{Maps}(X_q, \mathbb{R}) \) is surjective.

(iii) For any \( x \in X_{q+1}, y \in X_q \) the series

\[
\sum_{\hat{a} \in \mathcal{P}^I_{x,y}} \mathbb{I}_{x,y}^X,\xi(\hat{a}) e^{i\xi(\hat{a})} (\text{Int}^q_{X,\omega,\mathcal{O}}(t)(a))(y)
\]

is absolutely convergent and

\[
(\text{Int}^q_{X,\omega,\mathcal{O}}(t)(da))(x) = \sum_{y \in X_q} \sum_{\hat{a} \in \mathcal{P}^I_{x,y}} \mathbb{I}_{x,y}^X,\xi(\hat{a}) e^{i\xi(\hat{a})} (\text{Int}^q_{X,\omega,\mathcal{O}}(t)(a))(y).
\]

Recall that for an oriented \( n \)-dimensional manifold \( N \) and \( a \in \Omega^n(N) \) one has \( |a| := |a'| \text{ Vol} \in \Omega^n(M) \), where \( \text{Vol} \) is any volume form and \( a' \in C^\infty(N, \mathbb{R}) \) is the unique function satisfying \( a = a' \cdot \text{Vol} \). The integral \( \int_N a \) is called absolutely convergent, if \( \int_N |a| \) converges.

Proof. Proof of Proposition 9:

Item (i) follows from the definition of \( \rho(\xi, X) \). To prove (ii) it suffices to show that for any \( f \in \text{Maps}(X_q, \mathbb{R}) \) and any \( \epsilon > 0 \) one can construct a differential form \( a \in \Omega^n(M) \) so that \( |\text{Int}^q_{X,\omega,\mathcal{O}}(t)(a)(x) - f(x)| \leq \epsilon \). This is done in [7] section 5, (cf. Proposition 4, page 27). To prove (iii) a few additional considerations are necessary.

For simplicity we write \( \Gamma \) instead of \( \Gamma_\xi \) and denote \( V := \text{Maps}(X, \mathbb{C}) \) and by \( A := \text{End}_\mathbb{C}(V, V) \). \( V \) is a finite dimensional vector space of dimension the cardinality \( k \) of \( X \) while \( A \) is a \( \mathbb{C} \)-algebra isomorphic to the \( k \times k \) matrix algebra with coefficients in \( \mathbb{C} \). We regard \( A \) as a Banach algebra with the standard matrix norm.

We denote by \( L := L^1(\Gamma, A) \) the Banach space of the \( L^1 \)-maps from \( \Gamma \) to \( A \) with the obvious \( L^1 \) norm, and by \( N \) the vector space of maps \( a : \Gamma \to A \) with the property that the cardinality of the set \( \{ \gamma \in \Gamma, \bar{\omega}(\gamma) < R, a(\gamma) \neq 0 \} \) is finite for any real number \( R \).

Both \( L \) and \( N \) when equipped with the convolution product are \( \mathbb{C} \)-algebras with \( L \) a Banach algebra with unit. As the convolution product is associative whenever is defined we have the following Lemma.

Lemma 6. If \( I \in N \) and \( a \in N \cap L \) and \( \|1 - a\|_{L^1} < 1 \) then \( I \ast a \in L \) implies \( I \in L \).

The proof is straightforward.

Let \( s : X \to X \) be a section of \( \pi \), i.e. \( \pi \circ s = \text{id} \). Given \( x \in X_{q+1} \) and \( y \in X_q \) we identify the counting functions \( \mathbb{I}_{x,y}^X,\xi(\hat{a}) : \mathcal{P}^I_{x,y} \to \mathbb{Z} \subset \mathbb{C} \) with the maps \( I_{x,y} : \Gamma \to \mathbb{C} \) defined by

\[
I_{x,y}(\gamma) := \mathbb{I}_{x,y}^X,\xi(\hat{a}(\gamma)) e^{i\xi(\hat{a}(\gamma))}
\]
(14)
where $\alpha(\gamma)$ is the class represented by $\pi \circ \alpha$ with any path from $s(x)$ to $\gamma \cdot s(y)$. We write $I : \Gamma \to \mathbb{A}$ for the matrix valued map defined by the formula (14) if $\arg x - \arg y - 1 = 0$ and by $I_{x,y}(\gamma) = 0$ otherwise.

The series in (iii) is absolute convergent if $I \in L$. The proof that this is the case is achieved by using Lemma 6 above. We construct $\alpha \in L \cap N$, then show that $\|1 - \alpha\|_{L_1} < 1$ and finally that the convolution $I * a$ is in $L$. For this purpose we choose a smooth function $\chi : \bar{M} \to [0,1]$ with the properties that:

1. $\supp(\chi) \cap \supp(\chi^* \chi) = \emptyset$, for any $\gamma \neq e$ where $\gamma^* \chi$ denotes the composition of $\chi$ with the deck transformation determined by $\gamma$, and $\supp$ denotes the support of the function in parenthesis.

2. $\chi(s(x)) = 1$.

For any $x \in X$ denote by $\chi^x : \hat{W}_x^- \to [0,1]$ defined by $\chi^x := (\gamma^{-1})^* \chi \circ i^{-1}_s(x)$.

To construct $\alpha$ we start with a linear map $\beta : \mathcal{V} \to \mathcal{V}^*$ so that $\beta(E_s) \in \Omega(N(M))$, $E_s$ the characteristic function of $x \in X$, whose properties will be specified later and define $a^{\beta \cdot \chi} : \Gamma \to \mathbb{A}$ by the following formula

$$(a^{\beta \cdot \chi}(\gamma)(f))(x) := \int_{W_x^-} \chi^x \cdot e^{th_x} \cdot (i_{\gamma^{-1}})^* (\beta(f)), \ f \in V, \ x \in X$$

Note that $a^{\beta \cdot \chi}$ is in $N$ and we would like to be also in $L$. To check this fact we calculate its $L^1$ norm. Since the supports of $\chi^x$ are disjoint we have

$$\|a^{\beta \cdot \chi}\|_{L^1} = \sum_{\gamma \in \Gamma} \sup_{0 \neq f \in V} \max_{x \in X} \|\gamma \cdot (a^{\beta \cdot \chi}(\gamma)(f))(x)\|/\|f\| \leq$$

$$\sum_{\gamma \in \Gamma} \max_{x \in X} \left( \int_{W_x^-} \chi^x \cdot e^{th_x} \mu(g_x) \right) \|\beta\| \leq \sum_{\gamma \in \Gamma} \left( \int_{\supp(\chi^x)} e^{th_x} \mu(g_x) \right) \|\beta\| = C \|\beta\|$$

where $g_x$ denotes the pullback of the metric $g$ by $i_{\gamma^{-1}}^* \mu(g_x)$ the volume form with respect to $g_x$ and $C := \max_{x \in X} \int_{W_x^-} e^{th_x} \mu(g_x)$. Since $\rho(\xi, X) < \infty$, we have $a \in L$.

We can choose $\beta$ such that $a^{\beta \cdot \chi}(e) = 1$, where $e \in \Gamma$ denotes the neutral element. Moreover, given $\epsilon > 0$, we may assume $\beta$ so chosen that

$$\|a^{\beta \cdot \chi} - 1\|_{L^1} = \sum_{\gamma \in \Gamma} \sup_{0 \neq f \in V} \max_{x \in X} \|\gamma \cdot (a^{\beta \cdot \chi}(\gamma)(f))(x)\|/\|f\| \leq$$

$$\sum_{\gamma \in \Gamma} \max_{x \in X} \left( \int_{W_x^-} \chi^x \cdot e^{th_x} \mu(g_x) \right) \|\beta\| \leq \sum_{\gamma \in \Gamma} \max_{x \in X} \left( \int_{W_x^-} \chi^x \cdot e^{th_x} \mu(g_x) \right) \|\beta\| \leq$$

$$\sum_{x \in X} \left( \int_{K_x} e^{th_x} \mu(g_x) \right) \|\beta\| \leq \epsilon.$$

where $K_x$ is the support of $\chi^x$. (Hence $a^{\beta \cdot \chi}$ is invertible in the Banach algebra $L$ if $\beta$ is properly chosen.) To use Lemma 6 it remains to check that the convolution $I * a$ is in $L$. For this we observe that one can apply the Stokes theorem to the smooth form with compact support $d(\chi^x \cdot e^{th_x} ((i_{\gamma^{-1}})^* \beta(f)))$ on the manifold with corner $W_x^-$ and by Theorem 5 we have:

$$\int_{W_x^-} d(\chi^x \cdot e^{th_x} ((i_{\gamma^{-1}})^* \beta(f))) = \sum_{x \in X} \sum_{\sigma \in \Gamma} (\mathbb{I}_{x,y}(\sigma)a^{\beta \cdot \chi}(\sigma^{-1}(\sigma^{-1})(\gamma))(f)(x) = (I * a^{\beta \cdot \chi})(\gamma)(f)$$
On the other side we have
\[ \int_{W_x} d(\chi_x^l e^{th_x(\beta(f))}) = \int_{W_x} \chi_x^l e^{th_x(\beta(f))} + t \omega \wedge (\beta(f)) + \int_{W_x} d(\chi_x^l) e^{th_x(\beta(f))}. \]

Combining these two equalities one obtains
\[ \| I * a \beta \| \leq C(\| \delta(\beta) + t \omega \wedge \beta \| + \| d(\chi) \| \| \beta \|). \]

This finishes the proof of the first part of (iii). The second part follows from Stokes theorem. Some care is needed since this is applied to non compact manifolds with corners. See for example see [7] section 5, (cf. Proposition 4, page 28). \(\square\)

Observation 4 and Proposition 9 imply that

**Corollary 2.** Let \( X \) be a vector field with \( \xi \) as Lyapunov cohomology class , \( \rho(\xi, X) < \infty \), and suppose \( X \) satisfies MS . Let \( \omega \) be a closed one form representing \( \xi \) and \( t > \rho(\xi, X) \).

1. The finite dimensional vector spaces
   \[ C^q(X) := \text{Maps}(X_q, \mathbb{R}) \]
   and the linear maps
   \[ \delta^q_{X, \omega, \mathcal{O}}(t) : \text{Maps}(X_q, \mathbb{R}) \to \text{Maps}(X_{q+1}, \mathbb{R}) \]
   defined by
   \[ \delta^q_{X, \omega, \mathcal{O}}(t)(E_x) := \sum_{y \in X_{q+1}, \alpha \in \mathcal{P}_{\mathcal{O}, x}} I^{X, \mathcal{O}, \xi}_y(\alpha) e^{\Theta(\alpha)} E_y \]
   give rise to a cochain complex of finite dimensional vector spaces
   \[ C^*(X, \omega, \mathcal{O})(t) := \{ C^q(X), \delta^q_{X, \omega, \mathcal{O}}(t) \}, \]
   and the linear maps
   \[ \text{Int}^*_X, \omega, \mathcal{O}(t) : (\Omega^*(M), d_\omega(t)) \to C^*(X, \omega, \mathcal{O})(t) \]
   define a surjective morphism of cochain complexes.

2. Let \( \omega_1 \) and \( \omega_2 \) be two closed one forms representing the same cohomology class \( \xi \) and let \( h : M \to \mathbb{R} \) be a smooth function so that \( \omega_1 - \omega_2 = dh \). The collections of linear maps
   \[ m^q_h(t) : \Omega^q(M) \to \Omega^q(M), \quad m^q_h(t)(a) := e^{th} a, \]
   where \( a \in \Omega^q(M) \), and
   \[ s^q_h(t) : C^q(X, \omega_1, \mathcal{O}) \to C^q(X, \omega_2, \mathcal{O}), \quad s^q_h(t)(E_x) := e^{th(x)} E_x, \]
   where \( E_x \in \text{Maps}(X_q, \mathbb{R}) \) denotes the characteristic function of \( x \in X_q \), define morphisms of cochain complexes making the diagram

\[ \begin{array}{ccc}
\text{Int}^*_X, \omega_1(t) & \xrightarrow{m^q_h(t)} & \text{Int}^*_X, \omega_2(t) \\
\downarrow & & \downarrow \\
C^*(X, \omega_1)(t) & \xrightarrow{s^q_h(t)} & C^*(X, \omega_2)(t)
\end{array} \]

commutative.
4.1. Integration map for a bordism. In the case of a vector field $Y$ on bordism $(M, \partial_+, M, \partial_- M)$ Corollary 2 remains true as stated provided we replace $\Omega^*(M)$ by $\Omega^*(M, \partial_+ M)$, the space of differential forms which restrict to zero on $\partial_+ M$.

Since $Y$ is a vector field on a bordism there exists collar neighborhoods $\varphi_{\pm} : \partial_\pm M \times [0, \epsilon) \to M$ so that $\varphi^*(Y) = X_{\pm} + \pm s \partial_\pm$. Therefore $(X_+)_q = (\mathcal{Y}_+)_q$ and $(X_-)_q = (\mathcal{Y}_-)_q$.

Denote by $\omega_{\pm}$ the restriction of $\omega$ to $\partial_\pm M$. Let $u \in \Omega^1([0, \epsilon], \partial [0, \epsilon])$ with $\int_{[0, \epsilon]} u = 1$

We have the following commutative diagram

$$
\begin{array}{ccc}
\text{Int}_{X_+ \omega, \omega_+}^{-1}(t) & \xrightarrow{\Lambda u} & \text{Int}_{X_- \omega, \omega_-}^{-1}(t) \\
\downarrow & & \downarrow \\
\mathcal{C}^{* -1}(X_+, \mathcal{O}, \omega_+)(t) & \xrightarrow{i^*} & \mathcal{C}^{*}(Y, \mathcal{O}, \omega)(t) \\
\downarrow & & \downarrow \\
\mathcal{C}^{*}(X_-, \mathcal{O}, \omega_-)(t) & \xrightarrow{j^*} & \mathcal{C}^{*}(X_-, \mathcal{O}, \omega_-)(t)
\end{array}
$$

where $j^*$ is induced by the inclusion $\mathcal{X}_-$ in $\mathcal{Y}$ and $i^*$ is the linear map defined by $i^*(E_q) = E_y$ where $x \in (X_+)_q-1$ corresponds to $y \in \mathcal{Y}_q$ by the above identification.

Suppose $\mathcal{Y}_q$ the set of interior rest points of $Y$ is empty. Then $\mathcal{C}^{*}(Y, \mathcal{O}, \omega)(t) = C^0(Y, \mathcal{O}, \omega_+)(t) \oplus C^0(X_-, \mathcal{O}, \omega_-)(t)$ and with respect to this decomposition

$$
\delta_{\mathcal{Y}, \omega, \mathcal{O}}^q(t) = \begin{pmatrix}
\delta_{X_+ \omega_+, \mathcal{O}_+}^q(t) & -u_{X_+, \mathcal{O}_+}^q(t) \\
0 & \delta_{X_- \omega_-, \mathcal{O}_-}^q(t)
\end{pmatrix}
$$

The equivalence of Statement I and Statement II both in Case 1 and Case 2 is an algebraic tautology.

Case 1.

Statement I: $\delta_{\mathcal{Y}, \omega, \mathcal{O}}^{q+1}(t) \cdot \delta_{\mathcal{Y}, \omega, \mathcal{O}}^{-1}(t) = 0$

Statement II: $u_{\mathcal{Y}, \omega, \mathcal{O}}^q(t) : \mathcal{C}^{*}(X_-, \mathcal{O}, \omega_-)(t) \to \mathcal{C}^{*}(X_+, \mathcal{O}, \omega_+)(t)$ is a morphism of cochain complexes.

Case 2.

Statement I. $\text{Int}_{\mathcal{Y}, \omega, \mathcal{O}}^{-1}(t) : (\Omega^*(M, \partial_- M), d_\omega^*(t)) \to \mathcal{C}^{*}(Y, \mathcal{O}, \omega)(t)$ is a morphism of cochain complexes.

Statement II. $h^*(t) := j^* \cdot i^* : \Omega^{* + 1}(\partial_- M_+) \to \mathcal{C}^{*}(X_+, \mathcal{O}_+)(t)$ is an algebraic homotopy between $\text{Int}_{X_+ \omega_+, \mathcal{O}_+}(t)$ and $u_{\mathcal{Y}, \omega, \mathcal{O}}^q(t) \cdot \text{Int}_{X_- \omega_-, \mathcal{O}_-}(t)$.

5. The regularization $R(X, \omega, g)$

In this section we discuss the numerical invariant $R(X, \omega, g)$ associated to a vector field $X$, a closed one form $\omega$ and a Riemannian metric $g$. The invariant is defined by a possibly divergent integral but regularizable and is implicit in the work of [2]. More on this invariant is contained in [8].

In section 1.3 we have considered the angular momentum form $\Psi \in \Omega^{n-1}(TM \setminus M; \mathcal{O}_M)$ of an $n$-dimensional Riemannian manifold $(M, g)$. This form (see [14]) is actually associated to a pair $\nabla = (\nabla, \mu)$ consisting of a connection and a parallel Euclidean structure on a vector bundle $E \to M$. If $E$ is of rank $k$ it is a $k - 1$ form $\Psi_\varphi \in \Omega^{k-1}(E \setminus M; \mathcal{O}_E)$ with values in the pull back of the orientation bundle $\mathcal{O}_E$ of $E$ to the total space of $E$. Here $M$ is identified with the zero section in the bundle $E$. If $g$ is a Riemannian metric let $\nabla^\varphi := (\nabla^\varphi, g)$ denote the Levi–Civita pair associated to $g$ and write $\Psi_\varphi := \Psi_\nabla^\varphi$.

The angular momentum form form has the following properties:
(i) For the Euler form \( E_{\nabla} \in \Omega^k(M; \mathcal{O}_E) \) associated to \( \nabla \) we have \( d\Psi_{\nabla} = \pi^* E_{\nabla} \).

(ii) For two \( \nabla^1 \) and \( \nabla^2 \) we have \( \Psi_{\nabla^2} - \Psi_{\nabla^1} = \pi^* \text{cs}(\nabla^1, \nabla^2) \) modulo exact forms. Here \( \text{cs}(\nabla^1, \nabla^2) \in \Omega^{k-1}(M; \mathcal{O}_E)/d\Omega^{k-2}(M; \mathcal{O}_E) \) is the Chern–Simon invariant.

(iii) For every \( x \in M \) the form \(-\Psi_{\nabla}\) restricts to the standard generator of \( H^{k-1}(E_x \setminus \{0\}; \mathcal{O}_E) \), where \( E_x \) denotes the fiber over \( x \in M \). Note that the restriction of \(-\Psi_{\nabla}\) is closed by (i).

(iv) Suppose \( E = TM, \nabla^g \) is the Levi–Civita pair, and suppose that on the open set \( U \) we have coordinates \( x^1, \ldots, x^n \) in which the Riemannian metric \( g|_U \) is given by \( g_{ij} = \delta_{ij} \). Then, with respect to the induced coordinates \( x^1, \ldots, x^n, \xi^1, \ldots, \xi^n \) on \( TU \), the form \( \Psi_g \) is given by

\[
\Psi_g = \frac{1}{2\pi n/2} \sum_i (-1)^i \frac{\xi^i}{(\sum_j (\xi^j)^2)^{n/2}} \, d\xi^1 \wedge \cdots \wedge d\xi^i \wedge \cdots \wedge d\xi^n,
\]

cf. [14].

Let \( X \) be a vector field on \( M \), i.e. a section of the tangent bundle \( TM \). We suppose that it has only isolated zeros, that is its zero set \( \mathcal{X} \) is a discrete subset of \( M \). The vector field defines an integer valued map \( \text{IND} : \mathcal{X} \to \mathbb{Z} \), where \( \text{IND}(x) \) denotes the Hopf index of the vector field \( X \) at the zero \( x \in \mathcal{X} \). This integer \( \text{IND}(x) \) is the degree of the map \( (U, U \setminus x) \to (T_xM, T_xM \setminus 0) \) obtained by composing \( X : U \to TU \) with the projection \( p : TU \to T_xM \) induced by a local trivialization of the tangent bundle on a small disk \( U \subseteq M \) centered at \( x \).

Choose coordinates around \( x \) so that we can speak of the disk \( U_\epsilon \) with radius \( \epsilon > 0 \) centered \( x \). It is well known that we have:

\[
\text{IND}_x = -\lim_{\epsilon \to 0} \int_{\partial U_\epsilon} X^* \Psi_g
\]

Indeed, by (ii) we may assume that \( g \) is flat on \( U_\epsilon \). Thus \( E_g = 0 \) and \( \Psi_g \) is closed on \( U_\epsilon \) by (i). Using (iii) we see that \(-\Psi_g\) gives the standard generator of \( H^{n-1}(TU \setminus U_\epsilon; \mathcal{O}_{U_\epsilon}) \) and thus certainly \( \text{IND}(x) = -\int_{\partial U_\epsilon} X^* \Psi_g \).

The vector field \( X \) has its rest points (zeros) non-degenerate and in particular isolated, if the map \( X \) is transversal to the zero section in \( TM \). In this case \( \mathcal{X} \) is an oriented zero dimensional manifold, whose orientation is specified by \( \text{IND}(x) \). Moreover we have

\[
\text{IND}(x) = \text{sign} \det H \in \{ \pm 1 \},
\]

where \( H : T_xM \to T_xM \) denotes the Hessian. Particularly, if there exist coordinates \( x^1, \ldots, x^n \) centered at \( x \) so that

\[
X = -\sum_{1 \leq i \leq k} x^i \frac{\partial}{\partial x^i} + \sum_{i > k} x^i \frac{\partial}{\partial x^i}
\]

we get \( \text{IND}(x) = (-1)^k \).

Let \( X^1 \) and \( X^2 \) be two vector fields and \( \mathcal{X} := \{ X_s \}_{s \in [-1, 1]} \) a smooth homotopy from \( X^1 \) to \( X^2 \), i.e. \( X_s = X^1 \) for \( s \leq -1 + \epsilon \) and \( X_s = X^2 \) for \( s \geq 1 - \epsilon \). The homotopy is called non-degenerate if the map \( \mathcal{X} : [-1, 1] \times M \to TM \) defined by \( \mathcal{X}(s, x) := X_s(x) \) is transversal to the zero section of \( TM \). In this case necessarily \( X^1 \) and \( X^2 \) are vector fields with non-degenerate zeros and so are all but finitely many \( X_s \). Moreover all \( X_s \) have isolated zeros with indexes in \( \{ 0, 1, -1 \} \) and the
zero set \( \tilde{X} \) of \( X \) is an oriented one dimensional smooth submanifold of \([-1, 1] \times M \).

Note that we have
\[
\partial \tilde{X} = \sum_{y \in X^2} \text{IND}(y) y - \sum_{x \in X^1} \text{IND}(x) x.
\]

If \( X' \) is a second homotopy joining \( X^1 \) with \( X^2 \) then \( \tilde{X}' - \tilde{X} \) is the boundary of a smooth 2–cycle. Indeed, if we choose a homotopy of homotopies joining \( X \) with \( X' \) which is transversal to the zero section, then its zero set will do the job.

Given a closed one form \( \omega \) on \( M \) denote by
\[
I_{X, \omega} := \int_{\tilde{X}} p_2^* \omega,
\]
where \( p_2 : \tilde{X} \to M \) denotes the restriction of the projection \([-1, 1] \times M \to M \). It follows from the previous paragraph that \( I_{X, \omega} \) does not depend on the homotopy \( X \) — only on \( X^1, X^2 \) and \( \omega \), and therefor will be denoted from now on by \( I(X^1, X^2, \omega) \).

**Remark 6.** If there exists a simply connected open set \( V \subset M \) so that \( X_s \subset V \) for all \( s \in [-1, 1] \) then one can calculate \( I_{X, \omega} \) as follows: Choose a smooth function \( f : V \to \mathbb{R} \) so that \( \omega | V = df \). Then
\[
I_{X, \omega} = \sum_{y \in X^2} \text{IND}(y) f(y) - \sum_{x \in X^1} \text{IND}(x) f(x).
\]
The proof of this equality is a straight forward application of Stokes’ theorem.

With these considerations we will describe now the regularization referred to in Section 1.3, cf. (5). First note that for a non-vanishing vector field \( X \), a closed one form \( \omega \) and a Riemannian metric \( g \) the quantity
\[
R(X, \omega, g) := \int_M \omega \wedge X^* \Psi_g
\]
has the following two properties.

\[
R(X, \omega + df, g) - R(X, \omega, g) = -\int_M f E_g
\]
for every smooth function \( f \). If \( g^1 \) and \( g^2 \) are two Riemannian metrics then
\[
R(X, \omega, g^2) - R(X, \omega, g^1) = \int_M \omega \wedge \text{cs}(g^1, g^2)
\]
where \( \text{cs}(g^1, g^2) = \text{cs}(\nabla g^1, \nabla g^2) \). This follows from properties (i) and (ii) of the Mathai-Quillen form.

If \( X \) has zeros, then the form \( \omega \wedge X^* \Psi_g \) is well defined on \( M \setminus X \) but the integral \( \int_{M \setminus X} \omega \wedge X^* \Psi_g \) might be divergent unless \( \omega \) is zero on a neighborhood of \( X \).

We will define below a regularization of the integral \( \int_{M \setminus X} \omega \wedge X^* \Psi_g \) which in case \( X = \emptyset \) is equal to the integral (18). For this purpose we choose a smooth function \( f : M \to \mathbb{R} \) so that the closed 1–form \( \omega' := \omega - df \) vanishes on a neighborhood of \( X \), and put
\[
R(X, \omega; g; f) := \int_{M \setminus X} \omega' \wedge X^* \Psi_g - \int_M f E_g + \sum_{x \in X} \text{IND}(x) f(x)
\]
(19)

**Proposition 10.** The quantity \( R(X, \omega; g; f) \) is independent of \( f \).
Therefore $R(X,\omega,g;f)$ can be denoted by $R(X,\omega,g)$ and will be called the \textit{regularization} of $\int_{M\setminus X} \omega \wedge X^*\Psi_g$.

\textbf{Proof.} Suppose $f^1$ and $f^2$ are two functions such that $\omega^i := \omega - df^i$ vanishes in a neighborhood $U$ of $X$, $i = 1, 2$. For every $x \in X$ we choose a chart and let $D_\epsilon(x)$ denote the $\epsilon$-disk around $x$. Put $D_\epsilon := \bigcup_{x \in X} D_\epsilon(x)$. For $\epsilon$ sufficiently small $D_\epsilon \subseteq U$ and $f^2 - f^1$ is constant on each $D_\epsilon(x)$. From (19), Stokes’ theorem and (16) we conclude that

$$R(X,\omega;f^2) - R(X,\omega;f^1) - \sum_{x \in X} \text{IND}(x)\left(f^2(x) - f^1(x)\right) =$$

$$= - \int_{M\setminus X} \omega \wedge X^*\Psi_g$$

$$= - \lim_{\epsilon \to 0} \int_{M\setminus D_\epsilon} \omega \wedge X^*\Psi_g$$

$$= \sum_{x \in X} (f^2(x) - f^1(x)) \lim_{\epsilon \to 0} \int_{\partial D_\epsilon(x)} X^*\Psi_g$$

$$= - \sum_{x \in X} \text{IND}(x)\left(f^2(x) - f^1(x)\right)$$

and thus $R(X,\omega,g;f^1) = R(X,\omega,g;f^2)$. \qed

\textbf{Proposition 11.} Suppose that $X$ is a non-degenerate homotopy from the vector field $X^1$ to $X^2$ and $\omega$ is a closed one form. Then

$$R(X^2,\omega,g) - R(X^1,\omega,g) = I(X^1,X^2,\omega).$$

\textbf{Proof.} We may assume that there exists a simply connected $V \subseteq M$ with $X_s \subseteq V$ for all $s \in [-1,1]$. Indeed, since both sides of (20) do not depend on the homotopy $X$ we may first slightly change the homotopy and assume that no component of $X$ lies in a single $\{s\} \times M$. Then we find $-1 = t_0, \ldots, t_k = 1$ so that for every $0 \leq i < k$ we find a simply connected $V_i \subseteq M$ such that $X_s \subseteq V_i$ for all $s \in [t_i, t_{i+1}]$.

Assuming $V$ as above we choose a function $f$ so that $\omega' := \omega - df$ vanishes on a neighborhood of every $X_s$, i.e. $p_2^*\omega'$ vanishes on a neighborhood of $X$. Here $p_2 : [-1,1] \times M \to M$ denotes the canonical projection. Moreover let $\tilde{p}_2 : [-1,1] \times TM \to TM$ denote the canonical projection and note that $p_2^*\omega' \wedge X^*\tilde{p}_2^*\Psi_g$ is a globally defined form on $[-1,1] \times TM$. Using Stokes’ theorem and Remark 6 we then get:

$$R(X^2,\omega,g) - R(X^1,\omega,g) - I_{X,\omega} =$$

$$= \int_{[-1,1] \times M} d\left(p_2^*\omega' \wedge X^*\tilde{p}_2^*\Psi_g\right)$$

$$= \int_{[-1,1] \times M} p_2^*\omega' \wedge E_g$$

$$= 0$$

For the second equality we used $dX^*\tilde{p}_2^*\Psi_g = p_2^*E_g$. The integrand of the last integral vanishes because of dimensional reasons. \qed

With little effort, using Stokes’ theorem and the properties of the angular momentum form, one can prove

$$R(X,\omega + df,g) - R(X,\omega,g) = - \int_M f E_g + \sum_{x \in X} \text{IND}(x)f(x)$$
for every smooth function $f$, and
\[ R(X, \omega, g^2) - R(X, \omega, g^2) = \int_M \omega \wedge cs(g^1, g^2) \]
for any two Riemannian metrics $g^1$ and $g^2$. Its also not difficult to generalize the
regularization to vector fields with isolated singularities, cf. [8].

6. Proof of Theorems 2, 3 and WHS theory revisited

6.1. Novikov complex. Let $M$ be closed manifold $X$ smooth vector field which
satisfies MS and $\xi \in H^1(M; \mathbb{R})$ a cohomology class Lyapunov for $X$. Let $\mathcal{O}$ a
collection of orientations of the unstable manifolds and $\pi : \tilde{M} \to M$ the canonical
covering associated to $\Gamma \to \Gamma_\xi$ defined in introduction. Denote by $\Lambda_\xi$ the Novikov
field cf [7] and denote by $\Lambda_{\rho, \xi}$ the subring of elements representing Dirichlet series
with abscissa of convergence smaller than $\rho$, a positive real number. The Laplace
transform of these Dirichlet series can be evaluated at any $t > \rho$ providing a ring
homomorphism $ev : \Lambda_{\rho, \xi} \to \mathbb{C}$. Let $s : X \to \tilde{X}$, $\pi \cdot s = id$ be a lift of the rest points
of $X$ to $\tilde{M}$.

The Novikov complex $NC^*(X, \mathcal{O}, \xi)$ is a cochain complex of free $\Lambda_\xi$ modules
with a base indexed by $X$ and induced from the lift $s$, cf [7].

If $X$ has exponential growth then there exists a positive real number $\rho$ so that
this cochain complex is actually a complex of free $\Lambda_{\rho, \xi}$ modules with the same base
indexed by $X$ and induced from the lift $s$. Precisely this means that with respect to
the base, the boundary operator $\delta_{X, \mathcal{O}, \xi}$ is a matrix with entries in $\Lambda_{\rho, \xi} \subset \Lambda_\xi$ exactly
the counting functions $\|_{\rho, \xi}$ described in section 1. As a consequence, given closed
one form representing $\xi$, the cochain complex $NC^*(X, \mathcal{O}, s) \otimes_{ev} \mathbb{C}$ is isomorphic to
$C^*(X, \mathcal{O}, \omega)(t)$.

The above definitions routinely extend to bordisms. In particular if $X$ is homotopy
from $X_1$ to $X_2$ vector fields satisfying MS and having $\xi$ as Lyapunov cohomology class, $(\mathcal{O}_1, s_1)$ and $(\mathcal{O}_2, s_2)$ the additional data for $X_1$ and $X_2$ needed to
define Novikov complexes with base, by applying the above considerations to the
vector field $Y$ with the additional data $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ and $s = s_1 \cup s_2$ one obtain the
quasiisomorphism
\[ u := u_{\xi, \mathcal{O}_1, \mathcal{O}_2, \xi} : NC^*(X_1, \mathcal{O}_1, s_1) \to NC^*(X_2, \mathcal{O}_2, s_2) \]
as defined by the boundary operator
\[ \delta_{Y, \mathcal{O}, \xi}^q = \begin{pmatrix} \delta_{X_1, \mathcal{O}_1, \xi}^{q-1} & -u_{X_1, \mathcal{O}_1, \mathcal{O}_2, \xi}^q \\ 0 & \delta_{X_2, \mathcal{O}_2, \xi}^{q-1} \end{pmatrix} \]  
(21)
in the Novikov complex $NC^*(Y, \mathcal{O}, \xi)$.

We consider the Milnor torsion $\tau(u) \in \Lambda_\xi^\rho / \pm 1$ with $\Lambda_\xi^\rho$ the nonzero elements
of $\Lambda_\xi$, cf [15]. If the homotopy $X$ has exponential growth there exists $\rho > 0$, $\tau(u) \in \Lambda_{\rho, \xi}^\rho / \pm 1$ and therefore we have $\tau(u)(t) := ev_{\xi}(\tau(u))$. The absolute value of
this number is clearly torsion associated with the the mapping cone of $(\phi \otimes_{ev} \mathbb{C})$
eq  1 equipped with the scalar product which makes orthonormal the base provided by
$\Lambda_{\rho, \xi}$ and $\Lambda_\xi$. If we compare it with the torsion $T(u_{X_1, \mathcal{O}_1, \mathcal{O}_2, \omega}(t))$ of the mapping cone
of the quasiisomorphism $u_{\xi, \mathcal{O}_1, \mathcal{O}_2, \omega}(t) : C^*(X_1, \mathcal{O}_1, \omega)(t) \to C^*(X_1, \mathcal{O}_1, \omega)(t)$. we have on the nose
\[ \log |\tau(u)| + tI(X_1, X_2, \omega) = \log T(u_{X_1, \mathcal{O}_1, \mathcal{O}_2, \omega}(t)) \]
(22)
6.2. **Hutchings-Lee Pajitnov result.** Recall that for a base pointed manifold \((M, x_0)\) the set of Euler structures \(\mathcal{E}u(M, x_0)\) can be regarded as the set of connected components of the space \(\mathcal{V}(M, x_0)\) of vector fields which vanish only at \(x_0\) with the \(C^\infty\) topology. Given a vector field \(X\) and a lift \(\bar{s} : \mathcal{X} \to \mathcal{X}\) one can associated an Euler structure \(e(X, s)\) constructed as follows. Consider \(\bar{x}_0\) the canonical lifting of the basepoint \(x_0\) in the canonical covering \(\bar{M} \to M\). For any \(x \in \mathcal{X}\) choose a path \(\bar{\sigma}_x\) from \(\bar{x}_0\) to \(s(x)\) in \(\bar{M}\). Let \(\sigma_x\) be the path \(\pi \circ \bar{\sigma}_x\). Suppose the paths \(\sigma'_x\)'s are disjoint except for the base point \(x_0\). Use a regular neighborhood of the compact set \(\Sigma := \bigcup_{x \in \mathcal{X}} (\text{im}(\sigma_x))\) to modify \(X\) inside the regular neighborhood of \(\Sigma\) to have only one zero at \(x_0\). The vector field so obtained is always in the same connected component of \(\mathcal{V}(M, x_0)\) and gives rise to a well defined Euler structure denoted by \(e(X, s)\). One can reformulate Hutchings-Lee Pajitnov results as follows.

**Theorem 7.** Let \(\mathcal{X}\) be a MS homotopy from the vector field \(X_1\) to \(X_2\) which satisfy MS and have the same cohomology class \(\xi\) as Lyapunov class. Choose \((O_1, s_1)\) and \((O_2, s_2)\) to represent the same Euler structure and consider the quasiisomorphism \(u\) and its Milnor torsion \(\tau(u^*)\). Then

\[
\tau(u^*) = e^{Z_2 x_1 - Z_2 x_2}.
\]

Literary speaking such theorem is stated in [10] only for \(X_2\) a generalized triangulation, hence \(Z_2\) is 0, and only in the case the Novikov cohomology is trivial. However a word by word repetition of the arguments in [10] permits to extend the result to a bordism and when applied to the vector field \(Y\) on the bordism \((M \times [-1, 1], M \times +1, M \times -1)\), generated by a homotopy, Hutchings main result in [10] implies the result stated above. If the vector field \(Y\) has exponential growth which is the case if \(X = X_1\) has strong exponential growth (i.e. one has a homotopy \(\mathcal{X}\) with exponential growth property from \(X_1\) to \(X_2\), with \(X_2\) a generalized triangulation) the torsion \(\tau(u)\) lies in \(\Lambda^p_{\rho, \xi}/ \pm 1\) for some \(\rho\).

6.3. **The Bismut–Zhang theorem.** Suppose X is a smooth vector field which satisfies MS and is a generalized triangulation equivalently, \(0 \in H^1(M; \mathbb{R})\) is Lyapunov for \(X\) (in particular in view of Proposition 1 \(X = -\text{grad}' f\) for some Morse function \(f\) with respect to some Riemannian metric \(g\)'). Given a Riemannian metric \(g\) the elliptic complex \((\Omega^*(M), d^*_g(t))\) equipped with the scalar product induced from \(g\) is tautologically the same as the deRham complex associated to \((M, g)\) the trivial hermitian line bundle of rank one equipped with the flat connection defined by the closed one form \(\omega\). The Bismut–Zhang theorem applied to this flat bundle with this hermitian structure, generalized triangulation \(X\), and Riemannian metric \(g\) (cf [2] or [5]) reads in our notation:

\[
\log T_{\text{an}}(\omega)(t) = \log T_X(\omega)(t) + \log V_{X, g, \omega}(t) + t \Re(X, \omega, g)
\]

6.4. **The Witten–Helffer–Sjöstrand theory revisited.** Let \(M\) be a closed manifold and \((g, \omega)\) a pair consisting of a Riemannian metric \(g\) and a closed one form \(\omega\). We suppose that \(\omega\) is a Morse form i.e. locally \(\omega = dh\), \(h\) smooth function with all critical points non-degenerate. A critical point or a zero of \(\omega\) is a critical point of \(h\) and since non-degenerate, has an index, the index of the Hessian \(d^2_h\), denoted by \(\text{ind}(x)\). Denote by \(\mathcal{X}\) the set of critical points of \(\omega\) and by \(\mathcal{X}_q\) be the subset of critical points of index \(q\).
For $t \in \mathbb{R}$ consider the complex $(\Omega^q(M), d^q_\ast(t))$ with differential $d^q_\ast(t) : \Omega^q(M) \to \Omega^{q+1}(M)$ given by
\[
d^q_\ast(t)(\alpha) := da + t\omega \wedge \alpha.
\]
Using the Riemannian metric $g$ one constructs the formal adjoint of $d^q_\ast(t)$, $d^{q\ast}_\ast(t) : \Omega^{q+1}(M) \to \Omega^q(M)$, and one defines the Witten–Laplacian $\Delta^q_\ast(t) : \Omega^q(M) \to \Omega^q(M)$ associated to the closed 1–form $\omega$ and the metric $g$ by:
\[
\Delta^q_\ast(t) := d^q_\ast(t) \circ d^{q\ast}_\ast(t) \circ d^q_\ast(t)^\ast.
\]
Thus, $\Delta^q_\ast(t)$ is a second order differential operator, with $\Delta^q_\ast(0) = \Delta$, the Laplace–Beltrami operator. The operators $\Delta^q_\ast(t)$ are elliptic, selfadjoint and nonnegative, hence their spectra, $\text{Spect} \Delta^q_\ast(t)$, lie in the interval $[0, \infty)$. It is not hard to see that
\[
\Delta^q_\ast(t) = \Delta + t(L + L^2) + t^2||\omega||^2 \text{Id},
\]
where $L$ denotes the Lie derivative along the vector field $-\text{grad}_g \omega$, $L^2$ the formal adjoint of $L$ and $||\omega||^2$ is the fiber wise norm of $\omega$.

The following result extends a result due to E. Witten (cf. [23]) in the case that $\omega$ is exact and its proof was sketched in [7]. Part (iv) was established in BF96.

**Theorem 8.** Let $M$ be a closed manifold and $(g, \omega)$ be a pair as above. Then there exist constants $C_1, C_2, C_3, T > 0$ so that for $t > T$ we have:

(i) $\text{Spect} \Delta^q_\ast(t) \cap [C_1 e^{-C_2 t}, C_3 t] = \emptyset$.

(ii) $z(\text{Spect} \Delta^q_\ast(t) \cap [0, C_1 e^{-C_2 t}]) = z\mathcal{X}_q$.

(iii) $1 \in (C_1 e^{-C_2 t}, C_3 t)$.

(iv) For all but finitely many $t$ the dimension of $\ker \Delta^q_\ast(t)$ is constant in $t$.

Here $z\mathcal{X}$ denotes cardinality of the set $\mathcal{X}$.

Denote by $\Omega^q_{sm}(M)(t)$ the $\mathbb{R}$–linear span of the eigen forms which correspond to eigenvalues smaller than 1 referred bellow as the small eigenvalues. The collection of small smaller 1 for $t$ large enough eigenvalues can be indexed by $\mathcal{X}$ and since $\delta^q_\ast(t)$ is a real analytic family of elliptic operators one can extend it to a real analytic family of eigenvalues indexed by $\mathcal{X}$ and finally provide a real analytic family of cochain complexes of finite finite dimensional vector spaces $(\Omega^q_{sm}(M)(t), d_{\omega}(t))$. cf [?] Denote by $\Omega^q_{\infty}(M)(t)$ the orthogonal complement of $\Omega^q_{sm}(M)(t)$ which, by elliptic theory, is a closed subspace of $\Omega^q(M)$ with respect to $C^{\infty}$–topology, in fact with respect to any Sobolev topology. The space $\Omega^q_{\infty}(M)(t)$ is the closure of the span of the eigen forms which correspond to eigenvalues larger than one for $t$ large. As an immediate consequence of Theorem 8 we have for any $t$

\[
(\Omega^q(M), d_{\omega}(t)) = (\Omega^q_{sm}(M)(t), d_{\omega}(t)) \oplus (\Omega^q_{\infty}(M)(t), d_{\omega}(t))
\]

With respect to this decomposition the Witten–Laplacian is diagonalized
\[
\Delta^q_\ast(t) = \Delta^q_{\ast, sm}(t) \oplus \Delta^q_{\ast, \infty}(t).
\]

and by Theorem 8(ii), we have for $t > T$

\[
\dim \Omega^q_{sm}(M)(t) = z\mathcal{X}_q.
\]

Let $X$ be a smooth vector field which satisfies MS and EG and has $\omega$ as a Lyapunov form. Choose $\mathcal{O} = \{\mathcal{O}_x\}_{x \in X}$ a collection of orientations of the unstable manifolds with $\mathcal{O}_x$ orientation of $W^u_x$.

We know that there exists $T$ so that for $t > T$ the linear map:

\[
\text{Int}^q_{X, \mathcal{O}, \omega}(t) : \Omega^q(M) \to \text{Maps}(\mathcal{X}_q, \mathbb{R}).
\]
is well defined and we have the following

**Theorem 9.** There exists $T$ so that for $t > T$ Int\(^t\)\(_{\mathcal{X},\mathcal{O},r}(t)\) restricted to $\Omega(M)_{sm}(t)$ is an isomorphism and in fact an $O(1/t)$-isometry

For such $t$ consider the function
\[
\log V(t) := \sum (-1)^i \log Vol_q(t)
\]
where $Vol_q(t)$ denotes the volume of Int\(^t\)\(_{\mathcal{X},\mathcal{O},r}(t) : \Omega^q(M)_{sm}(t) \to Maps(X_q,\mathbb{R})\). When the first vector space is equipped with the scalar product provided by the Riemannian metric while the second with the scalar product which makes the base given by the characteristic functions orthonormal.

6.5. **Proof of Theorem 2, 3 and 4.** Proof of Theorem 2. Statements (i)-(iii) in Theorem 2 as stated are contained in Corollary 1 while statement (iv) follows from Theorem 9 above.

**Proof of Theorem 3.** We use the isomorphism Int\(_{\mathcal{X},\mathcal{O}}(t) : (\Omega^*(M)_{sm}, d_\omega(t)) \to \mathbb{C}^*(X,\omega,\mathcal{O})\) for $t$ large enough to provide the base $e_\omega(t) := (\text{Int}_{\mathcal{X},\mathcal{O}}(X,\omega,\mathcal{O})$ indexed by the rest points of $\mathcal{X}$ and observe that in view of the definition of Int\(_{\mathcal{X},\mathcal{O}}\) that with respect to this base the boundary operator $d_\omega(t)$ is given by a matrix whose entries are the Laplace transforms of the Dirichlet series associated to the counting functions $\|\mathcal{X}\|_{\omega}(1,\mathcal{O}_{x},\mathcal{O}_{y})$ and $\omega$ representing $\xi$, precisely,
\[
L(\mathcal{X}_{X,y}(\mathcal{O}_{x},\mathcal{O}_{y})(e^t) = (d_\omega(t))_{x,y}
\]
for $t > T$ stated in Theorem 2. This proves Theorem 3 (a). Part (b) follows from Theorem 7 when applied to the homotopy which defines strong exponential growth.

Indeed, as $u_{\omega,\mathcal{O}}$ is actually a morphism of cochain complexes of $\Lambda_{\rho,\xi}$ modules rather than $\Lambda_{\xi}$ modules. This makes the torsion $\tau(u^*)$ an element in $\Lambda_{\rho,\xi}/\{\pm 1\}$.

**Proof of Theorem 4.**

We begin with the homotopy commutative diagram
\[
\begin{array}{ccc}
\text{Int}_{\mathcal{X},\mathcal{O}}^*(M, d_\omega(t)) & \xrightarrow{\text{Id}} & \Omega^*(M, d_\omega(t)) \\
\text{Int}_{\mathcal{X},\mathcal{O}}^*(M, d_\omega(t)) & \xrightarrow{u^*_{\omega,\mathcal{O}}(\mathcal{O}_{x},\mathcal{O}_{y})} & \mathbb{C}^*(X^1,\mathcal{O}^1,\omega)(t)
\end{array}
\]
with all arrows inducing isomorphisms in cohomology. The homotopy being provided by $h^*$ defined in section ?? This implies
\[
\log \text{Vol} H_{X^1,\mathcal{O}^1,\omega}(t) = \log \text{Vol} H_{X^1,\mathcal{O}^1,\omega}(t) = \log \text{Vol} H(u^*_{\omega,\mathcal{O}}(\mathcal{O}_{x},\mathcal{O}_{y})(t)).
\]

**Theorem 4** follows combining the formulas (23), (22) (24) and (29).

Note that if $H^*(M; t\xi) = 0$ for $t$ large (or better say for $t$ generic) formula 29 is trivially satisfied and the formulas ?? and ?? together became
\[
\mathcal{Z}^\xi X_1 - \mathcal{Z}^\xi X_2 + t I(X_1, X_2, \omega) = \log T_{X_1}(\omega)(t) - \log T_{X_2}(\omega)(t)
\]
which remains true under the weaker hypothesis that $X_1$ and $X_2$ satisfy MS, NCT and EG provided $\xi$ is Lyapunov for $X$ and $\omega$ represents $\xi$. Combining again with (24) and (29) (for $X_2$ a generalized triangulation we get the same conclusion under the weaker assumption that $X$ satisfies EG only.)
Note also that the cochain complex \( (\Omega^*_{\text{la}}(M)(t), d_{\omega}(t)) \) is acyclic of finite codimension in the elliptic complex \( (\Omega^*(M), d_{\omega}(t)) \) therefore \( \log T_{\text{an,la}}(t) \) below is well defined. Consider the following functions:

\[
\log T_{\text{an}}(t) = \log T_{\text{an}}^{\omega,g}(t) := \frac{1}{2} \sum q (-1)^{q+1} q \log \det \Delta^q_{\omega}(t)
\]

\[
\log T_{\text{an,la}}(t) = \log T_{\text{an,la}}^{\omega,g}(t) := \frac{1}{2} \sum q (-1)^{q+1} q \log \det \Delta^q_{\omega,\text{la}}(t)
\]

(31)

where \( \det' \) denote the determinant calculated by ignoring the eigenvalue zero and in case of \( \Delta^q_{\omega}(t) \) and \( \Delta^q_{\omega,\text{la}}(t) \) the determinant is the zeta-regularized determinant à la Ray-Singer.

Theorem 3 and 4 actually state that for \( t \) large enough

\[
\log T_{\text{an},\text{sm}}(t) - \log V(t) = \log T_{X,\mathcal{O},\omega}(t)
\]

(32)

which implies that the “small” torsion is determined by the counting function of the instantons provided \( X \) is EG while the large torsion determines and is determined by the counting function of closed trajectories provided the vector field \( X \) is SEG.

The functions \( \log T_{\text{an}}(t), \log T_{\text{an},\text{sm}}(t), \log T_{\text{an,la}}(t) \log V(t) \) are not even continuous but for \( t \) large enough they are real analytic so it is natural to ask if they have analytic continuation to the entire \( \mathbb{R} \). Here is what can be derived from the previous consideration.

**Proposition 12.** 1. \( \log T_{\text{an}}(t), \log T_{\text{an},\text{sm}}(t), \log T_{\text{an,la}}(t) \) are quotients of entire real analytic functions.

2. Suppose \( X \) has exponential growth. The function \( \log T_{X,\mathcal{O},\omega}(t) \) has a meromorphic continuation to the full \( \mathbb{R} \) iff \( \log V(t) \) does

3. Suppose \( X \) has strong exponential growth. The function \( \log V(t) \) has a meromorphic continuation to the full \( \mathbb{R} \) iff \( L(Z^X_\xi)(t) \) does.

7. Appendix

**Appendix 1.** In this Appendix we will prove Proposition 1 and ??P:001. We begin with two lemmas.

Let \( M \) be a compact smooth manifold possibly with nonempty boundary, \( K \subset M \) a compact set and \( L \) the subset of \( M \times I, I = [0,1] \), defined by \( L := M \times \partial I \cup K \times I \).

**Lemma 7.** Suppose \( F \) is a real valued smooth function defined on an open neighborhood of \( L \) so that:

(i) \( \partial F/\partial t(x,t) < 0 \) for any \( x \in M \setminus K \) when defined,

(ii) \( F(x,0) < F(x,1) \) for any \( x \in M \).

Then there exists \( G : M \times I \to \mathbb{R} \) which agrees with \( F \) on a neighborhood of \( L \) and satisfies \( \partial G/\partial t(x,t) < 0 \).

The proof is obvious.
Lemma 8. Let $M$ be a smooth manifold, $X$ a vector field, $\omega$ a closed one form and $g_0$ a Riemannian metric. Suppose $K$ is a compact subset of $M$ and $U \subset M$ an open neighborhood of $K$. Suppose $X$ and $-\text{grad}_{g_0} \omega$ agree on $U$ and $\omega(X) < 0$ on a neighborhood of $M \setminus U$. Then there exists a Riemannian metric $g$ so that:

(i) $X = -\text{grad}_g \omega$

(ii) $g$ and $g_0$ agree on a neighborhood of $K$.

Proof. Let $N$ be an open neighborhood of $M \setminus U$ so that $\omega(X) < 0$ and therefore $X_x \neq 0, x \in N$. Suppose $N \cap K = \emptyset$. For $x \in N$ the tangent space $T_xM$ decomposes as the direct sum $T_xM = V_x \oplus [X_x]$, where $[X_x]$ denotes the one dimensional vector space generated by $X_x$ and $V_x = \ker(\omega(x) : T_xM \to \mathbb{R})$. Clearly on $U$ the function $-\omega(X)$ is the square of the length of $X_x$ with respect to the metric $g$ and $X_x$ is orthogonal to $V_x$ and on $N$ it is strictly negative. Consider a Riemannian metric $g_N$ on $N$ defined as follows: For $x \in N$ the scalar product on $T_xM$ agrees to the one defined by $g$ but make $V_x$ and $[X_x]$ perpendicular and the length of $X_x$ equal to $\sqrt{-\omega(X)(x)}$. Consider the Riemannian metric $g_U := g$. Use a partition of unity of the covering $\{U, N\}, \{\rho_U, \rho_N\}$ and then the Riemannian metric $g := \rho_U g_U + \rho_N g_N$. This satisfies the requirements. $\square$

For $\rho \geq \epsilon > 0$ denote by $\mathbb{D}_{\rho,\epsilon} := \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^{n-k} | -\rho \leq -|x|^2 + |y|^2 \leq \rho, |x| \cdot |y| \leq \epsilon\}$

Denote by $D^k$ the unit disc in $\mathbb{R}^k$ and by $C^k$ the corona $\{x \in \mathbb{R}^k | 1 \leq |x| \leq 2\}$.

Observe that for $\rho > \epsilon > 0$

(i) $\overline{\mathbb{D}_{\rho,\epsilon} \setminus \mathbb{D}_{\epsilon,\epsilon}}$ is a disjoint union of two cylinders $M \times I$ where for the first $M$ is diffeomorphic to $D^{n-k} \times S^{k-1}$ and for the second to $D^k \times S^{n-k-1}$.

(ii) $\overline{\mathbb{D}_{\rho} \setminus \mathbb{D}_{\rho,\epsilon}}$ is diffeomorphic with one cylinder $M \times I$ where $M$ is diffeomorphic to $C^k \times S^{n-k-1}$.

Proof of Proposition 1. 1: Start with $\omega$ and modify it in given neighborhood of the $X$ given by the disk of radius $2\rho$ in the charts the vector field has the form (1). Of course one should pick up $\rho$ so that $X$ has the form 1 in the disk of radius $4\rho$. Since the modification will be done only in the domain $\mathbb{D}_\rho$ we can suppose that the form $\omega$ is exact hence is the differential of a smooth function with the properties that it is strictly increasing on any non constant trajectory. So start with such function $F$ on a neighborhood of $\mathbb{D}_\rho$ and consider its restriction to a neighborhood of $\partial \mathbb{D}_{\rho,\rho}$.

Consider the function $f_0 = F(0) + 1/2 \sum_1^n t_i^2 + 1/2 \sum 1 \sum t_i^2$ on a neighborhood of some $\mathbb{D}_{\epsilon,\epsilon}$ with $\epsilon \leq \rho$ and so small that $|F - F(0)|_{\partial \mathbb{D}_\rho} > \epsilon$. We want to extend this function to the full $\mathbb{D}_{\rho,\rho}$ in a way that the extension is strictly decreasing on non constant trajectories. We do this in two steps; first we extend it to a neighborhood of $\mathbb{D}_{\rho,\epsilon}$ applying Lemma 7 for each cylinder with $K$ and then to $\mathbb{D}_{\rho,\rho}$ applying again Lemma 7 for the cylinder of $C^k \times S^{n-k-1}$ with $K = \partial C^k \times S^{n-k-1}$.

2: Follows immediately from Lemma 8

Proof of Proposition 2 Choose $f$ a smooth Morse function which is Lyapunov for $X$ which exists by (1.) Choose an arbitrary closed one form $\omega_0$ which vanishes in a neighborhood of $X$ and representing $\xi$. Consider the closed one form $\omega = \omega_0 + Cd(f)$ with $C$ very large. Clearly $\omega$ is a Lyapunov form for $X$. q.e.d.

References


Dan Burghelea, Dept. of Mathematics, The Ohio State University, 231 West Avenue, Columbus, OH 43210, USA.

E-mail address: burghele@mps.ohio-state.edu

Stefan Haller, Department of Mathematics, University of Vienna, Nordbergstrasse 15, A-1090, Vienna, Austria.

E-mail address: stefan.haller@univie.ac.at