Abstract. We consider a system \((M, g, \omega, X)\) where \((M, g)\) is a closed Riemannian manifold, \(\omega\) a closed one form whose all zeros are nondegenerate and \(X\) an \((-\omega, g)\) gradient like vector field. For \(t \in \mathbb{R}_+\) large enough, the Witten complex \((\Omega^*, d^*_t = d^* + t\omega \wedge)\) decomposes canonically as a direct sum of two one parameter family of cochain complexes \((\Omega_{sm}(t), d^*_t)\) and \((\Omega_{la}(t), d^*_t)\).

We show that under a mild hypotheses, generically satisfied, the inverse Laplace transform of \((\Omega_{sm}(t), d^*_t))\) determines completely the Novikov’s instantons counting (organized as a Dirichlet series), Theorem 3, and the inverse Laplace transform of the analytic torsion of \((\Omega_{la}(t), d^*_t))\) properly corrected, determines the counting function of closed trajectories of \(X\).

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1. Introduction

Let \((M, \omega, g)\) be a system consisting of a closed manifold \(M\), a Riemannian metric \(g\) and a closed one form \(\omega\). We suppose that \(\omega\) is a Morse form. This means that locally \(\omega = dh\), \(h\) smooth function with all critical points nondegenerate. A critical point \(x \in M\), or equivalently, a zero of \(\omega\) is a critical point of \(h\) and since nondegenerate, has an index, the index of the Hessian \(d^2h\), denoted by \(\text{ind}(x)\).

Denote by \(X\) be the set of critical points of \(\omega\) and by \(X_q\) be the subset of critical points of index \(q\).

For \(t \in [0, \infty)\) consider the complex \((\Omega^*(M), d^*_t(t))\) with differential \(d^*_t(t) : \Omega^q(M) \to \Omega^{q+1}(M)\) given by

\[
(1) \quad d^*_t(t)(α) := dα + tω \wedge α.
\]

Clearly \(d^*_0(0) = d^*\) is the usual exterior differential. Using the Riemannian metric \(g\) one constructs the formal adjoint of \(d^*_t(t)\), \((d^*_t(t))^\dagger : Ω^{q+1}(M) \to Ω^q(M)\), and defines the Witten Laplacian \(\Delta^*_t(t) : Ω^q(M) \to Ω^q(M)\) (associated to the closed
1-form \( \omega \) by:

\[
\Delta_\omega^q(t) := (d\omega^q(t))^2 \cdot d\omega^q + d\omega^{q-1}(t) \cdot (d\omega^{q-1}(t))^2.
\]

Thus, \( \Delta_\omega^q(t) \) is a second order differential operator, with \( \Delta_\omega^q(0) = \Delta^q \), the Riemannian metric \( \ast \) topology. The space \( \Omega^q \) where \( q \) (3) \( \Delta \)

the set \( W \) the stable/unstable set of \( X \)

consider rest points or zeros of \( X \).

Thus, \( \Delta^q \) lie in the interval \([0, \infty)\).

It is not hard to see that

\[
\Delta^q = \Delta^q + t(L + L^2) + t^2||\omega||^2 Id
\]

where \( L \) denotes the Lie derivative along the vector field \( -\text{grad}_g \omega \), \( L^2 \) the formal adjoint of \( L \) and \( ||\omega||^2 \) is the fiberwise norm of \( \omega \).

The following result extends a result due to E. Witten (cf [?]) in the case that \( \omega \) is exact and its proof was sketched in [?].

**Theorem 1.** Suppose \( \omega \) is a Morse form. There exists the constants \( C_1, C_2, C_3, T > 0 \) so that

(i) \( \text{Spect}(\Delta_\omega^q(t)) \cap [C_1 e^{-C_2 t}, C_3 t] = \emptyset \)

(ii) \( \sharp(\text{Spect}(\Delta_\omega^q(t)) \cap [0, C_1 e^{-C_2 t}]) = \sharp(X_q) \)

(iii) \( 1 \in (C_1 e^{-C_2 t}, C_3 t) \)

Denote by \( \Omega_{\text{sm}}^q(M)(t) \) the \( \mathbb{R} \)-linear span of the eigenforms which correspond to eigenvalues smaller than 1 and referred to bellow as the small eigenvalues. Denote by \( \Omega_{\text{sm}}^q(M)(t) \) the orthogonal complement of \( \Omega_{\text{sm}}^q(M)(t) \) which, by elliptic theory, is a closed subspace of \( \Omega^q(M) \) with respect to \( C^\infty \) (in fact, with any Sobolev) topology. The space \( \Omega_{\text{sm}}^q(M)(t) \) is the closure of the span of the eigenforms which correspond to eigenvalues larger than one.

The immediate consequence of Theorem 1 is that for \( t > T \) we have

\[
(\Omega^q(M), d\omega(t)) = (\Omega_{\text{sm}}^q(M)(t), d\omega(t)) \oplus (\Omega_{\text{sm}}^q(M)(t), d\omega(t))
\]

With respect to this decomposition the Witten Laplacian is diagonalized

\[
\Delta_\omega^q(t) = \Delta_{\omega,\text{sm}}^q(t) \oplus \Delta_{\omega,\text{la}}^q(t).
\]

Clearly by Theorem 1 (ii), we have

\[
\dim(\Omega^q(M)_{\text{sm}}(t)) = \sharp(X_q)
\]

for any \( t, t > T \).

Consider \( X \) an \((-\omega, g)\) gradient like vector field, i.e. \( X := -\text{grad}_g \omega \) in some neighborhood of \( \mathcal{X} \) and \( \omega(X)(x) < 0 \) for any \( x \in M \setminus \mathcal{X} \). Hence \( \mathcal{X} \) is also the set of rest points or zeros of \( X \). (Note that if \( X \) is \((-\omega, g)\) gradient like one can modify the Riemannian metric \( g \) into \( g' \) with \( g' \) equal to \( g \) in a small neighborhood of \( \mathcal{X} \) so that \( X = -\text{grad}_{g'} \omega \), cf Appendix.) Let \( \Phi_t \) be the flow of \( X \). Then for any \( x \in \mathcal{X}_q \) consider

\[
W_x^\pm := \{ y \in M | \lim_{t \to \pm \infty} \Phi_t(y) = x \}
\]

the stable/unstable set of \( X \) associated with the critical point \( x \). Observe that the set \( W_x^\pm \) has an unique structure of smooth manifold so that the inclusion
$i^+_x : W^+_x \to M$ is a smooth one to one immersion. The smooth manifold $W^+_x$ resp. $W^+_x$ is diffeomorphic to $\mathbb{R}^{n-q}$ resp. $\mathbb{R}^q$.

Denote by $g_x$ the pullback $g_x = (i^-_x)^*(g)$ of $g$ on $W^-_x$ and by $h_x : W^-_x \to (-\infty, a]$ the unique smooth function which satisfies $h_x(x) = 0$ and $dh_x = (i^-_x)^*(\omega)$.

Introduce $\rho(\omega, X, g) \in [0, \infty]$ defined by

$$\rho(\omega, X, g) := \inf\{t \in (0, \infty) \mid \int_{W^-_x} e^{th_x} d\text{vol}_{g_x} < \infty\}$$

and observe that it is independent on $g$ and independent on the form $\omega$, up to its cohomology class (see section 5). So $\rho$ defined by (??) provides an invariant $\rho(\omega, X) \in [0, \infty)$, cf Section 5.

Choose $\mathcal{O} = \{O_x, x \in X\}$ a collection of orientations of the unstable manifolds of the critical points, with $O_x$ orientation of $W^-_x$.

When $\rho(\omega, X) < \infty$ and for $t > \rho(\omega, g)$ denote by

$$\text{Int}^t_{X, \omega, g}(t) : \Omega^q(M) \to \text{Maps}(\mathcal{X}_q, \mathbb{R})$$

the linear map defined by

$$\text{Int}^t_{X, \omega, g}(t)(a) = \int_{W^-_x} e^{th_x}(i^-_x)^*(a), \ a \in \Omega^q(M)$$

**Definition 1. (Hypothesis MS)** The vector field $X$, satisfies Hypothesis MS (Morse Smale) if for any $x, y \in X$ the maps $i^-_x$ and $i^+_y$ are transversal.

It is well known that generically this is always the case; the proof is similar with the proof of Proposition 1 in [?] and left to the reader.

**Theorem 2.** Suppose $X$ is an $(-\omega, g)$ gradient like vector field which satisfies the Hypothesis MS and suppose $\mathcal{O}$ is a set of orientations as above.

There exists $T' > \rho(\omega, g)$ \footnote{and larger than $T$ provided by Theorem 1} so that for $t > T'$ and any $q$ the linear map $\text{Int}^t_{X}(t)$ defined in (??) restricted to $\Omega^q(M)_{\text{sm}}(t)$ is an isomorphism and an $O(1/t)$ isometry (w.r. to the scalar product induced by $g$ on $\Omega^* (M)$ and the "obvious scalar product" \footnote{the unique scalar product which makes the canonical base, provided by the characteristic functions of the critical points, orthonormal} on $\text{Maps}(\mathcal{X}_q, \mathbb{R})$). In particular $(\Omega^q_{\text{sm}}(M)(t), x \in X$ which depends on $\omega, g, \mathcal{O}$ and $X$.

As a consequence we have

$$dq^{-1}(E^\mathcal{O}_x(t)) := \sum_{x \in \mathcal{X}_q} I^\mathcal{O}(x, y)(t) E^\mathcal{O}_x(t)$$

where $I^\mathcal{O}(x, y) : [T, \infty) \to \mathbb{R}$ are smooth functions (actually analytic cf Theorem 3 below).

If $\rho(\omega, X) = \infty$ Theorem 2 is an empty statement, however we conjecture that this is never the case of Section 5.

**A few relevant smooth real valued functions:** In addition to the functions $I^\mathcal{O}(x, y)(t)$ defined for $t \geq T'$ cf (??), we consider also the function

$$\log \mathbb{V}(t) = \log \mathbb{V}(\omega, g, X)(t) := \sum_q (-1)^q \log \text{Vol}\{E_q(t) \mid x \in \mathcal{X}_q\}$$
Note that the change in the orientations $\mathcal{O}$ does not change the right side of (3), so $\mathcal{O}$ does not appear in the notation $V(t)$. The cochain complex $(\Omega^*(M)(t), d_\omega(t))$ is acyclic and by Theorem 1 of finite codimension in the elliptic complex $(\Omega^*(M), d_\omega)$. Therefore we can define the function

$$\log T_{an, la}(t) = \log T_{an, la}(\omega, g, t) := \frac{1}{2} \sum_q (-1)^{q+1} q \log \det \Delta^q_{\omega, la}(t)$$

where $\det \Delta^q_{\omega, la}(t)$ is the zeta-regularized product of all eigenvalues of $\Delta^q_{\omega, la}(t)$ larger than one.  

A geometric invariant associated with $(\omega, g)$: Recall that Quillen-Mathai (cf also [?]) have introduced a form $\Phi \in \Omega^{n-1}(TM \setminus M)$ for any $(M, g)$ an $n$-dimensional Riemannian manifold. We will consider the $n$-differential form $\omega \wedge X^*(\Phi) \in \Omega^n(M \setminus X)$; here $X$ is regarded as a map $X : M \setminus X \to TM \setminus M$.

If $\omega$ is a Morse form one consider

$$\int_{M \setminus X} \omega \wedge X^*(\Phi),$$

The above integral is in general divergent however it does have a regularization defined by the following formula:

$$R(X, \omega, g) = (-1)^{n+1} \int_M \omega_0 \wedge X^*(\Psi(g)) + \int_M f e_g - \sum (-1)^{\ind f} f(x)$$

where:

(i) $\omega = \omega_0 + df$ with $f$ is a smooth function whose differential $df$ is equal to $\omega$ in a small neighborhood of $X$.

(ii) $E_g \in \Omega^n(M)$ is the Euler form associated with $g$.

It will be shown below, in section 6 that the definition is independent of the choice of $f$. Finally we introduce the function

(14) $\log T_{an}(t) = \log T_{an}(X, \omega, g, t) := \log T_{an, la}(\omega, g, t) - \log V(t) + (-1)^{n+1} t R(X, \omega, g)$

The main results of this paper, Theorems 3 and 4 show that if $\rho(\omega, X) < \infty$ and under mild hypotheses (always satisfied generically) the smooth functions $T^C(x, y)(t)$ resp. $\log T_{an}(\omega, t)$ are restrictions of holomorphic functions on a half plane $\{z \in \mathbb{C} | \Re z > R\}$ where $R$ a positive real number (larger than $\rho(\omega, X)$). which have inverse Laplace transforms. Their inverse Laplace transforms are the counting functions for the instantons \(^4 \text{from } x \in X_q \text{ to } y \in X_{q-1} \text{, resp. for the closed trajectories of } X\).

To formulate precisely these results we need few additional definitions, namely of Dirichlet series, and counting functions for instantons and closed trajectories.

**Topology.**

\(^3\text{which by the ellipticity are all eigenvalues of } \Delta^q_{\omega}(t) \text{ but finitely many}\)

\(^4\text{a trajectory } \gamma(t) \text{ from } x \text{ to } y, x, y \in X \text{ is called instanton if it is isolated in the space of trajectories from } x \text{ to } y \text{. Under the Hypothesis MS the existence of an instanton from } x \text{ to } y \text{ implies that } \ind x = \ind y + 1\)
The closed one form $\omega$ induces the homomorphism $[\omega] : H_1(M, \mathbb{Z}) \to \mathbb{R}$ and then the injective group homomorphism

$$[\omega] : \Gamma := H_1(M, \mathbb{Z})/\ker[\omega] \to \mathbb{R}. $$

For any two points $x, y \in M$ denote by $\mathcal{P}_{x,y} := \{t \in [0, 1] \mapsto \gamma(t) \in \mathcal{X} \}$ the space of continuos paths from $x$ to $y$ and by $\varpi : \mathcal{P}_{x,y} \to \mathbb{R}$ the map defined by

$$\varpi(\alpha) := \int_{[0,1]} \alpha^*(\omega), \quad \alpha \in \mathcal{P}_{x,y}. $$

We says that $\alpha$ is equivalent to $\beta$, $\alpha, \beta \in \mathcal{P}_{x,y}$, iff the closed path $\beta^{-1} \ast \alpha$ represents an element in $\ker[\omega]$ or equivalently iff $\int_{[0,1]} \alpha^*(\omega) = \int_{[0,1]} \beta^*(\omega)$. We denote by $\hat{\mathcal{P}}_{x,y}$ the set of equivalence classes of elements in $\mathcal{P}_{x,y}$ and observe that $\varpi$ induces the injective map

$$\varpi : \hat{\mathcal{P}}_{x,y} \to \mathbb{R}. $$

One can see that $\Gamma$ acts freely and transitively (both from the left and from the right) on $\hat{\mathcal{P}}_{x,y}$. The action $\ast$ is defined by juxtaposing at $x$ resp. $y$ a closed curve representing an element $\gamma \in \Gamma$ to a path representing the element $\hat{\alpha} \in \hat{\mathcal{P}}_{x,y}$ and we have

$$\varpi(\gamma \ast \hat{\alpha}) = \varpi(\gamma) + \varpi(\hat{\alpha}),$$

$$\varpi(\hat{\alpha} \ast \gamma) = \varpi(\hat{\alpha}) + \varpi(\gamma).$$

Recall that a trajectory $\gamma$ from $x$ to $y$, $x, y \in \mathcal{X}$ is called instanton if it is isolated in the space of trajectories from $x$ to $y$.

**Proposition 1.** Suppose $\mathcal{X}$ satisfies the Hypothesis MS. If instantons from $x \in \mathcal{X}_q$ to $y \in \mathcal{X}_r$ exists then $q - r = 1$ and the set of instantons from $x$ to $y$ in each class $\hat{\alpha} \in \hat{\mathcal{P}}_{x,y}$ is finite.

Once a collection of orientations $\mathcal{O}$ is given any instanton $\gamma$ from $x \in \mathcal{X}_q$ to $y \in \mathcal{X}_{q-1}$ has a sign $\epsilon(\gamma) = \pm 1$ defined as follows:

The orientations $\mathcal{O}_x$ and $\mathcal{O}_y$ induce an orientation on $\gamma$. Take $\epsilon(\gamma) = +1$ if this orientation is compatible with the orientation " from $x$ to $y"$ and $\epsilon(\gamma) = -1$ otherwise. One defines the integer

$$\|[\gamma](x, y)\| = \sum_{\gamma \in \hat{\alpha}} \epsilon(\gamma).$$

**Definition 2.** A trajectory $\theta(t) := \Psi_t(x)$ is closed iff there exists $T \in \mathbb{R^+}$ so that $\theta(t + T) = \theta(t)$. (This means that $\theta(t) = \Psi_t(x)$ defines a smooth map $\theta : S^1 = \mathbb{R}/\mathbb{T} \to M$.) A closed trajectory is nondegenerate iff $D_x(\Psi_T) : T_xM \to T_xM$ is invertible with the eigenvalue 1 of multiplicity one.

**Definition 3.** (Hypothesis NCT) The vector field $\mathcal{X}$ satisfies the Hypothesis NCT if all closed trajectories of $\mathcal{X}$ are nondegenerate.

**Proposition 2.** If $\mathcal{X}$ satisfies both Hypotheses MS and NCT then for any $\gamma \in \Gamma = H_1(M : \mathbb{Z})/\ker[\omega]$, the set of closed trajectories representing the class $\gamma$, is finite.

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5Here $\ast$ denotes the juxtaposition of paths, precisely $\beta \ast \alpha : [0, 1] \to M, \alpha, \beta : [0, 1] \to M, \beta(0) = \alpha(1)$ is given by $\alpha(2t)$ for $0 \leq t \leq 1/2$ and $\beta(1 - 2t)$ for $1/2 \leq t \leq 1$.

6A smooth map $\alpha : (-\infty, \infty) \to M$ is a trajectory from $x$ to $y$ if of the form $\alpha(t) = \Phi_t(\gamma(0))$ with $\lim_{t \to -\infty} \gamma(t) = x/y$. 

Any nondegenerate closed trajectory $\theta$ has a period $p(\theta) \in \mathbb{N}$ and a sign $\epsilon(\theta) := \pm 1$ defined as follows:

i): $p(\theta)$ := the largest positive integer $p$ such that $\theta: S^1 \to M$ factors through a self map of $S^1$ of degree $p$.

ii): $\epsilon(\theta) := \text{sign } \det(D_x(\Psi_T))$.

We write

\[ Z_X(\gamma) := \sum_{\theta \in \gamma} (-1)^{\epsilon(\theta)} p(\theta) \in \mathbb{Q}. \]

**Dirichlet series and their Laplace transform:** A Dirichlet series $f$ is given by a pair of finite or infinite sequences

\[ \lambda_1 < \lambda_2 < \lambda_3 < \lambda_k < \lambda_{k+1} \cdots \quad \text{and} \quad a_1 a_2 a_3 a_k a_{k+1} \cdots \]

with the first sequence, a sequence of real numbers, the second sequence, a sequence of nonzero complex numbers, with the property that $\lambda_k \to \infty$ if the sequences are infinite. The associated series

\[ L(f)(z) = \sum_i e^{-z\lambda_i} a_i \]

has an Abscissa of Convergence $\rho(f) \leq \infty$, characterized by the following properties:

(i) If $\Re z > \rho(f)$ then $f(z)$ is convergent and defines a holomorphic function.

(ii) If $\Re z < \rho(f)$ then $f(z)$ is divergent. cf [?], [?].

A Dirichlet series can be regarded as a complex valued measure with support on the discrete set

\[ \{ \lambda_1 < \lambda_2 < \lambda_3 < \lambda_k < \lambda_{k+1} \cdots \} \]

where the "measure" of $\lambda_i$ equal to $a_i$, in which case the above series is the Laplace transform of this measure cf [?]. The following results are reformulation of basic results which were at the basis of the Novikov theory and the work of Hutchings-Lee Pazhitnov etc.

**Proposition 3.** (Novikov) If $X$ satisfies the Hypothesis MS then for any $x \in X_q$ and $y \in X_{q-1}$ the collection of pairs of numbers

\[ \Pi^0(x,y) = \left\{ [\omega](\hat{\alpha}), \Pi^0(x,y)(\hat{\alpha}) \mid [\omega](\hat{\alpha}) \neq 0, \hat{\alpha} \in \hat{\mathcal{P}}_{x,y} \right\} \]

defines a Dirichlet series whose first sequence of real numbers, the $\lambda$'s, consists of the numbers $[\omega](\hat{\alpha})$ when $\Pi^0(x,y)(\hat{\alpha})$ is nonzero and the second sequence, the $a$'s, consists of the numbers $\Pi^0(x,y)(\hat{\alpha}) \in \mathbb{Z}$.

2. (D.Fried, M.Hutchings) If $X$ satisfies Hypothesis NCT then the collection of pairs of numbers

\[ Z = \{ [\omega](\gamma), Z_X(\gamma) \mid \gamma \in \Gamma, [\omega](\gamma) \neq 0 \} \]

defines a Dirichlet series whose first sequence of real numbers, the $\lambda$'s, consists of $[\omega](\gamma)$ when $Z_X(\gamma)$ is nonzero and the second sequence, the $a$'s, consists of the numbers $Z_X(\gamma) \in \mathbb{Q}$.

The main results of this paper are Theorems 3 and 4 below.
Theorem 3. Suppose \( \rho([\omega], X) < \infty \). Then if \( X \) is an \((-\omega, g)\) gradient like vector field which satisfies the Hypothesis MS then there exists a positive real number \( R, R > \rho(\omega, g) \), so that for any \( x \in X_q, y \in X_{q-1} \) the function \( I_{x,y}^O(t) \) is the restriction of a holomorphic function on \( \{ z \in \mathbb{C} \mid |\mathbb{R}z > R \} \) which has inverse Laplace transform. The inverse Laplace transform is exactly the Dirichlet series \( I^O(x,y) \) defined above.

Remark 1. 1. The proof of the above theorem implies that when \( \rho([\omega], X) < \infty \) and \( t \) large enough, \( (t > R) \), the sum

\[
\sum_{\alpha \in P_{n,x}} I^O(y,x)(\hat{\alpha}) e^{i|\omega|(\hat{\alpha})}
\]

is convergent for any \( x \in X_q, y \in X_{q+1} \). Moreover, the vector spaces Maps\( (X_q, \mathbb{R}) \), together with the linear maps

\[
\delta_x^\omega : \text{Maps}(X_q, \mathbb{R}) \to \text{Maps}(X_{q+1}, \mathbb{R})
\]

defined by the linear extension of

\[
\delta_{x,\omega}^t(E_x) := \sum_{\alpha \in P_{n,x}} I^O(y,x), (\hat{\alpha}) e^{i|\omega|(\hat{\alpha})} E_y
\]

(\( E_x \) the characteristic function of the critical point \( x \)) defines a cochain complex of finite dimensional vector spaces denoted by \( \mathbb{C}^\ast(X, \omega, t) \). This complex will be equipped with the "obvious scalar product" which makes the base provided by the characteristic functions \( E_x \) orthonormal.

2. Observe that a cochain complex \( \mathbb{C}^\ast(X, \omega', t) \) can be defined for any closed one form \( \omega' \), and \( t \) large enough by taking \( \mathbb{C}^O := \text{Maps}(X_q, \mathbb{R}) \) and \( \delta_{x,\omega'}^t \) given by (23) (with \( \omega \) replaced by \( \omega' \)) provided the cohomology class \( [\omega'] \) contains a representative \([\omega]\) so that \( X \) is an \((-\omega, g)\) gradient like vector field which satisfies the Hypothesis MS and \( \rho([\omega], X) < \infty \).

3. If \( \omega' = \omega + df \), then the linear maps \( S_f^O(t) : \text{Maps}(X_q, \mathbb{R}) \to \text{Maps}(X_q, \mathbb{R}) \)

defined by

\[
S_f^O(t)(E_x) = e^{tf(x)} E_x
\]

define an isomorphism of cochain complexes

\[
S_f^O(t) : \mathbb{C}^\ast(X, \omega', t) \to \mathbb{C}^\ast(X, \omega, t)
\]

Theorem 4. Suppose \( \rho([\omega], X) < \infty \). If \( X \) is \((-\omega, g)\) gradient like and satisfies Hypotheses MS and NCT, then there exists a positive real number \( R, R > \rho(\omega, g) \) so that \( \log T_{an}(t) \) is the restriction of a holomorphic function on \( \{ z \in \mathbb{C} \mid |\mathbb{R}z > R \} \) which has inverse Laplace transform. The inverse Laplace transform is exactly the Dirichlet series \( Z \) defined above.

Remark 2. : 1. Both collections of numbers \( I^O(x,y) \) and \( Z \) can be defined in much weaker conditions than the Hypothesis MS and NCT, however without these hypotheses we are unable so far to check that they are Dirichlet series.

2. The Dirichlet series \( I^O(x,y) \) and \( Z \) are quite special; the numbers \( a's \) for the first are integers and for the second rational numbers. It will be interesting to understand the qualitative implications these facts have on the functions \( I^O(x,y)(t) \) and \( Z(t) \).
Corollary 1. (J. Marsick cf [?] ) Suppose $(M, \omega, g, X)$ is a system consisting of a closed manifold $M$, a closed one form $\omega$ with no zeros, a Riemannian metric $g$ and vector field $X$ satisfying $X(\omega) < 0$.

Suppose all closed trajectories of $X$ are nondegenerate and denote by

$$\log T_n(t) := 1/2 \sum (-1)^{q+1} q \log \text{Det}(\Delta_q(t)).$$

Then

$$\log T_n(t) + (-1)^{n+1} t \int_M \omega \wedge X^*(\Psi(g))$$

is the Laplace transform of the Dirichlet series $Z_X$ which counts the set of closed trajectories of $X$.

Remark 3. 1. J. Marsick proved the above result in the case $X = -\text{grad}_g \omega$. The same arguments yield the result as stated above.

2. In case that $M$ is the mapping torus of a diffeomorphism $\phi : N \to N, M = N_\phi$, whose all periodic points are nondegenerate, the Laplace transform of the Dirichlet series $Z_X$ is the Lefschetz zeta function of $\phi$, $\text{Lef}(Z)$, with the variable $Z$ replaced by $e^{-t}$.

The proof of Theorem 1 as stated is contained in [?] and so is the proof of Theorem 2 but in a slightly different formulation and apparent less generality. For this reason as well as for the sake of completeness we will review the arguments the proof, making the necessary references to [?], in section 3. The proof of Theorem 2 is based on a fundamental topological result, Theorem 5 below, which is also proven in [?]. However a significant short cut in the proof and a slightly more general formulation makes appropriate to have it reconsidered. This will be done in section 2. Section 5 contains a discussion of the invariant $\rho(\omega, X)$ and Section 6 of the regularization $R(X, \omega, g)$.

2. Topology of the space of trajectories and stable/unstable sets

Let $X$ be an $(-\omega, g)$ gradient like vector field. Let $\pi : \tilde{M} \to M$ be a regular covering with deck transformations group $\Gamma$ (not necessary equal to $\Gamma = H_1(M, \mathbb{Z})/\ker(\omega)$) but satisfying the following property (P).

(P): $\tilde{M}$ is connected and $\tilde{\omega} = \pi^*\omega$ is exact.

Denote by $\tilde{X}$ the vector field $\tilde{X} := \pi^*(X)$, and by $h$ a smooth function so that $dh = \tilde{\omega}$. The function $h$ is proper. We write $\tilde{X} = h^{-1}(X)$ and $\tilde{X}_q = h^{-1}(X_q)$. Clearly $Cr(h) = \pi^{-1}(Cr(\omega))$ are the zeros of $\tilde{X}$.

Given $\tilde{x} \in \tilde{X}$ let $i^+_{\tilde{x}} : W^+_{\tilde{x}} \to \tilde{M}$ and $i^-_{\tilde{x}} : W^-_{\tilde{x}} \to \tilde{M}$, be the one to one immersions whose image defines the stable and the unstable sets of $\tilde{x}$ with respect to the vector field $\tilde{X}$. The maps $i^\pm_{\tilde{x}}$ are actually smooth embeddings because $\tilde{X}$ is gradient like for the proper function $h$, and the manifold topology on $W^\pm_{\tilde{x}}$ coincide with the topology induced from $\tilde{M}$. Clearly for any $\tilde{x}$ with $\pi(\tilde{x}) = x$ one has $\pi \circ i^\pm_{\tilde{x}} = i^\pm_x$ and the maps $\pi : W^\pm_{\tilde{x}} \to W^\pm_x$ are diffeomorphisms.

We will suppose that $\tilde{X}$ satisfies Hypothesis MS, i.e. for any $x, y \in X$ the maps $i^\pm_{\tilde{x}}$ and $i^\pm_y$ are transversal, or equivalently, for any $\tilde{x}, \tilde{y} \in \tilde{X}$ the maps $i^\pm_{\tilde{x}}$ and $i^\pm_{\tilde{y}}$ are transversal. This implies that $\mathcal{M}(\tilde{x}, \tilde{y}) := W^-_{\tilde{x}} \cap W^+_{\tilde{y}}$ is a submanifold of $\tilde{M}$ of dimension $\text{ind}(x) - \text{ind}(y)$. The manifold $\mathcal{M}(\tilde{x}, \tilde{y})$ is equipped with the action $\mu : \mathbb{R} \times \mathcal{M}(\tilde{x}, \tilde{y}) \to \mathcal{M}(\tilde{x}, \tilde{y})$, defined by the flow generated by $\tilde{X}$. If $\tilde{x} \neq \tilde{y}$ the action
There exists a finite collection of real numbers $a_i$ such that

$$h^{-1}(c) \cap M(x, y)$$

is a smooth manifold of dimension $\text{ind}(x) - \text{ind}(y) - 1$, possibly empty, which, in view of the fact that $X(x) = X(y) < 0$ is diffeomorphic to the submanifold $h^{-1}(c) \cap M(x, y)$, where $c$ is any regular value of $h$ with $h(x) > c > h(y)$.

Note that if $\text{ind}(x) \leq \text{ind}(y)$, and $x \neq y$, in view the transversality required by the Hypothesis MS, the manifold $M(x, y)$ is empty.

An unparametrized broken trajectory from $x$ to $y$ $\tilde{x}, \tilde{y} \in \tilde{X}$, is an element of the set

$$B(\tilde{x}, \tilde{y}) := \bigcup_{k \geq 0, \tilde{y}_0, \ldots, \tilde{y}_{k+1} \in \tilde{X}} \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_k, \tilde{y}_{k+1}).$$

For any $\tilde{x} \in \tilde{X}$ introduce the completed unstable set $\tilde{W}^-_{\tilde{x}}$ defined by

$$\tilde{W}^-_{\tilde{x}} := \bigcup_{k \geq 0, \tilde{y}_0, \ldots, \tilde{y}_k \in \tilde{X}} \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_{k-1}, \tilde{y}_k) \times W^-_{\tilde{y}_k}.$$

To study $\tilde{W}^-_{\tilde{x}}$ we introduce the set $B(\tilde{x}; \lambda)$ of unparametrized broken trajectory from $\tilde{x} \in \tilde{X}$ to the level $\lambda \in \mathbb{R}$.

$$B(\tilde{x}; \lambda) := \bigcup_{k \geq 0, \tilde{y}_0, \ldots, \tilde{y}_k \in \tilde{X}} \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_{k-1}, \tilde{y}_k) \times (W^-_{\tilde{y}_k} \cap h^{-1}(\lambda)).$$

Clearly, if $\lambda > h(\tilde{x})$ then $B(\tilde{x}; \lambda) = \emptyset$.

One can view the set $B(\tilde{x}, \tilde{y})$ resp. $B(\tilde{x}; \lambda)$ as a subset of $C^0([h(\tilde{y}), h(\tilde{x})], \tilde{M})$ resp. $C^0([\lambda, h(\tilde{x})], \tilde{M})$, by parametrizing a broken trajectory by the value of $h$. This leads to the following characterization (and implicitly to a canonical parametrization) of an unparametrized broken trajectories.

**Remark 4.** Let $\tilde{x}, \tilde{y} \in \tilde{X}$ and set $a := h(\tilde{y})$, $b := h(\tilde{x})$. The parameterization above defines a one to one correspondence between $B(\tilde{x}, \tilde{y})$ and the set of continuous mappings $\gamma : [a, b] \rightarrow \tilde{M}$, which satisfy the following two properties:

1. $h(\gamma(s)) = a + b - s$, $\gamma(a) = \tilde{x}$ and $\gamma(b) = \tilde{y}$.

2. There exists a finite collection of real numbers $a = s_0 < s_1 < \cdots < s_{r-1} < s_r = b$, so that $\gamma(s) \in \tilde{X}$ and $\gamma$ restricted to $(s_i, s_{i+1})$ has derivative at any point in the interval $(s_i, s_{i+1})$, and the derivative satisfies

$$\frac{d\gamma}{ds}(s) = \frac{X}{\|X\|^2}(\gamma(s)).$$
Similarly the elements of \(B(\tilde{x}; \lambda)\) correspond to continuous mappings \(\gamma : [\lambda, b] \to \tilde{M}\), which satisfies (??) and (??) with "a" replaced by "\(\lambda\".  

We have the following result  

Proposition 4. For any \(\tilde{x}, \tilde{y} \in \tilde{X}\) and \(\lambda \in \mathbb{R}\), the spaces \(B(\tilde{x}, \tilde{y})\) with the topology induced from \(C^0([h(\tilde{y}), h(\tilde{x})])\), and \(B(\tilde{x}; \lambda)\) with the topology induced from \(C^0([\lambda, h(\tilde{x})], \tilde{M})\) are compact.  

Let \(\dot{i}_x : \tilde{W}_x^- \to \tilde{M}\) denote the map whose restriction to \(T(\tilde{y}_0, \tilde{y}_1) \times \cdots \times T(\tilde{y}_{k-1}, \tilde{y}_k)\) is the composition of the projection on \(W_{\tilde{y}_k}^-\) with \(\dot{i}_{\tilde{y}_k}\), and let \(\dot{h}_x := h^x \circ \dot{i}_x : \tilde{W}_x^- \to \mathbb{R}\), where \(h^x = h - h(\tilde{x})\).  

Recall that an \(n\)-dimensional manifold with corners \(P\), is a paracompact Hausdorff space equipped with a maximal smooth atlas with charts \(\varphi : U \to \varphi(U) \subseteq \mathbb{R}^k_+\), where \(\mathbb{R}^k_+ = \{(x_1, x_2, \ldots, x_n) \mid x_i \geq 0\}\). The collection of points of \(P\) which correspond (by some chart and then by any other chart) to points in \(\mathbb{R}^n\) with exactly \(k\) coordinates equal to zero is a well defined subset of \(P\) and it will be denoted by \(P_k\). It has a structure of a smooth \((n-k)\)-dimensional manifold. \(\partial P = P_1 \cup P_2 \cup \cdots \cup P_n\) is a closed subset which is a topological manifold and \((P, \partial P)\) is a topological manifold with boundary \(\partial P\). The following theorem was proven in [2]. In this paper we will shorten the proof.  

Theorem 5. Let \(X\) be an \((-\omega, g)\) a gradient like vector field which satisfies the Hypothesis MS and let \(\pi : \tilde{M} \to M\) be a principal \(\Gamma\)-covering\(^9\) so that \(\pi^*(\omega) = dh\).  

1. For any two critical points \(\tilde{x}, \tilde{y} \in \tilde{X}\) the smooth manifold \(T(\tilde{x}, \tilde{y})\) has \(B(\tilde{x}, \tilde{y})\) as a canonical compactification. Moreover \(B(\tilde{x}, \tilde{y})\) has the structure of a compact smooth manifold with corners, and  

\[
B(\tilde{x}, \tilde{y})_k = \bigcup_{\tilde{y}_0, \ldots, \tilde{y}_{k+1} \in \tilde{X}} T(\tilde{y}_0, \tilde{y}_1) \times \cdots \times T(\tilde{y}_{k}, \tilde{y}_{k+1}),
\]

in particular \(B(\tilde{x}, \tilde{y})_0 = T(\tilde{x}, \tilde{y})\).  

2. For any critical point \(\tilde{x} \in \tilde{X}\), \(\tilde{W}_\tilde{x}^-\) has a canonical structure of a smooth manifold with corners, and  

\[
(\tilde{W}_\tilde{x}^-)_k = \bigcup_{\tilde{y}_0, \ldots, \tilde{y}_k \in \tilde{X}} T(\tilde{y}_0, \tilde{y}_1) \times \cdots \times T(\tilde{y}_{k-1}, \tilde{y}_k) \times W_{\tilde{y}_k}^-,
\]

in particular \((\tilde{W}_\tilde{x}^-)_0 = W_{\tilde{x}}^-\). Moreover \(\dot{i}_x\) is a smooth proper map whose image is a closed set in \(\tilde{M}\).  

3. The maps \(\dot{h}_x\) are smooth and proper.  

\(^7\)the condition \(\gamma(a) = \tilde{x}\) has to be ignored  

\(^8\)In [2] this result was proven for \(X = -\text{grad}_g \omega\). As already noticed and proven cf Appendix, if \(X\) is \((-\omega, g)\) gradient like one can change \(g\) in \(g'\) so that \(X = -\text{grad}_{g'} \omega\).  

\(^9\)the group \(\Gamma\) is always supposed to be finitely generated
Proof. First we verify Theorem 5 in the case that \( \omega \) has degree of rationality one in which case all critical values of \( h \) form a discrete subset of \( \mathbb{R} \). This is done in all details in [?] section 4 paragraphs 4.1-4.3.

Next, we observe that as long as part 1) and 2) of Theorem 5 are concerned the result depends on \( X \) and not on the form \( \omega \). We also observe that if Theorem 5, 1) and 2) are true for one covering \( \pi : \tilde{M} \to M \) which satisfies (P) then they are true for the universal covering (which obviously satisfies (P) ) which in turn implies they are true other covering which satisfies (P).

It is therefore sufficient to show that if \( X \) is an \((-\omega, g)\) gradient like, then is also \((-\omega', g)\) gradient like for some \( \omega' \), closed one form of degree of rationality one. Indeed, choose \( U \) a neighborhood of \( X \) so that \( \omega|_U = -\text{grad}_g \omega|_U \) and \( N \) a compact neighborhood of \( M \setminus U \). As \( X \) is \((-\omega, g)\) gradient like, it remains \((-\omega', g)\) gradient like for any \( \omega' \) with \( \omega' \) equal to \( \omega \) on \( U \) and \( C^0 \)-closed enough from \( \omega \) on \( N \). In view of the fact that closed one forms of degree of rationality one are dense in the space of all closed one form with respect any topology \( C^r \), \( r \geq 0 \), one can choose \( \omega' \) of degree of rationality one so that \( X \) is an \((-\omega', g)\) gradient like.

Note that once 1) and 2) are verified, the map \( \hat{h} \) is clearly smooth, (being equal to \( \hat{h} \cdot \hat{i}_x \)) and the inverse image of each closed interval \([a, b] \subset \mathbb{R} \) is compact hence 3) is true as well.

References


Dept. of Mathematics, O.S.U, 231 West Avenue, Columbus, OH 43210, USA and Institute of Mathematics, Univ.of Vienna, Srudlehofgasse 4, A-1090, Vienna, Austria