TORSION, AS A FUNCTION ON THE SPACE OF REPRESENTATIONS

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ABSTRACT. For a closed manifold M we introduce the set of co-Euler structures as Poincaré dual of Euler structures. The Riemannian Geometry, Topology and Dynamics on manifolds permit to introduce partially defined holomorphic functions on $\mathrm{Rep}^M(\Gamma; V)$ called in this paper complex valued Ray–Singer torsion, Milnor–Turaev torsion, and dynamical torsion. They are associated essentially to M plus an Euler (or co-Euler structure) and a Riemannian metric plus additional geometric data in the first case, a smooth triangulation in the second case and a smooth flow of type described in section 2 in the third case. In this paper we define these functions, show they are essentially equal and have analytic continuation to rational functions on $\mathrm{Rep}^M(\Gamma; V)$, describe some of their properties and calculate them in some case. We also recognize familiar rational functions in topology (Lefschetz zeta function of a diffeomorphism, dynamical zeta function of closed trajectories, Alexander polynomial of a knot) as particular cases of our torsions. A numerical invariant derived from Ray–Singer torsion and associated to two homotopic acyclic representations is discussed in the last section.

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1. Introduction

For a finitely presented group Γ denote by $\mathrm{Rep}(\Gamma;V)$ the algebraic set of all complex representations of Γ on the complex vector space V. For a closed base pointed manifold (M,x_0) with $\Gamma=\pi_1(M,x_0)$ denote by $\mathrm{Rep}^M(\Gamma;V)$ the algebraic closure of $\mathrm{Rep}^M_0(\Gamma;V)$, the Zariski open set of representation $\rho\in\mathrm{Rep}(\Gamma;V)$ so that $H^*(M;\rho)=0$. The manifold M is called V-acyclic iff $\mathrm{Rep}^M(\Gamma;V)$, or equivalently $\mathrm{Rep}^M_0(\Gamma;V)$, is non-empty. If M is V-acyclic then the Euler–Poincaré characteristic $\chi(M)$ vanishes. There are plenty of V-acyclic manifolds.

In this paper to a V-acyclic manifold and an Euler or co-Euler structure:

We associate three partially defined holomorphic functions on $\mathrm{Rep}^M(\Gamma;V)$, the complex valued Ray–Singer torsion, the Milnor–Turaev torsion, and the dynamical torsion, and describe some of their properties.

We show they are essentially equal and have analytic continuation to rational functions on $\mathrm{Rep}^M(\Gamma;V)$.

We calculate them in some cases and recognize familiar rational functions in topology (Lefschetz zeta function of a diffeomorphism, dynamical zeta function of some flows, Alexander polynomial of a knot) as particular cases of our torsions, cf. section 7.

The complex Ray–Singer torsion is associated to a Riemannian metric, a complex vector bundle E equipped with a non-degenerate symmetric bilinear form and to a flat connection on E. The square of the standard Ray–Singer torsion, slightly modified, is the absolute value of the complex Ray–Singer torsion.

The Milnor–Turaev torsion is associated to a smooth triangulation, and the dynamical torsion to a vector field with the properties listed in section 2.4. There are plenty of such vector fields. The Milnor–Turaev torsion is a particular case of dynamical torsion but the only of these three functions which, on the nose, is a rational function on the full space $\mathrm{Rep}^M(\Gamma;V)$; this is why it receives independent attention.

The results answer the question

(Q) *Is the Ray–Singer torsion the absolute value of an univalent holomorphic function on the space of representations?*

(for a related result consult [BK05]) and establish the analytic continuation of the dynamical torsion. Both issues are very subtle when formulated inside the field of spectral geometry or dynamical systems.

In section 2, for the reader's convenience, we recall some less familiar characteristic forms used in this paper and define the class of vector fields we use to define the dynamical torsion. These vector fields have finitely many rest points but infinitely many instantons and closed trajectories. However, despite their infiniteness, they can be counted by appropriate counting functions related to topology and analysis of the underlying manifold cf. [BH04a]. The dynamical torsion is derived from these counting functions.

In section 3 we define Euler and co-Euler structures and discuss some of their properties. Although they can be defined for arbitrary base pointed manifolds (M,x_0) we present the theory only in the case $\chi(M)=0$ when the base point is irrelevant.

In section 4 we present a few facts about the algebraic geometry of the variety of cochain complexes of finite dimensional vector spaces and introduce the concept of *holomorphic closed one m-form* as opposed to *holomorphic closed one form* to best formulate the answer to the question (Q).

The first function we discuss, cf. section 5.1, is the Ray–Singer torsion slightly modified with the help of a co-Euler structure. This is a positive real valued function defined on $\mathrm{Rep}_0^M(\Gamma; V)$, the variety of the acyclic representations. We show that this function is independent of the Riemannian metric and locally is the absolute value of holomorphic functions, more precisely a *holomorphic closed one m-form*.

Next we introduce the complex Ray–Singer torsion, cf. section 5.2, and show the relation to the first; its absolute value is the square of the modified Ray–Singer torsion. The complex Ray–Singer torsion is a meromorphic function on the space of representations and is defined analytically using regularized determinants of elliptic operators but not self adjoint in general.

The Milnor–Turaev torsion, defined in section 6.1, is associated with a smooth manifold, a given Euler structure and a homology orientation and is constructed using a smooth triangulation. Its square is equal to the *square of the complex Ray–Singer torsion* as defined in section 5.2, when the co-Euler structure for Ray–Singer corresponds, by Poincaré duality map, to the Euler structure for Milnor–Turaev.

Up to sign the dynamical torsion, introduced in section 6.2, is associated to a smooth manifold and a given Euler structure and is constructed using a smooth vector field in the class described in section 2.4. The sign is fixed with the help of an equivalence class of orderings of the rest points of X, cf. section 6.2. A priori the dynamical torsion is only a partially defined holomorphic function on $\mathrm{Rep}^M(\Gamma; V)$ and is defined using the instantons and the closed trajectories of X. For a representation ρ the dynamical torsion is expressed as a series which might not be convergent for each ρ but is certainly convergent for ρ in a subset U of $\mathrm{Rep}^M(\Gamma; V)$ with non-empty interior. The existence of U is guaranteed by the exponential growth property (EG) (cf. section 2.4 for the definition) required from the vector field.

The main results, Theorems 1, 2 and 3, establish essentially the equality of these functions. Note that if V has dimension one then $\mathrm{Rep}^M(\Gamma; V)$ identifies to finitely many copies of $(\mathbb{C} \setminus 0)^k$, k the first Betti number, and therefore our (rational / holomorphic / meromorphic) functions are functions of k complex variables.

One can calculate the Milnor–Turaev torsion when M has a structure of mapping torus of a diffeomorphism ϕ as the "twisted Lefschetz zeta function" of the diffeomorphism ϕ, cf. section 7.1. The Alexander polynomial as well as the twisted Alexander polynomials of a knot can also be recovered from these torsions cf. section 7.3. If the vector field has no rest points but admits a closed Lyapunov cohomology class, cf. section 7.2, the dynamical torsion can be expressed in terms of closed trajectories only, and the dynamical zeta function of the vector field (including all its twisted versions) can be recovered from the dynamical torsion.

In section 8.1 we express the phase difference of the Milnor–Turaev torsion of two representations in the same connected component of $\mathrm{Rep}_0^M(\Gamma; V)$ in terms of the Ray–Singer torsion. This invariant is analogous to the Atiyah–Patodi–Singer spectral flow but has not been investigated so far.

2. Characteristic forms and vector fields

2.1. **Euler, Chern–Simons, and the global angular form.** Let M be smooth closed manifold of dimension n. Let $\pi : TM \to M$ denote the tangent bundle, and $\mathcal{O}_M$ the orientation bundle, which is a flat real line bundle over M. For a Riemannian metric g denote by

$$\mathrm{e}(g) \in \Omega^n(M;\mathcal{O}_M)$$

its Euler form, and for two Riemannian metrics g_1 and g_2 by

$$\mathrm{cs}(g_1,g_2) \in \Omega^{n-1}(M;\mathcal{O}_M)/d(\Omega^{n-2}(M;\mathcal{O}_M))$$

their Chern–Simons class. The definition of both quantities is implicit in the formulas (4) and (5) below. The following properties follow from (4) and (5) below.

$$\begin{aligned} d\,\mathrm{cs}(g_1,g_2) &= \mathrm{e}(g_2)-\mathrm{e}(g_1) && (1)\\ \mathrm{cs}(g_2,g_1) &= -\,\mathrm{cs}(g_1,g_2) && (2)\\ \mathrm{cs}(g_1,g_3) &= \mathrm{cs}(g_1,g_2)+\mathrm{cs}(g_2,g_3) && (3)\end{aligned}$$

Denote by ξ the Euler vector field on TM which assigns to a point $x\in TM$ the vertical vector $-x\in T_xTM$. A Riemannian metric g determines the Levi–Civita connection in the bundle $\pi: TM\to M$. There is a canonic n-form $\mathrm{vol}(g)\in\Omega^n(TM;\pi^*\mathcal{O}_M)$, which assigns to an n-tuple of vertical vectors *their volume times their orientation* and vanishes when contracted with horizontal vectors. The global angular form, see for instance [BT82], is the differential form

$$\Psi(g) := \frac{\Gamma(n/2)}{(2\pi)^{n/2}|\xi|^n} i_\xi \,\mathrm{vol}(g) \in \Omega^{n-1}(TM\setminus M;\pi^*\mathcal{O}_M)$$

and is the pull back of a form on $(TM\setminus M)/\mathbb{R}_+$. Moreover, we have the equalities

$$\begin{aligned} d\Psi(g) &= \pi^*\,\mathrm{e}(g). && (4)\\ \Psi(g_2)-\Psi(g_1) &= \pi^*\,\mathrm{cs}(g_1,g_2) \mod \pi^* d\Omega^{n-2}(M;\mathcal{O}_M). && (5)\end{aligned}$$

2.2. **Euler and Chern–Simons chains.** For a vector field X with non-degenerate rest points we have the singular 0-chain $\mathrm{e}(X)\in C_0(M;\mathbb{Z})$ defined by $\mathrm{e}(X) := \sum_{x\in\mathcal{X}}\mathrm{IND}(x)x$, with $\mathrm{IND}(x)$ the Hopf index.

For two vector fields X_1 and X_2 with non-degenerate rest points we have the singular 1-chain rel. boundaries $\mathrm{cs}(X_1,X_2)\in C_1(M;\mathbb{Z})/\partial C_2(M;\mathbb{Z})$ defined from the zero set of a homotopy from X_1 to X_2 cf. [BH04c]. They are related by the formulas, see [BH04c],

$$\begin{aligned} \partial\,\mathrm{cs}(X_1,X_2) &= \mathrm{e}(X_2)-\mathrm{e}(X_1) && (6)\\ \mathrm{cs}(X_2,X_1) &= -\,\mathrm{cs}(X_1,X_2) && (7)\\ \mathrm{cs}(X_1,X_3) &= \mathrm{cs}(X_1,X_2)+\mathrm{cs}(X_2,X_3). && (8)\end{aligned}$$

2.3. **Kamber–Tondeur one form.** Let E be a real or complex vector bundle over M. For a connection ∇ and a Hermitian structure μ on E we define a real valued one form $\omega(\nabla,\mu) := -\frac{1}{2}\mathrm{tr}_\mu(\nabla\mu)\in\Omega^1(M;\mathbb{R})$. More explicitly, for a tangent vector $Y\in TM$

$$\omega(\nabla,\mu)(Y) := -\frac{1}{2}\,\mathrm{tr}_\mu(\nabla_Y\mu). \qquad (9)$$

Note that if ∇ is flat then $\omega(\nabla,\mu)$ will be a closed. Moreover, $\omega(\nabla^{\det E},\mu^{\det E}) = \omega(\nabla,\mu)$ where $\nabla^{\det E}$ and $\mu^{\det E}$ denote the induced connection and Hermitian structure on the determinant line $\det E := \Lambda^{\mathrm{rank}\,E}E$.

A similar form (closed for a flat connections) can be associated to a symmetric non-degenerate bilinear form b on E instead of a Hermitian structure by a similar formula

$$\omega(\nabla, b)(Y) := -\frac{1}{2}\operatorname{tr}_b(\nabla_Y b). \tag{10}$$

Note that $\omega(\nabla, b) \in \Omega^1(M; \mathbb{C})$ depends holomorphically on ∇ and if b_1 and b_2 are homotopic then $\omega(\nabla, b_2) - \omega(\nabla, b_1)$ is exact.

2.4. **Vector fields, instantons and closed trajectories.** Consider a vector field X which satisfies the following properties:

- (H) All rest points are of hyperbolic type.
- (EG) The vector field has exponential growth at all rest points.
- (L) The vector field is of Lyapunov type.
- (MS) The vector field satisfies Morse–Smale condition.
- (NCT) The vector field has all closed trajectories non-degenerate.

Precisely this means that:

- (H) In the neighborhood of each rest point the differential of X has all eigenvalues with non-trivial real part; the number of eigenvalues with negative real part is called the index and denoted by $\operatorname{ind}(x)$; as a consequence the stable and unstable stable sets are images of one-to-one immersions $i_x^\pm : W_x^\pm \to M$ with $W_x^\pm$ diffeomorphic to $\mathbb{R}^{n-\operatorname{ind}(x)}$ resp. $\mathbb{R}^{\operatorname{ind}(x)}$.
- (EG) With respect to one and then any Riemannian metric g on M, the volume of the disk of radius r in W_x^- (w.r. to the induced metric) is $\leq e^{Cr}$, $C > 0$.
- (L) There exists a real valued closed one form ω so that $\omega(X)_x < 0$ for x not a rest point.[1]
- (MS) For any two rest points x and y the mappings i_x^- and i_y^+ are transversal and therefore the space of non-parameterized trajectories form x to y, $\mathcal{T}(x, y)$, is a smooth manifold of dimension $\operatorname{ind}(x) - \operatorname{ind}(y) - 1$. Instantons are exactly the elements of $\mathcal{T}(x, y)$ when this is a smooth manifold of dimension zero, i.e. $\operatorname{ind}(x) - \operatorname{ind}(y) - 1 = 0$.
- (NCT) Any closed trajectory is non-degenerate, i.e. the differential of the return map in normal direction at one and then any point of a closed trajectory does not have non-zero fixed points.

Property (L) implies that for any real number R the set of instantons θ from x to y as well as the set of closed trajectories $\hat\theta$ with $-\omega([\theta]) \leq R$ resp. $-\omega([\hat\theta]) \leq R$ is finite.[2]

Denote by $\hat{\mathcal{P}}_{x,y}$ the set of homotopy classes of paths from x to y and by $\mathcal{X}_q$ the set of rest points of index q. Suppose a collection $\mathcal{O} = \{\mathcal{O}_x \mid x \in \mathcal{X}\}$ of orientations of the unstable manifolds is given and (MS) is satisfied. Then any instanton θ has a sign $\epsilon(\theta) = \pm 1$ and therefore, if (L) is also satisfied, for any two rest points $x \in \mathcal{X}_{q+1}$ and $y \in \mathcal{X}_q$ we have the counting function of instantons $\mathbb{I}_{x,y}^{X,\mathcal{O}} : \hat{\mathcal{P}}_{x,y} \to \mathbb{Z}$ defined by

$$\mathbb{I}_{x,y}^{X,\mathcal{O}}(\hat\alpha) := \sum_{\theta \in \hat\alpha} \epsilon(\theta). \tag{11}$$

Under the hypothesis (NCT) any closed trajectory $\hat\theta$ has a sign $\epsilon(\hat\theta) = \pm 1$ and a period $p(\hat\theta) \in \{1, 2, \dots\}$, cf. [H02]. If (H), (L), (MS), (NCT) are satisfied, as the

[1] This ω has nothing in common with $\omega(\nabla, b)$ notation used in the previous section.

[2] Here $[\hat\theta]$ denotes the homotopy class of the closed trajectory $\hat\theta$, and similarly for instantons θ.

set of closed trajectories in a fixed homotopy class $\gamma \in [S^1, M]$ is compact, we have the counting function of closed trajectories $\mathbb{Z}_X : [S^1, M] \to \mathbb{Q}$ defined by

$$\mathbb{Z}_X(\gamma) := \sum_{\hat\theta \in \gamma} \epsilon(\hat\theta)/p(\hat\theta). \tag{12}$$

Here are a few properties about vector fields which satisfy (H) and (L).

Proposition 1. *1. Given a vector field X which satisfies (H) and (L) arbitrary close in the C^r-topology for any $r \geq 0$ there exists a vector field Y which agrees with X on a neighborhood of the rest points and satisfies (H), (L), (MS) and (NCT).*

2. Given a vector field X which satisfies (H) and (L) arbitrary close in the C^0-topology there exists a vector field Y which agrees with X on a neighborhood of the rest points and satisfies (H), (EG), (L), (MS) and (NCT).

3. If X satisfies (H), (L) and (MS) and a collection $\mathcal{O}$ of orientations is given then for any $x \in \mathcal{X}_q$, $z \in \mathcal{X}_{q-2}$ and $\gamma \in \hat{\mathcal{P}}_{x,z}$ one has[3]

$$\sum_{\hat\alpha * \hat\beta = \gamma,\ y \in \mathcal{X}_{q-1}} \mathbb{I}^{X,\mathcal{O}}_{x,y}(\hat\alpha) \cdot \mathbb{I}^{X,\mathcal{O}}_{y,z}(\hat\beta) = 0. \tag{13}$$

This proposition is a recollection of some of the main results in [BH04a]; 1), 2) and 3) correspond to Proposition 3, Theorem 1 and Theorem 5 in [BH04a].

3. Euler and co-Euler structures

3.1. **Euler structures.** Euler structures have been introduced by Turaev [T90] for manifolds M with $\chi(M) = 0$. If one removes the hypothesis $\chi(M) = 0$ the concept of Euler structure can still be considered for any connected base pointed manifold (M, x_0) cf. [B99] and [BH04c]. Here we will consider only the case $\chi(M) = 0$. The set of Euler structures, denoted by $\mathfrak{Eul}(M;\mathbb{Z})$, is equipped with a free and transitive action

$$m : H_1(M;\mathbb{Z}) \times \mathfrak{Eul}(M;\mathbb{Z}) \to \mathfrak{Eul}(M;\mathbb{Z})$$

which makes $\mathfrak{Eul}(M;\mathbb{Z})$ an affine version of $H_1(M;\mathbb{Z})$. If $\mathfrak{e}_1, \mathfrak{e}_2$ are two Euler structure we write $\mathfrak{e}_2 - \mathfrak{e}_1$ for the unique element in $H_1(M;\mathbb{Z})$ with $m(\mathfrak{e}_2 - \mathfrak{e}_1, \mathfrak{e}_1) = \mathfrak{e}_2$.

To define the set $\mathfrak{Eul}(M;\mathbb{Z})$ we consider pairs (X, c) with X a vector field with non-degenerate zeros and $c \in C_1(M;\mathbb{Z})$ so that $\partial c = \mathrm{e}(X)$. We make (X_1, c_1) and (X_2, c_2) equivalent iff $c_2 - c_1 = \mathrm{cs}(X_1, X_2)$ and write $[X, c]$ for the equivalence class represented by (X, c). The action m is defined by $m([c'], [X, c]) := [X, c' + c]$.

Observation 1. *Suppose X is a vector field with non-degenerate zeros, $x_0 \in M$ and $\mathfrak{e} \in \mathfrak{Eul}(M;\mathbb{Z})$ an Euler structure. Then there exists a collection of paths $\{\sigma_x \mid x \in \mathcal{X}\}$ with $\sigma_x(0) = x_0$, $\sigma_x(1) = x$ and such that $\mathfrak{e} = [X, c]$ where $c = \sum_{x \in \mathcal{X}} \mathrm{IND}(x)\sigma_x$.*

A remarkable source of Euler structures is the set of homotopy classes of nowhere vanishing vector fields. Any nowhere vanishing vector field X provides an Euler structure $[X, 0]$ which only depends on the homotopy class of X. Still assuming $\chi(M) = 0$, every Euler structure can be obtained in this way provided $\dim(M) > 2$. Be aware, however, that different homotopy classes may give rise to the same Euler structure.

[3]It is understood that only finitely many terms from the left side of the equality are not zero. Here $*$ denotes juxtaposition.

To construct such a homotopy class one can proceed as follows. Represent the Euler structure $\mathfrak{e}$ by a vector field X and a collection of paths $\{\sigma_x \mid x \in \mathcal{X}\}$ as in Observation 1. Since $\dim(M) > 2$ we may assume that the interiors of the paths are mutually disjoint. Then the set $\bigcup_{x\in\mathcal{X}} \sigma_x$ is contractible. A smooth regular neighborhood of it is the image by a smooth embedding $\varphi : (D^n, 0) \to (M, x_0)$. Since $\chi(M) = 0$, the restriction of the vector field X to $M\backslash\mathrm{int}(D^n)$ can be extended to a non-vanishing vector field $\tilde{X}$ on M. It is readily checked that $[\tilde{X}, 0] = \mathfrak{e}$. For details see [BH04c].

If M is a compact connected smooth manifold of dimension larger than 2 an alternative description of $\mathfrak{Eul}(M;\mathbb{Z})$ with respect to a base point x_0 is $\mathfrak{Eul}(M;\mathbb{Z}) = \pi_0(\mathfrak{X}(M, x_0))$, where $\mathfrak{X}(M, x_0)$ denotes the space of vector fields of class C^r, $r \geq 0$, which vanish at x_0 and are non-zero elsewhere. We equip this space with the C^r-topology and note that the result $\pi_0(\mathfrak{X}(M, x_0))$ is the same for all r, and since $\chi(M) = 0$, canonically identified for different base points.

Let τ be a smooth triangulation of M and consider the function $f_\tau : M \to \mathbb{R}$ linear on any simplex of the first barycentric subdivision and taking the value $\dim(s)$ on the barycenter x_s of the simplex $s \in \tau$. A smooth vector field X on M with the the barycenters as hyperbolic rest points and f_τ strictly decreasing on non-constant trajectories is called an Euler vector field of τ. By an argument of convexity two Euler vector fields are homotopic by a homotopy of Euler vector fields.[4] Therefore, a triangulation τ, a base point x_0 and a collection of paths $\{\sigma_s \mid s \in \tau\}$ with $\sigma_s(0) = x_0$ and $\sigma_s(1) = x_s$ define an Euler structure $[X_\tau, c]$, where $c := \sum_{s\in\tau}(-1)^{\dim(s)}\sigma_s$, X_τ is any Euler vector field for τ, and this Euler structure does not depend on the choice of X_τ. Clearly, for fixed τ and x_0, every Euler structure can be realized in this way by an appropriate choice of $\{\sigma_s \mid s \in \tau\}$, cf. Observation 1.

3.2. **Co-Euler structures.** Again, suppose $\chi(M) = 0$.[5] Consider pairs (g, α) where g is a Riemannian metric on M and $\alpha \in \Omega^{n-1}(M; \mathcal{O}_M)$ with $d\alpha = \mathrm{e}(g)$ where $\mathrm{e}(g) \in \Omega^n(M; \mathcal{O}_M)$ denotes the Euler form of g, see section 2.1. We call two pairs (g_1, α_1) and (g_2, α_2) equivalent if

$$\mathrm{cs}(g_1, g_2) = \alpha_2 - \alpha_1 \in \Omega^{n-1}(M; \mathcal{O}_M)/d\Omega^{n-2}(M; \mathcal{O}_M).$$

We will write $\mathfrak{Eul}^*(M;\mathbb{R})$ for the set of equivalence classes and $[g, \alpha]$ for the equivalence class represented by the pair (g, α). Elements of $\mathfrak{Eul}^*(M;\mathbb{R})$ are called *co-Euler structures.*

There is a natural action

$$m^* : H^{n-1}(M; \mathcal{O}_M) \times \mathfrak{Eul}^*(M;\mathbb{R}) \to \mathfrak{Eul}^*(M;\mathbb{R})$$

given by

$$m^*([\beta], [g, \alpha]) := [g, \alpha - \beta]$$

[4]Any Euler vector field X satisfies (H), (EG), (L) and has no closed trajectory, hence also satisfies (NCT). The counting functions of instantons are exactly the same as the incidence numbers of the triangulation hence take the values 1, -1 or 0.

[5]The hypothesis is not necessary and the theory of co-Euler structure can be pursued for an arbitrary base pointed smooth manifold (M, x_0), cf. [BH04c].

for $[\beta] \in H^{n-1}(M;\mathcal{O}_M)$. This action is obviously free and transitive. In this sense $\mathfrak{Eul}^*(M;\mathbb{R})$ is an affine version of $H^{n-1}(M;\mathcal{O}_M)$. If $\mathfrak{e}_1^*$ and $\mathfrak{e}_2^*$ are two co-Euler structures we write $\mathfrak{e}_2^* - \mathfrak{e}_1^*$ for the unique element in $H^{n-1}(M;\mathcal{O}_M)$ with $m^*(\mathfrak{e}_2^* - \mathfrak{e}_1^*, \mathfrak{e}_1^*) = \mathfrak{e}_2^*$.

Observation 2. *Given a Riemannian metric g on M any co-Euler structure can be represented as a pair (g,α) for some $\alpha \in \Omega^{n-1}(M;\mathcal{O}_M)$ with $d\alpha = \mathrm{e}(g)$.*

There is a map $\mathrm{PD} : \mathfrak{Eul}(M;\mathbb{Z}) \to \mathfrak{Eul}^*(M;\mathbb{R})$ which combined with the Poincaré duality map $D : H_1(M;\mathbb{Z}) \to H_1(M;\mathbb{R}) \to H^{n-1}(M;\mathcal{O}_M)$, the composition of the coefficient homomorphism for $\mathbb{Z} \to \mathbb{R}$ with the Poincaré duality isomorphism [6], makes the diagram below commutative:

$$\begin{tikzcd} H_1(M;\mathbb{Z}) \times \mathfrak{Eul}(M;\mathbb{Z}) \arrow[r, "m"] \arrow[d, "D\times \mathrm{PD}"'] & \mathfrak{Eul}(M;\mathbb{Z}) \arrow[d, "\mathrm{PD}"] \\ H^{n-1}(M;\mathcal{O}_M) \times \mathfrak{Eul}^*(M;\mathbb{R}) \arrow[r, "m^*"] & \mathfrak{Eul}^*(M;\mathbb{R}) \end{tikzcd}$$

There are many ways to define the map PD, cf. [BH04c]. For example, assuming $\chi(M) = 0$ and $\dim M > 2$ one can proceed as follows. Represent the Euler structure by a nowhere vanishing vector field $\mathfrak{e} = [X, 0]$. Choose a Riemannian metric g, regard X as mapping $X : M \to TM \setminus M$, set $\alpha := X^*\Psi(g)$, put $\mathrm{PD}(\mathfrak{e}) := [g,\alpha]$ and check that this does indeed only depend on $\mathfrak{e}$.

A co-Euler structure $\mathfrak{e}^* \in \mathfrak{Eul}^*(M;\mathbb{R})$ is called *integral* if it belongs to the image of PD. Integral co-Euler structures constitute a lattice in the affine space $\mathfrak{Eul}^*(M;\mathbb{R})$.

4. Complex representations and cochain complexes

4.1. **Complex representations.** Let Γ be a finitely presented group with generators $g_1, \dots, g_r$ and relations $R_i(g_1, g_2, \dots, g_r) = e$, $i = 1, \dots, p$, and V be a complex vector space of dimension N. Let $\mathrm{Rep}(\Gamma;V)$ be the set of linear representations of Γ on V, i.e. group homomorphisms $\rho : \Gamma \to \mathrm{GL}_{\mathbb{C}}(V)$. By identifying V to $\mathbb{C}^N$ this set is, in a natural way, an algebraic set inside the space $\mathbb{C}^{rN^2+1}$ given by $pN^2 + 1$ equations. Precisely if $A_1, \dots, A_r, z$ represent the coordinates in $\mathbb{C}^{rN^2+1}$ with $A := (a^{ij})$, $a^{ij} \in \mathbb{C}$, so $A \in \mathbb{C}^{N^2}$ and $z \in \mathbb{C}$, then the equations defining $\mathrm{Rep}(\Gamma;V)$ are

$$\begin{aligned} z \cdot \det(A_1) \cdot \det(A_2) \cdots \det(A_r) &= 1 \\ R_i(A_1, \dots, A_r) &= \mathrm{id}, \qquad i = 1, \dots, p \end{aligned}$$

with each of the equalities R_i representing N^2 polynomial equations.

Suppose $\Gamma = \pi_1(M, x_0)$, M a closed manifold. Denote by $\mathrm{Rep}_0^M(\Gamma;V)$ the set of representations ρ with $H^*(M;\rho) = 0$ and notice that they form a Zariski open set in $\mathrm{Rep}(\Gamma;V)$. Denote the closure of this set by $\mathrm{Rep}^M(\Gamma;V)$. This is an algebraic set which depends only on the homotopy type of M, always a union of irreducible components of $\mathrm{Rep}(\Gamma;V)$.

Recall that every representation $\rho \in \mathrm{Rep}(\Gamma;V)$ induces a canonical vector bundle F_ρ equipped with a canonical flat connection ∇_ρ. They are obtained from the trivial

[6] We will use the same notation D for the Poincaré duality isomorphism $D : H_1(M;\mathbb{R}) \to H^{n-1}(M;\mathcal{O}_M)$.

bundle $\tilde{M} \times V \to \tilde{M}$ and the trivial connection by passing to the Γ quotient spaces. Here $\tilde{M}$ is the canonical universal covering provided by the base point x_0. The Γ-action is the diagonal action of deck transformations on $\tilde{M}$ and of the action ρ on V. The fiber of F_ρ over x_0 identifies canonically with V. The holonomy representation determines a right Γ-action on the fiber of F_ρ over x_0, i.e. an anti homomorphism $\Gamma \to \mathrm{GL}(V)$. When composed with the inversion in $\mathrm{GL}(V)$ we get back the representation ρ. The pair (F_ρ, ∇_ρ) will be denoted by $\mathbb{F}_\rho$.

If ρ_0 is a representation in the connected component $\mathrm{Rep}_\alpha(\Gamma; V)$ one can identify $\mathrm{Rep}_\alpha(\Gamma; V)$ to the connected component of ∇_{ρ_0} in the complex analytic space of flat connections of the bundle F_{ρ_0} modulo the group of bundle isomorphisms of F_{ρ_0} which fix the fiber above x_0.

4.2. **Closed one m-forms.** Recall that in view of Poincaré's lemma a closed one form on a (topological / smooth manifold / complex analytic) space X is an equivalence class of systems $(U_\alpha, h_\alpha : U_\alpha \to \mathbb{C})_{\alpha\in\mathcal{A}}$, with $\{U_\alpha\}$ an open cover of X and h_α (continuous / smooth / holomorphic) maps so that $h_{\alpha_2} - h_{\alpha_1}$ is locally constant on $U_{\alpha_1} \cap U_{\alpha_2}$ The concept of *closed one m-form* is the multiplicative analog of the closed one form; we will consider it only in the complex analytic category.

Let $\mathbb{C}_* := \mathbb{C} \setminus \{0\}$ denote the multiplicative group of non-zero complex numbers and let X be a complex analytic space. A holomorphic closed one m-form is an equivalence class of systems $(U_\alpha, h_\alpha : U_\alpha \to \mathbb{C}_*)_{\alpha\in\mathcal{A}}$, with h_α holomorphic maps so that $h_{\alpha_2}/h_{\alpha_1}$ is locally constant on $U_{\alpha_1} \cap U_{\alpha_2}$. Two such systems (U_α, h_α) and (U'_β, h'_β) are equivalent if together, they form a system as above. We denote by $\mathcal{Z}(X; \mathbb{C}_*)$ the set of holomorphic closed one m-forms. This is an abelian group w.r. to multiplication. A holomorphic map $\varphi : Y \to X$ defines the pullback $\varphi^* : \mathcal{Z}(X; \mathbb{C}_*) \to \mathcal{Z}(Y; \mathbb{C}_*)$.

For $u \in \mathcal{Z}(X; \mathbb{C}_*)$ and s a real number one can define u^s as follows. Choose a system $(U_\alpha, h_\alpha)_{\alpha\in\mathcal{A}}$ representing u such that for all α there exists $\lambda_\alpha : U_\alpha \to \mathbb{R}_+$ and $\theta_\alpha : U_\alpha \to \mathbb{R}$ with $h_\alpha(\rho) = \lambda_\alpha(\rho) e^{i\theta_\alpha(\rho)}$. Define u^s by $u^s := \{U_\alpha, h^s_\alpha : U_\alpha \to \mathbb{C}_*\}$ with $h^s_\alpha(\rho) = \lambda^s_\alpha(\rho) e^{is\theta_\alpha(\rho)}$. This is independent of the choice of U_α, h_α, θ_α and satisfies $u^{s_1+s_2} = u^{s_1} \cdot u^{s_2}$, $(u^{s_1})^{s_2} = u^{s_1 s_2}$ and $(uv)^s = u^s v^s$.

Example 1. An element $a \in H_1(M; \mathbb{Z})$ defines a holomorphic function

$$\det{}_a : \mathrm{Rep}^M(\Gamma; V) \to \mathbb{C}_*.$$

The complex number $\det_a(\rho)$ is the evaluation on $a \in H_1(M; \mathbb{Z})$ of $\det(\rho) : \Gamma \to \mathbb{C}_*$ which factors through $H_1(M; \mathbb{Z})$. Note that for $a, b \in H_1(M; \mathbb{Z})$ we have $\det_{a+b} = \det_a \det_b$. If a is a torsion element, then $\det_a$ is constant equal to a root of unity of order, the order of a. Therefore any $\hat{a} \in H_1(M; \mathbb{Z})/\operatorname{Tor}(H_1(M; \mathbb{Z}))$ defines a holomorphic closed one m-form denoted $\det_{\hat{a}}$, and so is $(\det_{\hat{a}})^s$, $s \in \mathbb{R}$.

Example 2. An element $\hat{b} \in H_1(M; \mathbb{R})$ defines a holomorphic closed one m-form, $\det_{\hat{b}}$ in the following way. Choose a representation of $\hat{b} = \mu_1 a_1 + \mu_2 a_2 + \cdots + \mu_r a_r$ where μ_k are real numbers and $a_k \in H_1(M; \mathbb{Z})$, $k = 1, \ldots, r$, and consider the closed one m-form

$$\det{}_{\hat{b}} := \prod_{k=1,\ldots,r} (\det{}_{a_k})^{\mu_k}.$$

It is easy to see that this independent of the representation of $\hat{b}$.

4.3. **The space of cochain complexes.** Let $k = (k_0, k_1, \dots, k_n)$ be a string of non-negative integers. The string is called admissible, and will write $k \geq 0$ in this case, if the following requirements are satisfied

$$k_0 - k_1 + k_2 \mp \cdots + (-1)^n k_n = 0 \tag{14}$$

$$k_i - k_{i-1} + k_{i-2} \mp \cdots + (-1)^i k_0 \geq 0 \quad \text{for any } i \leq n-1. \tag{15}$$

Denote by $\mathbb{D}(k) = \mathbb{D}(k_0, \dots, k_n)$ the collection of cochain complexes of the form

$$C = (C^*, d^*) : 0 \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} C^n \to 0$$

with $C^i := \mathbb{C}^{k_i}$, and by $\mathbb{D}_{\mathrm{ac}}(k) \subseteq \mathbb{D}(k)$ the subset of acyclic complexes. Note that $\mathbb{D}_{\mathrm{ac}}(k)$ is non-empty iff $k \geq 0$. The cochain complex C is determined by the collection $\{d^i\}$ of linear maps $d^i : \mathbb{C}^{k_i} \to \mathbb{C}^{k_{i+1}}$. If regarded as the subset of those $\{d^i\} \in \bigoplus_{i=0}^{n-1} L(\mathbb{C}^{k_i}, \mathbb{C}^{k_{i+1}})$, with $L(V, W)$ the space of linear maps from V to W, which satisfy the quadratic equations $d^{i+1} \cdot d^i = 0$, the set $\mathbb{D}(k)$ is an affine algebraic set given by degree two homogeneous polynomials and $\mathbb{D}_{\mathrm{ac}}(k)$ is a Zariski open set. The map $\pi_0 : \mathbb{D}_{\mathrm{ac}}(k) \to \mathrm{Emb}(C^0, C^1)$ which associates to $C \in \mathbb{D}_{\mathrm{ac}}(k)$ the linear map d^0, is a bundle whose fiber is isomorphic to $\mathbb{D}_{\mathrm{ac}}(k_1 - k_0, k_2, \dots, k_n)$.

This can be easily generalized as follows. Consider a string $b = (b_0, \dots, b_n)$. We will write $k \geq b$ if $k - b = (k_0 - b_0, \dots, k_n - b_n)$ is admissible, i.e. $k - b \geq 0$. Denote by $\mathbb{D}_b(k) = \mathbb{D}_{(b_0, \dots, b_n)}(k_0, \dots, k_n)$ the subset of cochain complexes $C \in \mathbb{D}(k)$ with $\dim(H^i(C)) = b_i$. Note that $\mathbb{D}_b(k)$ is non-empty iff $k \geq b$. The obvious map $\pi_0 : \mathbb{D}_b(k) \to L(C^0, C^1; b_0)$, $L(C^0, C^1; b_0)$ the space of linear maps in $L(C^0, C^1)$ whose kernel has dimension b_0, is a bundle whose fiber is isomorphic to $\mathbb{D}_{b_1, \dots, b_n}(k_1 - k_0 + b_0, k_2, \dots, k_n)$. Note that $L(C^0, C^1; b_0)$ is the total space of a bundle $\mathrm{Emb}(\underline{\mathbb{C}}^{k_0}/L, \underline{\mathbb{C}}^{k_1}) \to \mathrm{Gr}_{b_0}(k_0)$ with $L \to \mathrm{Gr}_{b_0}(k_0)$ the tautological bundle over $\mathrm{Gr}_{b_0}(k_0)$ and $\underline{\mathbb{C}}^{k_0}$ resp. $\underline{\mathbb{C}}^{k_1}$ the trivial bundles over $\mathrm{Gr}_{b_0}(k_0)$ with fibers of dimension k_0 resp. k_1. As a consequence we have

Proposition 2. *1. $\mathbb{D}_{\mathrm{ac}}(k)$ and $\mathbb{D}_b(k)$ are connected smooth quasi affine algebraic sets whose dimension is*

$$\dim \mathbb{D}_b(k) = \sum_j (k^j - b^j) \cdot \Big(k^j - \sum_{i \leq j} (-1)^{i+j} (k^i - b^i) \Big).$$

2. The closures $\hat{\mathbb{D}}_{\mathrm{ac}}(k)$ and $\hat{\mathbb{D}}_b(k)$ are irreducible algebraic sets, hence affine algebraic varieties, and $\hat{\mathbb{D}}_b(k) = \bigsqcup_{k \geq b' \geq b} \mathbb{D}_{b'}(k)$.

For any cochain complex in $C \in \mathbb{D}_{\mathrm{ac}}(k)$ denote by $B^i := \mathrm{img}(d^{i-1}) \subseteq C^i = \mathbb{C}^{k_i}$ and consider the short exact sequence $0 \to B^i \xrightarrow{\mathrm{inc}} C^i \xrightarrow{d^i} B^{i+1} \to 0$. Choose a base $\mathfrak{b}_i$ for each B_i, and choose lifts $\overline{\mathfrak{b}}_{i+1}$ of $\mathfrak{b}_{i+1}$ in C^i using d^i, i.e. $d^i(\overline{\mathfrak{b}}_{i+1}) = \mathfrak{b}_{i+1}$. Clearly $\{\mathfrak{b}_i, \overline{\mathfrak{b}}_{i+1}\}$ is a base of C^i. Consider the base $\{\mathfrak{b}_i, \overline{\mathfrak{b}}_{i+1}\}$ as a collection of vectors in $C^i = \mathbb{C}^{k_i}$ and write them as columns of a matrix $[\mathfrak{b}_i, \overline{\mathfrak{b}}_{i+1}]$. Define the torsion of the acyclic complex C, by

$$\tau(C) := (-1)^{N+1} \prod_{i=0}^{n} \det[\mathfrak{b}_i, \overline{\mathfrak{b}}_{i+1}]^{(-1)^i}$$

where $(-1)^N$ is Turaev's sign, see [FT00]. The result is independent of the choice of the bases $\mathfrak{b}_i$ and of the lifts $\overline{\mathfrak{b}}_i$ cf. [M66] [FT00], and leads to the function

$$\tau : \mathbb{D}_{\mathrm{ac}}(k) \to \mathbb{C}_*.$$

Turaev provided a simple formula for this function, cf. [T01], which permits to recognize τ as the restriction of a rational function on $\hat{\mathbb{D}}_{\mathrm{ac}}(k)$, although this is rather obvious from general reasons.

For $C \in \hat{\mathbb{D}}_{\mathrm{ac}}(k)$ denote by $(d^i)^t : \mathbb{C}^{k_{i+1}} \to \mathbb{C}^{k_i}$ the transpose of $d^i : \mathbb{C}^{k_i} \to \mathbb{C}^{k_{i+1}}$, and define $P_i = d^{i-1} \cdot (d^{i-1})^t + (d^i)^t \cdot d^i$. Define $\Sigma(k)$ as the subset of cochain complexes in $\hat{\mathbb{D}}_{\mathrm{ac}}(k)$ where $\ker P \neq 0$, and consider $S\tau : \hat{\mathbb{D}}_{\mathrm{ac}}(k) \setminus \Sigma(k) \to \mathbb{C}_*$ defined by

$$S\tau(C) := \Big(\prod_{i \text{ even}} (\det P_i)^i / \prod_{i \text{ odd}} (\det P_i)^i \Big)^{-1}.$$

One can verify

Proposition 3. *Suppose $k = (k_0, \dots, k_n)$ is admissible.*

1. $\Sigma(k)$ is a proper subvariety containing the singular set of $\hat{\mathbb{D}}_{\mathrm{ac}}(k)$.

2. $S\tau = \tau^2$ and implicitly $S\tau$ has an analytic continuation to $\mathbb{D}_{\mathrm{ac}}(k)$.

In particular τ defines a square root of $S\tau$. We will not use explicitly $S\tau$ in this writing however it justifies the definition of complex Ray–Singer torsion.

5. Analytic torsion

Let M be a closed manifold, g Riemannian metric and (g, α) a representative of a co-Euler structure $\mathfrak{e}^* \in \mathfrak{Eul}^*(M; \mathbb{R})$. Suppose $E \to M$ is a complex vector bundle and denote by $\mathcal{C}(E)$ the space of connections and by $\mathcal{F}(E)$ the subset of flat connections. $\mathcal{C}(E)$ is a complex affine (Fréchet) space while $\mathcal{F}(E)$ a closed complex analytic subset (Stein space) of $\mathcal{C}(E)$. Let b be a non-degenerate symmetric bilinear form and μ a Hermitian (fiber metric) structure on E. While Hermitian structures always exist, non-degenerate symmetric bilinear forms exist iff the bundle is the complexification of some real vector bundle, and in this case $E \simeq E^*$.

The connection $\nabla \in \mathcal{C}(E)$ can be interpreted as a first order differential operator $d^\nabla : \Omega^*(M; E) \to \Omega^{*+1}(M; E)$ and g and b resp. g and μ can be used to define the formal b-adjoint resp. μ-adjoint $\delta^\nabla_{q;g,b}$ resp. $\delta^\nabla_{q;g,\mu} : \Omega^{q+1}(M; E) \to \Omega^q(M; E)$ and therefore the Laplacians

$$\Delta^\nabla_{q;g,b} \text{ resp. } \Delta^\nabla_{q;g,\mu} : \Omega^q(M; E) \to \Omega^q(M; E).$$

They are elliptic second order differential operators with principal symbol $\sigma_\xi = |\xi|^2$. Therefore they have a unique well defined zeta regularized determinant (modified determinant) $\det(\Delta^\nabla_{q;g,b}) \in \mathbb{C}$ ($\det'(\Delta^\nabla_{q;g,b}) \in \mathbb{C}_*$) resp. $\det(\Delta^\nabla_{q;g,\mu}) \in \mathbb{R}_{\geq 0}$ ($\det'(\Delta^\nabla_{q;g,\mu}) \in \mathbb{R}_{>0}$) calculated with respect to the Agmon angle π. Recall that the zeta regularized determinant (modified determinant) is the zeta regularized product of all (non-zero) eigenvalues.

Denote by

$$\Sigma(E, g, b) := \big\{ \nabla \in \mathcal{C}(E) \bigm| \ker(\Delta^\nabla_{*;g,b}) \neq 0 \big\}$$
$$\Sigma(E, g, \mu) := \big\{ \nabla \in \mathcal{C}(E) \bigm| \ker(\Delta^\nabla_{*;g,\mu}) \neq 0 \big\}$$

and by

$$\Sigma(E) := \big\{ \nabla \in \mathcal{F}(E) \bigm| H^*(\Omega^*(M; E), d^\nabla) \neq 0 \big\}.$$

Note that $\Sigma(E, g, \mu) \cap \mathcal{F}(E) = \Sigma(E)$ for any μ, and $\Sigma(E, g, b) \cap \mathcal{F}(E) \supseteq \Sigma(E)$. Both, $\Sigma(E)$ and $\Sigma(E, g, b) \cap \mathcal{F}(E)$, are closed complex analytic subsets of $\mathcal{F}(E)$, and $\det(\Delta^\nabla_{q;g,\dots}) = \det'(\Delta^\nabla_{q;g,\dots})$ on $\mathcal{F}(E) \setminus \Sigma(E, g, \cdots)$.

We consider the real analytic functions: $T^{\mathrm{even}}_{g,\mu} : \mathcal{C}(E) \to \mathbb{R}_{\geq 0}$, $T^{\mathrm{odd}}_{g,\mu} : \mathcal{C}(E) \to \mathbb{R}_{\geq 0}$, $R_{\alpha,\mu} : \mathcal{C}(E) \to \mathbb{R}_{>0}$ and the holomorphic functions $T^{\mathrm{even}}_{g,b} : \mathcal{C}(E) \to \mathbb{C}$, $T^{\mathrm{odd}}_{g,b} : \mathcal{C}(E) \to \mathbb{C}$, $R_{\alpha,b} : \mathcal{C}(E) \to \mathbb{C}_*$ defined by:

$$\begin{aligned} T^{\mathrm{even}}_{g,\cdots}(\nabla) &:= \prod_{q\ \mathrm{even}} (\det \Delta^{\nabla}_{q;g,\cdots})^q, \\ T^{\mathrm{odd}}_{g,\cdots}(\nabla) &:= \prod_{q\ \mathrm{odd}} (\det \Delta^{\nabla}_{q;g,\cdots})^q, \\ R_{\alpha,\cdots}(\nabla) &:= e^{\int_M \omega(\cdots,\nabla)\wedge\alpha}. \end{aligned}$$

We also write $T'^{\,\mathrm{even}}_{g,\cdots}$ resp. $T'^{\,\mathrm{odd}}_{g,\cdots}$ for the same formulas with $\det'$ instead of det. These functions are discontinuous on $\Sigma(E, g, \cdots)$ and coincide with $T^{\mathrm{even}}_{g,\cdots}$ resp. $T^{\mathrm{odd}}_{g,\cdots}$ on $\mathcal{F}(E) \setminus \Sigma(E, g, \cdots)$. Here $\cdots$ stands for either b or μ.

Let $E_r \to M$ be a smooth real vector bundle equipped with a non-degenerate symmetric positive definite bilinear form b_r. Let $\mathcal{C}(E_r)$ resp. $\mathcal{F}(E_r)$ the space of connections resp. flat connections in E_r. Denote by $E \to M$ the complexification of E_r, $E = E_r \otimes \mathbb{C}$, and by b resp. μ the complexification of b_r resp. the Hermitian structure extension of b_r. We continue to denote by $\mathcal{C}(E_r)$ resp. $\mathcal{F}(E_r)$ the subspace of $\mathcal{C}(E)$ resp. $\mathcal{F}(E)$ consisting of connections which are complexification of connections resp. flat connections in E_r, and by ∇ the complexification of the connection $\nabla \in \mathcal{C}(E_r)$. If $\nabla \in \mathcal{C}(E)$, then

$$\mathrm{Spect}\, \Delta^{\nabla}_{q;g,b} = \mathrm{Spect}\, \Delta^{\nabla}_{q;g,\mu} \in \mathbb{R}_{\geq 0}$$

and therefore

$$\begin{aligned} T^{\mathrm{even/odd}}_{g,b}(\nabla) &= \left|T^{\mathrm{even/odd}}_{g,b}(\nabla)\right| = T^{\mathrm{even/odd}}_{g,\mu}(\nabla), \\ T'^{\,\mathrm{even/odd}}_{g,b}(\nabla) &= \left|T'^{\,\mathrm{even/odd}}_{g,b}(\nabla)\right| = T'^{\,\mathrm{even/odd}}_{g,\mu}(\nabla), \\ R_{\alpha,b}(\nabla) &= \left|R_{\alpha,b}(\nabla)\right| = R_{\alpha,\mu}(\nabla). \end{aligned} \tag{16}$$

Observe that $\Omega^*(M; E)(0)$ the (generalized) eigenspace of $\Delta^{\nabla}_{*,g,b}$ corresponding to the eigenvalue zero is a finite dimensional vector space of dimension the multiplicity of 0. The restriction of the symmetric bilinear form induced by b remains non-degenerate and defines for each component $\Omega^q(M; E)(0)$ an equivalence class of bases. Since d^{∇} commutes with $\Delta^{\nabla}_{*,g,b}$, $\left(\Omega^*(M; E)(0), d^{\nabla}\right)$ is a finite dimensional complex. When acyclic, i.e. $\nabla \in \mathcal{F}(E) \setminus \Sigma(E)$, denote by

$$T_{\mathrm{an}}(\nabla, g, b)(0) \in \mathbb{C}_*$$

the Milnor torsion associated to the equivalence class of bases induced by b.

5.1. **The modified Ray–Singer torsion.** Let $E \to M$ be a complex vector bundle, and let $\mathfrak{e}^* \in \mathfrak{Eul}^*(M; \mathbb{R})$ be a co-Euler structure. Choose a Hermitian structure (fiber metric) μ on E, a Riemannian metric g on M and $\alpha \in \Omega^{n-1}(M; \mathcal{O}_M)$ so that $[g, \alpha] = \mathfrak{e}^*$, see section 3.2. For $\nabla \in \mathcal{F}(E) \setminus \Sigma(E)$ consider the quantity

$$T_{\mathrm{an}}(\nabla, \mu, g, \alpha) := \left(T^{\mathrm{even}}_{g,\mu}(\nabla)/T^{\mathrm{odd}}_{g,\mu}(\nabla)\right)^{-1/2} \cdot R_{\alpha,\mu}(\nabla) \in \mathbb{R}_{>0}$$

referred to as the *modified Ray–Singer torsion.* The following proposition is a reformulation of one of the main theorems in [BZ92], cf. also [BFK01] and [BH04c].

Proposition 4. *If $\nabla \in \mathcal{F}(E) \setminus \Sigma(E)$, then $T_{\mathrm{an}}(\nabla, \mu, g, \alpha)$ is gauge invariant and independent of μ, g, α.*

When applied to $\mathbb{F}_\rho$ the number $T_{\rm an}^{\mathfrak{e}^*}(\rho) := T_{\rm an}(\nabla_\rho, \mu, g, \alpha)$ defines a real analytic function $T_{\rm an}^{\mathfrak{e}^*} : \mathrm{Rep}_0^M(\Gamma; V) \to \mathbb{R}_{>0}$. It is natural to ask if $T_{\rm an}^{\mathfrak{e}^*}$ is the absolute value of a holomorphic function. The answer will be provided by Corollary 1 to Theorem 1 in section 5.2.

5.2. **Complex Ray–Singer torsion.** Let E be a complex vector bundle equipped with a non-degenerate symmetric bilinear for b. Suppose (g, α) is a pair consisting of a Riemannian metric g and form $\alpha \in \Omega^{n-1}(M; \mathcal{O}_M)$ with $d\alpha = \mathrm{e}(g)$. For any $\nabla \in \mathcal{F}(E) \setminus \Sigma(E)$ consider the complex number

$$\mathcal{ST}_{\rm an}(\nabla, b, g, \alpha) := \left(T'^{\,\rm even}_{g,b}(\nabla)/T'^{\,\rm odd}_{g,b}(\nabla)\right)^{-1} \cdot R_{\alpha,b}(\nabla)^2 \cdot T_{\rm an}(\nabla, g, b)(0)^2 \in \mathbb{C}_* \quad (17)$$

referred to as the *square of the complex Ray–Singer torsion.* The idea of considering b-Laplacians for torsion was brought to the attention of the first author by W. Müller [M]. The second author came to it independently.

Proposition 5. *1. $\mathcal{ST}_{\rm an}(\nabla, b, g, \alpha)$ is a holomorphic function on $\mathcal{F}(E) \setminus \Sigma(E)$ and the restriction of a meromorphic function on $\mathcal{F}(E)$ with poles and zeros in $\Sigma(E)$.*

2. If b_1 and b_2 are two non-degenerate symmetric bilinear forms which are homotopic then $\mathcal{ST}_{\rm an}(\nabla, b_1, g, \alpha) = \mathcal{ST}_{\rm an}(\nabla, b_2, g, \alpha)$.

3. If (g_1, α_1) and (g_2, α_2) are two pairs representing the same co-Euler structure then $\mathcal{ST}_{\rm an}(\nabla, b, g_1, \alpha_1) = \mathcal{ST}_{\rm an}(\nabla, b, g_2, \alpha_2)$.

4. If γ is a gauge transformation of E then $\mathcal{ST}_{\rm an}(\gamma\nabla, \gamma b, g, \alpha) = \mathcal{ST}_{\rm an}(\nabla, b, g, \alpha)$.

5. We have $\mathcal{ST}_{\rm an}(\nabla_1 \oplus \nabla_2, b_1 \oplus b_2, g, \alpha) = \mathcal{ST}_{\rm an}(\nabla_1, b_1, g, \alpha) \cdot \mathcal{ST}_{\rm an}(\nabla_2, b_2, g, \alpha)$.

6. If (g, α) defines an integral co-Euler structure then $\mathcal{ST}_{\rm an}(\nabla, b, g, \alpha)$ is independent of b.

The proof of 2) and 3) is derived from the formulas for $d/dt(\mathcal{ST}_{\rm an}(\nabla, b(t), g, \alpha))$ resp. $d/dt(\mathcal{ST}_{\rm an}(\nabla, b, g(t), \alpha)$ which are similar to such formulas for Ray–Singer torsion in the case of a Hermitian structure instead of a non-degenerate symmetric bilinear form. One of the less obvious reason that such formulas can be obtained is a sort of Hodge theory for $\Delta^{\nabla}_{*,g,b}$ which continues to hold. The proof of 1) and 5) require a careful inspection of the definitions. The proof of 6) is more involved. It requires the study of the Witten deformation for the Laplacians based on b rather than μ. The full arguments are contained in [BH05].

As a consequence to each homotopy class of non-degenerate symmetric bilinear forms $[b]$ and co-Euler structure $\mathfrak{e}^*$ we can associate a meromorphic function on $\mathcal{F}(E)$. Changing the co-Euler structure our function changes by multiplication with a non-vanishing holomorphic function as one can see from (17). Changing the homotopy class $[b]$ our function changes by multiplication with a constant which is equal to 1 when the co-Euler structure is integral.

Denote by $\mathrm{Rep}^{M,E}(\Gamma; V)$ the union of components of $\mathrm{Rep}^M(\Gamma; V)$ which consists of representations equivalent to holonomy representations of flat connections in the bundle E. Suppose E admits non-degenerate symmetric bilinear forms and let $[b]$ be a homotopy class of such forms. Let $x_0 \in M$ be a base point and denote by $\mathcal{G}(E)_{x_0,[b]}$ the group of gauge transformations which leave fixed E_{x_0} and the class $[b]$. In view of Proposition 5, $\mathcal{ST}_{\rm an}(\nabla, b, g, \alpha)$ defines a meromorphic function $\mathcal{ST}_{\rm an}^{\mathfrak{e}^*,[b]}$ on $\pi^{-1}(\mathrm{Rep}^{M,E}(\Gamma; V) \subseteq \mathcal{F}(E)/\mathcal{G}_{x_0,[b]}$. Note that $\pi : \mathcal{F}(E)/\mathcal{G}_{x_0,[b]} \to \mathrm{Rep}(\Gamma; V)$ is an principal holomorphic covering of its image which contains $\mathrm{Rep}^{M,E}(\Gamma; V)$. The absolute value of this function is the square of modified Ray–Singer torsion. We summarize these in the following theorem.

Theorem 1. *With the hypotheses above we have.*

1. If $\mathfrak{e}_1^*$ and $\mathfrak{e}_2^*$ are two co-Euler structures then

$$\mathcal{ST}_{\mathrm{an}}^{\mathfrak{e}_1^*,[b]} = \mathcal{ST}_{\mathrm{an}}^{\mathfrak{e}_2^*,[b]} \cdot e^{([\omega(\nabla,b)],D^{-1}(\mathfrak{e}_1^*-\mathfrak{e}_2^*))}$$

with $D : H_1(M;\mathbb{R}) \to H^{n-1}(M;\mathcal{O}_M)$ the Poincaré duality isomorphism.

2. If $\mathfrak{e}^*$ is integral then $\mathcal{ST}_{\mathrm{an}}^{\mathfrak{e}^*,[b]}$ is independent of $[b]$ and descends to a meromorphic function on $\mathrm{Rep}^{M,E}(\Gamma;V)$ denoted $\mathcal{ST}_{\mathrm{an}}^{\mathfrak{e}^*}$.

3. We have

$$\left|\mathcal{ST}_{\mathrm{an}}^{\mathfrak{e}^*,[b]}\right| = (T_{\mathrm{an}}^{\mathfrak{e}^*}\cdot\pi)^2. \tag{18}$$

Property 5) in Proposition 5 shows that up to multiplication with a root of unity the complex Ray–Singer torsion can be defined on all components of $\mathrm{Rep}^M(\Gamma;V)$, since $F=\oplus_k E$ is trivial for sufficiently large k.

Corollary 1. *For $\mathfrak{e}^* \in \mathfrak{Eul}^*(M;\mathbb{R})$ the function $T_{\mathrm{an}}^{\mathfrak{e}^*}$ is the absolute value of a holomorphic closed one m-form on $\mathrm{Rep}_0^M(\Gamma;V)$. Moreover (in view of Theorem 2 from section 6.1) this holomorphic closed one m-form can be written as product of a holomorphic closed one m-form on $\mathrm{Rep}^M(\Gamma;V)$ and a rational function on $\mathrm{Rep}^M(\Gamma;V)$ (the function $\mathcal{ST}_{\mathrm{an}}^{\mathfrak{e}^*}$ with $\mathfrak{e}^*$ any integral co-Euler structure).*

The above corollary can be also derived pursuing arguments similar to those in [Q85] where a similar result for the absolute value of determinant of $\bar\partial$-operator is established.

6. Milnor–Turaev and dynamical torsion

6.1. **Milnor–Turaev torsion.** Consider a smooth triangulation τ of M, and choose a collection of orientations $\mathcal{O}$ of the simplices of τ. Let $x_0 \in M$ be a base point, and set $\Gamma := \pi_1(M,x_0)$. Let V be a finite dimensional complex vector space. For a representation $\rho \in \mathrm{Rep}(\Gamma;V)$, consider the chain complex $(C_\tau^*(M;\rho), d_\tau^{\mathcal{O}}(\rho))$ associated with the triangulation τ which computes the cohomology $H^*(M;\rho)$.

Denote the set of simplexes of dimension q by $\mathcal{X}_q$, and set $k_i := \sharp(\mathcal{X}_i)\cdot\dim(V)$. Choose a collection of paths $\sigma := \{\sigma_s \mid s\in\tau\}$ from x_0 to the barycenters of τ as in section 3.1. Choose an ordering o of the barycenters and a framing ϵ of V. Using σ, o and ϵ one can identify $C_\tau^q(M;\rho)$ with $\mathbb{C}^{k_q}$. We obtain in this way a map

$$t_{\mathcal{O},\sigma,o,\epsilon} : \mathrm{Rep}(\Gamma;V) \to \mathbb{D}(k_0,\dots,k_n)$$

which sends $\mathrm{Rep}_0^M(\Gamma;V)$ to $\mathbb{D}_{\mathrm{ac}}(k_0,\dots,k_n)$. A look at the explicit definition of $d_\tau^{\mathcal{O}}(\rho)$ implies that $t_{\mathcal{O},\sigma,o,\epsilon}$ is actually a regular map between two algebraic sets. Change of $\mathcal{O},\sigma,o,\epsilon$ changes the map $t_{\mathcal{O},\sigma,o,\epsilon}$.

Recall that the triangulation τ determines Euler vector fields X_τ which together with σ determine an Euler structure $\mathfrak{e}\in\mathfrak{Eul}(M;\mathbb{Z})$, see section 3.1. Note that the ordering o induces a cohomology orientation $\mathfrak{o}$ in $H^*(M;\mathbb{R})$. In view of the arguments of [M66] or [T86] one can conclude (cf. [BH04c]):

Proposition 6. *If $\rho\in\mathrm{Rep}_0^M(\Gamma;V)$ different choices of $\tau,\mathcal{O},\sigma,o,\epsilon$ provide the same composition $\tau\cdot t_{\mathcal{O},\sigma,o,\epsilon}(\rho)$ provided they define the same Euler structure $\mathfrak{e}$ and homology orientation $\mathfrak{o}$.*

We therefore obtain a well defined complex valued rational function on $\mathrm{Rep}^M(\Gamma;V)$ called the Milnor–Turaev torsion and denoted from now on by $\mathcal{T}_{\mathrm{comb}}^{\mathfrak{e},\mathfrak{o}}$.

Theorem 2. *1. The poles and zeros of* $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$ *are contained in* $\Sigma(M)$*, the subvariety of representations* ρ *with* $H^*(M;\rho)\neq 0$.

2. The absolute value of $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}(\rho)$ *calculated on* $\rho\in \mathrm{Rep}^M_0(\Gamma;V)$ *is the modified Ray–Singer torsion* $T^{\mathfrak{e}^*}_{\mathrm{an}}(\rho)$*, where* $\mathfrak{e}^*=\mathrm{PD}(\mathfrak{e})$.

3. If $\mathfrak{e}_1$ *and* $\mathfrak{e}_2$ *are two Euler structures then* $\mathcal{T}^{\mathfrak{e}_2,\mathfrak{o}}_{\mathrm{comb}} = \mathcal{T}^{\mathfrak{e}_1,\mathfrak{o}}_{\mathrm{comb}}\cdot \det_{\mathfrak{e}_2-\mathfrak{e}_1}$ *and* $\mathcal{T}^{\mathfrak{e},-\mathfrak{o}}_{\mathrm{comb}} = (-1)^{\dim V}\cdot \mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$ *where* $\det_{\mathfrak{e}_2-\mathfrak{e}_1}$ *is the regular function on* $\mathrm{Rep}^M(\Gamma;V)$ *defined in section 4.2.*

4. When restricted to $\mathrm{Rep}^{M,E}(\Gamma;V)$*,* E *a complex vector bundle equipped with a non-degenerate symmetric bilinear form* b*,* $(\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}})^2 = \mathcal{S}\mathcal{T}^{\mathfrak{e}^*}_{\mathrm{an}}$*, where* $\mathfrak{e}^*=\mathrm{PD}(\mathfrak{e})$.

Part 1) and 3) follow from the definition and the general properties of τ, 2) can be derived from the work of Bismut–Zhang [BZ92] cf. also [BFK01], and 4) is a particular case of the main result of [BH05]. One observes that the arguments in [BFK01] (for the proof of analytic torsion equal to Reidemeister torsion via Witten deformation) hold when properly modified in the case of b instead of μ.

6.2. **Dynamical torsion.** Let X be a vector field on M satisfying (H), (EG), (L), (MS) and (NCT) from section 2.4. Choose orientations $\mathcal{O}$ of the unstable manifolds. Let $x_0\in M$ be a base point and set $\Gamma := \pi_1(M,x_0)$. Let V be a finite dimensional complex vector space. For a representation $\rho\in\mathrm{Rep}(\Gamma;V)$ consider the associated flat bundle (F_ρ,∇_ρ), and set $C^q_X(M;\rho) := \Gamma(F_\rho|_{\mathcal{X}_q})$, where $\mathcal{X}_q$ denotes the set of zeros of index q. Recall that for every path θ from $x\in\mathcal{X}$ to $y\in\mathcal{X}$ the parallel transport provides an isomorphism $(\mathrm{pt}^\rho_\theta)^{-1} : (F_\rho)_y \to (F_\rho)_x$. For $x\in\mathcal{X}_q$ and $y\in\mathcal{X}_{q-1}$ consider the expression:

$$\delta^{\mathcal{O}}_X(\rho)_{x,y} := \sum_{\theta\in\mathcal{T}(x,y)} \epsilon(\theta)(\mathrm{pt}^\rho_\theta)^{-1} \tag{19}$$

If the right hand side of (19) is absolutely convergent for all x and y they provide a linear mapping $\delta^{\mathcal{O}}_X(\rho) : C^{q-1}_X(M;\rho)\to C^q_X(M;\rho)$ which, in view of Proposition 1(3), makes $\big(C^*_X(M;\rho),\delta^{\mathcal{O}}_X(\rho)\big)$ a cochain complex. There is an integration homomorphism $\mathrm{Int}^{\mathcal{O}}_X(\rho) : \big(\Omega^*(M;F_\rho), d^{\nabla_\rho}\big)\to\big(C^*_X(M;\rho),\delta^{\mathcal{O}}_X(\rho)\big)$ which does not always induce an isomorphism in cohomology.

Let us also consider the expression

$$P_X(\rho) := \sum_{\hat\theta}(\epsilon(\hat\theta)/p(\hat\theta))\,\mathrm{tr}(\rho(\hat\theta)^{-1}) \tag{20}$$

where the sum is over all closed trajectories $\hat\theta$ of X. Again, the right hand side of (20) will in general not converge.

Proposition 7. *The set* U *of representations* ρ *in* $\mathrm{Rep}^M_0(\Gamma;V)$ *for which the right side of the formula (19) and (20) is absolutely convergent has an open interior in any connected component of* $\mathrm{Rep}^M(\Gamma;V)$*. The subset* $\Sigma\subset U$ *consisting of the representations* $\rho\in U$ *where* $\mathrm{Int}^{\mathcal{O}}_X(\rho)$ *does not induce an isomorphism in cohomology is a closed proper complex analytic subset of* U*.*

This Proposition is a consequence of exponential growth property (EG). A proof is presented in [BH04b].

As in section 6.1 we choose a collection of paths $\sigma := \{\sigma_x \mid x\in\mathcal{X}\}$ from x_0 to the zeros of X, an ordering o' of $\mathcal{X}$, and a framing ϵ of V. Using σ, o, ϵ we can identify $C^q_X(M;\rho)$ with $\mathbb{C}^{k_q}$, where $k_q := \sharp(\mathcal{X}_q)\cdot\dim(V)$. As in the previous section we obtain in this way a holomorphic map

$$t_{\mathcal{O},\sigma,o',\epsilon} : U \to \hat{\mathbb{D}}_{\mathrm{ac}}(k_0,\dots,k_n).$$

An ordering o' of $\mathcal{X}$ is given by orderings o'_q of $\mathcal{X}_q$, $q=0,1,\dots,n$. Two orderings o'_1 and o'_2 are equivalent if $o'_{1,q}$ is obtained from $o'_{2,q}$ by a permutation π_q so that $\prod_q \operatorname{sgn}(\pi_q)=1$. We call an equivalence class of such orderings a *rest point orientation.* Let us write $\mathfrak{o}'$ for the rest point orientation determined by o'. Moreover, let $\mathfrak{e}$ denote the Euler structure represented by X and σ, see Observation 1. As in the previous section, the composition $\tau\cdot t_{\mathcal{O},\sigma,o',\epsilon} : U\setminus\Sigma \to \mathbb{C}_*$ is a holomorphic map which only depends on $\mathfrak{e}$ and $\mathfrak{o}'$, and will be denoted by $\tau_X^{\mathfrak{e},\mathfrak{o}'}$. Consider the holomorphic map $P_X : U \to \mathbb{C}$ defined by formula (20). The *dynamical torsion* is the partially defined holomorphic function

$$\mathcal{T}_X^{\mathfrak{e},\mathfrak{o}'} := \tau_X^{\mathfrak{e},\mathfrak{o}'}\cdot e^{P_X} : U\setminus\Sigma \to \mathbb{C}_*.$$

The following result is based on a non-commutative version of a theorem of Hutchings–Lee and Pajitnov [H02] which will be elaborated in subsequent work [BH05b].

Theorem 3. *The partially defined holomorphic function $\mathcal{T}_X^{\mathfrak{e},\mathfrak{o}'}$ has an analytic continuation to a rational function equal to $\pm\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$.*

7. Examples

7.1. **Milnor–Turaev torsion for mapping tori and twisted Lefschetz zeta function.** Let Γ_0 be a group, $\alpha:\Gamma_0\to\Gamma_0$ an isomorphism and V a complex vector space. Denote by $\Gamma := \Gamma_0\times_\alpha\mathbb{Z}$ the group whose underlying set is $\Gamma_0\times\mathbb{Z}$ and group operation $(g',n)*(g'',m) := (\alpha^m(g')\cdot g'', n+m)$. A representation $\rho:\Gamma\to\mathrm{GL}(V)$ determines a representation $\rho_0(\rho):\Gamma_0\to\mathrm{GL}(V)$ the restriction of ρ to $\Gamma_0\times 0$ and an isomorphism of V, $\theta(\rho)\in\mathrm{GL}(V)$.

Let (X,x_0) be a based point compact space with $\pi_1(X,x_0)=\Gamma_0$ and $f:(X,x_0)\to(X,x_0)$ a homotopy equivalence. For any integer k the map f induces the linear isomorphism $f^k : H^k(X;V)\to H^k(X;V)$ and then the standard Lefschetz zeta function

$$\zeta_f(z) := \frac{\prod_{k\ \mathrm{even}}\det(I-zf^k)}{\prod_{k\ \mathrm{odd}}\det(I-zf^k)}.$$

More general if ρ is a representation of Γ then f and $\rho=(\rho_0(\rho),\theta(\rho))$ induce the linear isomorphisms $f^k_\rho : H^k(X;\rho_0(\rho))\to H^k(X;\rho_0(\rho))$ and then the ρ-twisted Lefschetz zeta function

$$\zeta_f(\rho,z) := \frac{\prod_{k\ \mathrm{even}}\det(I-zf^k_\rho)}{\prod_{k\ \mathrm{odd}}\det(I-zf^k_\rho)}.$$

Let N be a closed connected manifold and $\varphi:N\to N$ a diffeomorphism. Without loss of generality one can suppose that $y_0\in N$ is a fixed point of φ. Define the mapping torus $M=N_\varphi$, the manifold obtained from $N\times I$ identifying $(x,1)$ with $(\varphi(x),0)$. Let $x_0=(y_0,0)\in M$ be a base point of M. Set $\Gamma_0 := \pi_1(N,n_0)$ and denote by $\alpha:\pi_1(N,y_0)\to\pi_1(N,y_0)$ the isomorphism induced by φ. We are in the situation considered above with $\Gamma=\pi_1(M,x_0)$. The mapping torus structure on M equips M with a canonical Euler structure $\mathfrak{e}$ and canonical homology orientation

$\mathfrak{o}$. The Euler structure $\mathfrak{e}$ is defined by any vector field X with $\omega(X) < 0$ where $\omega := p^*dt \in \Omega^1(M;\mathbb{R})$; all are homotopic. The Wang sequence

$$\cdots \to H^*(M;\mathbb{F}_\rho) \to H^*(N;i^*(\mathbb{F}_\rho)) \xrightarrow{\varphi_\rho^*-\mathrm{id}} H^*(N;i^*(\mathbb{F}_\rho)) \to H^{*+1}(M;\mathbb{F}_\rho) \to \cdots \tag{21}$$

implies $H^*(M;\mathbb{F}_\rho) = 0$ iff $\det(I - \varphi_\rho^k) \neq 0$ for all k. The cohomology orientation is derived from the Wang long exact sequence for the trivial one dimensional real representation. For details see [BH04c]. We have

Proposition 8. *With these notations* $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}(\rho) = \zeta_\varphi(\rho,1)$.

This result is known cf. [BJ96]. A proof can be also derived easily from [BH04c].

7.2. **Vector fields without rest points and Lyapunov cohomology class.** Let X be a vector field without rest points, and suppose X satisfies (L) and (NCT). As in the previous section X defines an Euler structure $\mathfrak{e}$. Consider the expression (20). By Theorem 3 we have:

Observation 3. *With the hypothesis above there exists an open set* $U \subseteq \mathrm{Rep}^M(\Gamma;V)$ *so that* $P_X(\rho) := \sum_{\hat\theta}(\epsilon(\hat\theta)/p(\hat\theta))\,\mathrm{tr}(\rho(\hat\theta)^{-1})$ *is absolutely convergent for* $\rho \in U$. *The function* e^{P_X} *has an analytic continuation to a rational function on* $\mathrm{Rep}^M(\Gamma;V)$ *equal to* $\pm\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$. *The set* U *intersects non-trivially each connected component of* $\mathrm{Rep}^M(\Gamma;V)$.

7.3. **The Alexander polynomial.** If M is obtained by surgery on a framed knot, and $\dim V = 1$, then $\pi_1(M)/[\pi_1(M),\pi_1(M)] = \mathbb{Z}$, and the function $(z-1)^2\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$ equals the Alexander polynomial of the knot, see [T02]. Any twisted Alexander polynomial of the knot can be also recovered from $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$ for V of higher dimension. One expects that passing to higher dimensional representations $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$ captures even more subtle knot invariants.

8. Applications

8.1. **The invariant $A^{\mathfrak{e}^*}(\rho_1,\rho_2)$.** Let M be a V-acyclic manifold and $\mathfrak{e}^*$ a co-Euler structure. Using the modified Ray–Singer torsion we define a $\mathbb{R}/\pi\mathbb{Z}$ valued invariant (which resembles the Atiyah–Patodi–Singer spectral flow) for two representations ρ_1, ρ_2 in the same component of $\mathrm{Rep}^M_0(\Gamma;V)$.

A holomorphic path in $\mathrm{Rep}^M_0(\Gamma;V)$ is a holomorphic map $\tilde\rho : U \to \mathrm{Rep}^M_0(\Gamma;V)$ where U is an open neighborhood of the segment of real numbers $[1,2]\times\{0\} \subset \mathbb{C}$ in the complex plane. For a co-Euler structure $\mathfrak{e}^*$ and a holomorphic path $\tilde\rho$ in $\mathrm{Rep}^M_0(\Gamma;V)$ define

$$\arg^{\mathfrak{e}^*}(\tilde\rho) := \Re\left(2/i\int_1^2 \frac{\partial(T^{\mathfrak{e}^*}_{\mathrm{an}}\circ\tilde\rho)}{T^{\mathfrak{e}^*}_{\mathrm{an}}\circ\tilde\rho}\right) \mod \pi. \tag{22}$$

Here, for a smooth function φ of complex variable z, $\partial\varphi$ denotes the complex valued 1-form $(\partial\varphi/\partial z)dz$ and the integration is along the path $[1,2]\times 0 \subset U$. Note that

Observation 4. *1. Suppose E is a complex vector bundle with a non-degenerate bilinear form b, and suppose $\tilde\rho$ is a holomorphic path in* $\mathrm{Rep}^{M,E}_0(\Gamma;V)$. *Then*

$$\arg^{\mathfrak{e}^*}(\tilde\rho) = \arg\Big(\mathcal{S}\mathcal{T}^{\mathfrak{e}^*,[b]}_{\mathrm{an}}(\tilde\rho(2))/\mathcal{S}\mathcal{T}^{\mathfrak{e}^*,[b]}_{\mathrm{an}}(\tilde\rho(1))\Big) \mod \pi.$$

As consequence

2. If $\tilde\rho'$ and $\tilde\rho''$ are two holomorphic paths in $\mathrm{Rep}_0^M(\Gamma;V)$ with $\tilde\rho'(1)=\tilde\rho''(1)$ and $\tilde\rho'(2)=\tilde\rho''(2)$ then

$$\arg^{\mathfrak{e}^*}(\tilde\rho')=\arg^{\mathfrak{e}^*}(\tilde\rho'')\mod \pi.$$

3. If $\tilde\rho'$, $\tilde\rho''$ and $\tilde\rho'''$ are three holomorphic paths in $\mathrm{Rep}_0^M(\Gamma;V)$ with $\tilde\rho'(1)=\tilde\rho'''(1)$, $\tilde\rho'(2)=\tilde\rho''(1)$ and $\tilde\rho''(2)=\tilde\rho'''(2)$ then

$$\arg^{\mathfrak{e}^*}(\tilde\rho''')=\arg^{\mathfrak{e}^*}(\tilde\rho')+\arg^{\mathfrak{e}^*}(\tilde\rho'')\mod \pi.$$

Observation 4 permits to define a $\mathbb{R}/\pi\mathbb{Z}$ valued numerical invariant $A^{\mathfrak{e}^*}(\rho_1,\rho_2)$ associated to a co-Euler structure $\mathfrak{e}^*$ and two representations ρ_1,ρ_2 in the same connected component of $\mathrm{Rep}_0^M(\Gamma;V)$. If there exists a holomorphic path with $\tilde\rho(1)=\rho_1$ and $\tilde\rho(2)=\rho_2$ we set

$$A^{\mathfrak{e}^*}(\rho_1,\rho_2):=\arg^{\mathfrak{e}^*}(\tilde\rho)\mod\pi.$$

Given any two representations ρ_1 and ρ_2 in the same component of $\mathrm{Rep}_0^M(\Gamma;V)$ one can always find a finite collection of holomorphic paths $\tilde\rho_i$, $1\leq i\leq k$, in $\mathrm{Rep}_0^M(\Gamma;V)$ so that $\tilde\rho_i(2)=\tilde\rho_{i+1}(1)$ for all $1\leq i<k$, and such that $\tilde\rho_1(1)=\rho_1$ and $\tilde\rho_k(2)=\rho_2$. Then take

$$A^{\mathfrak{e}^*}(\rho_1,\rho_2):=\sum_{i=1}^k\arg^{\mathfrak{e}^*}(\tilde\rho_i)\mod\pi.$$

In view of Observation 4 the invariant is well defined, and if $\mathfrak{e}^*$ is integral it is actually well defined in $\mathbb{R}/2\pi\mathbb{Z}$. This invariant was first introduced when the authors were not fully aware of "the complex Ray–Singer torsion." The formula (22) is a more or less obvious expression of the phase of a holomorphic function in terms of its absolute value, the Ray–Singer torsion, as positive real valued function. By Theorem 2 the invariant can be computed with combinatorial topology and by section 7 quite explicitly in some cases. If the representations ρ_1,ρ_2 are unimodular then the co-Euler structure is irrelevant. It is interesting to compare this invariant to the Atiyah–Patodi–Singer spectral flow; it is not the same but are related.

8.2. **A question in dynamics.** Let Γ be a finitely presented group, V a complex vector space and $\mathrm{Rep}(\Gamma;V)$ the variety of complex representations. Consider triples $\underline{a}:=\{a,\epsilon_-,\epsilon_+\}$ where a is a conjugacy class of Γ and $\epsilon_\pm\in\{\pm1\}$. Define the rational function $\mathrm{let}_{\underline{a}}:\mathrm{Rep}(\Gamma;V)\to\mathbb{C}$ by

$$\mathrm{let}_{\underline{a}}(\rho):=\Big(\det\big(\mathrm{id}-(-1)^{\epsilon_-}\rho(\alpha)^{-1}\big)\Big)^{(-1)^{\epsilon_-+\epsilon_+}}$$

where $\alpha\in\Gamma$ is a representative of a.

Let (M,x_0) be a V-acyclic manifold and $\Gamma=\pi_1(M,x_0)$. Note that $[S^1,M]$ identifies with the conjugacy classes of Γ. Suppose X is a vector field satisfying (L) and (NCT). Every closed trajectory $\hat\theta$ gives rise to a conjugacy class $[\hat\theta]\in[S^1,M]$ and two signs $\epsilon_\pm(\hat\theta)$. These signs are obtained from the differential of the return map in normal direction; $\epsilon_-(\hat\theta)$ is the parity of the number of real eigenvalues larger than $+1$ and $\epsilon_-(\hat\theta)$ is the parity of the number of real eigenvalues smaller than -1. For a simple closed trajectory, i.e. of period $p(\hat\theta)=1$, let us consider the triple $\underline{\hat\theta}:=\big([\hat\theta],\epsilon_-(\hat\theta),\epsilon_+(\hat\theta)\big)$. This gives a (at most countable) set of triples as in the previous paragraph.

Let $\xi \in H^1(M;\mathbb{R})$ be a Lyapunov cohomology class for X. Recall that for every R there are only finitely many closed trajectories $\hat\theta$ with $-\xi([\hat\theta]) \leq R$. Hence, we get a rational function $\zeta_R^{X,\xi} : \mathrm{Rep}(\Gamma;V) \to \mathbb{C}$

$$\zeta_R^{X,\xi} := \prod_{-\xi([\hat\theta])\leq R} \mathrm{let}_{\underline{\hat\theta}}$$

where the product is over all triples $\underline{\hat\theta}$ associated to simple closed trajectories with $-\xi([\hat\theta]) \leq R$. It is easy to check that formally we have

$$\lim_{R\to\infty} \zeta_R^{X,\xi} = e^{P_X}.$$

It would be interesting to understand in what sense (if any) this can be made precise. We conjecture that there exists an open set with non-empty interior in each component of $\mathrm{Rep}(\Gamma;V)$ on which we have true convergence. In fact there exist vector fields X where the sets of triples are finite in which case the conjecture is obviously true.

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