TOPOLOGY OF REAL AND ANGLE VALUED MAPS
AND GRAPH REPRESENTATIONS.

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ABSTRACT. Using graph representations a new class of computable topological invariants associated with a tame real or angle valued map were recently introduced, providing a theory which can be viewed as an alternative to Morse-Novikov theory for real or angle valued Morse maps. The invariants are "barcodes" and "Jordan cells". From them one can derive all familiar topological invariants which can be derived via Morse-Novikov theory, like the Betti numbers and in the case of angle valued maps also the Novikov Betti numbers and the monodromy. Stability results for bar codes and the homotopy invariance of the Jordan cells are the key results, and two new polynomials for any $r$ associated to a continuous nonzero complex valued map provide potentially interesting refinements of the Betti numbers and of the Novikov Betti numbers. In our theory the bar codes which are intervals with ends critical values/angles, the Jordan cells and the "canonical long exact sequence" of a tame map are the analogues of instantons between rest points, closed trajectories and of the Morse-Smale complex of the gradient of a Morse function in the Morse-Novikov theory.

1. Introduction

This is essentially the lecture delivered at the Congress of the Romanian mathematicians, Brasov, June 2011 under the title "New topological invariants for angle valued maps".

The presentation summarizes work done in [1], [2] and [3]. Using graph representations and inspired by persistence theory [7] [4] and [1] a new class of computable topological invariants associated with a tame real or angle valued maps were recently introduced, providing a theory which can be viewed as an alternative to Morse-Novikov theory for real or angle valued Morse maps. The invariants are "barcodes" and "Jordan cells" and, when the underlying space is a simplicial complex and the map is simplicial, can be calculated by algorithms of the same complexity as the ones which calculate the Betti numbers. From them one can derive all familiar topological invariants which can be derived via Morse-Novikov theory and a few more. Stability results for bar codes, Theorem 7.3, and homotopy invariance of the Jordan cells and of the cardinality of some sets of bar codes, Theorems 7.1, 7.2, are the key results, and two new polynomials associated to a continuous nonzero complex value map provide potentially interesting refinements of the Betti numbers and of the Novikov-Betti numbers.

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In our theory the bar codes which are intervals with ends critical values/angles, the Jordan cells and the canonical long exact sequence of a tame map are the analogues of the instantons, the closed trajectories and the Morse Smale complex of the gradient of a Morse function in the Morse Novikov theory.

Note that almost all maps are tame in particular all Morse maps on a smooth manifold or on a stratified space and all simplicial maps on a simplicial complex. In the case of angle (circle) valued maps the space of tame maps have the same homotopy type as the space of all continuous maps. This is not the case of Morse maps. Note also that in case of Morse angle valued maps the cardinality of instants and closed trajectories might not be finite, but the cardinality of the set of bar codes and Jordan cells is always finite. This presentation contains Theorem 7.3 which was not present in my lecture at the Romanian congress. The organization of the material is also slightly different.

2. Topology

Let $\kappa$ be a field and $\overline{\kappa}$ its algebraic closure. Denote by $\kappa[t, t^{-1}]$ the ring of the Laurent polynomials and by $\kappa[[t, t^{-1}]]$ the field of Laurent power series with coefficients in $\kappa$. Clearly $\kappa[t, t^{-1}] \subset \kappa[[t, t^{-1}]]$.

Let $X$ be a compact ANR. Denote by $H_r(X)$ the singular homology with coefficients in $\kappa$ and call $\beta_r(X) = \dim H_r(X)$ the $r$–th Betti number of $X$.

Let $\xi \in H^1(X, \mathbb{Z})$. For the pair $(X, \xi)$ consider $\tilde{X} \to X$ the infinite cyclic cover associated with $\xi$, precisely the pull back of the canonical infinite cyclic cover $\mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$ by a map $f : X \to S^1$ representing $\xi$. Let $T : \tilde{X} \to \tilde{X}$ be the deck transformation.

Consider $NH_r(X, \xi) = H_r(\tilde{X}) \otimes_{\kappa[[t, t^{-1}]]} \kappa[[t, t^{-1}]]$. This is the Novikov homology which is a vector space over the field $\kappa[[t, t^{-1}]]$. We call $\beta_r(X; \xi) = \dim_{\kappa[[t, t^{-1}]]} NH_r(X; \xi)$ the $r$–th Novikov-Betti number of $(X, \xi)$.

Let $V(\xi) := \ker\{H_r(\tilde{X}) \to NH_r(X; \xi)\}$ be the kernel of the linear map induced by tensoring $H_r(\tilde{X})$ with $\kappa[[t, t^{-1}]]$ over the ring $\kappa[[t, t^{-1}]]$. The $\kappa[[t, t^{-1}]]$ module $V(\xi)$ is a finite dimensional vector space over the field $\kappa$. The multiplication by $t$ can be viewed as a $\kappa$– linear isomorphism $T(\xi) : V(\xi) \to V(\xi)$. The pair $(V(\xi), T(\xi))$ will be referred to as the monodromy associated with $\xi$.

3. Tame maps

**Definition 3.1.** A continuous map $f : X \to \mathbb{R}$ resp. $f : X \to S^1$, $X$ a compact ANR, is tame if the following hold:

1. Any fiber $X_\theta = f^{-1}(\theta)$ is the deformation retract of an open neighborhood.
(2) Away from a finite set of numbers resp. angles $\Sigma = \{s_1, s_2, \cdots, s_r\} \subset \mathbb{R}$, resp. $S^1$ the restriction of $f$ to $X \setminus f^{-1}(\Sigma)$ is a fibration.

For any real resp. angle valued tame map we have the finite set of numbers $s_1 < s_2 < \cdots < s_{N-1} < s_N$ resp. $0 < \theta_1 < \theta_2 < \cdots < \theta_{m-1} < \theta_m < 2\pi$ where the homotopy type of the fibers change. The numbers $s_i$ resp. $\theta_i$ are the critical values of the tame map $f$.

In the case of a real valued map the Betti numbers and in the case of an angle valued map the Novikov-Betti numbers and the monodromy can be recovered from the invariants associated with the tame map. These invariants are the bar codes and the Jordan cells and are computable, cf section 10.

4. Bar codes and Jordan cells. The invariants of a tame map $f$.

Bar codes are finite sets $I$ of real numbers of four types:

1. Type 1, closed, $[a, b]$ with $a \leq b$,
2. Type 2, open, $(a, b)$ with $a < b$,
3. Type 3, left open right closed, $(a, b]$ with $a < b$,
4. Type 4, left closed right open, $[a, b)$ with $a < b$.

Jordan cells are pairs $J = (\lambda \in \mathbb{R}, n \in \mathbb{Z}_{\geq 0})$. A Jordan cell should be interpreted as a a matrix

$$T(\lambda, k) = \begin{pmatrix}
\lambda & 1 & 0 & \cdots & 0 \\
0 & \lambda & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & \cdots & \lambda \\
0 & \cdots & 0 & 0 & \lambda
\end{pmatrix}$$

(1)

For a tame map $f : X \to \mathbb{R}$ and any integer $r$ we associate cf. section 6 a collection of of bar codes

$$B_r(f) = B_{r}^{\circ}(f) \cup B_{r}^{\circ,\circ}(f) \cup B_{r}^{\circ,\circ,\circ}(f) \cup B_{r}^{\circ,\circ,\circ,\circ}(f)$$

with $B_{r}^{\circ}, B_{r}^{\circ,\circ}, B_{r}^{\circ,\circ,\circ}$ of type 1,2,3,4. whose ends $a, b$ are critical values.

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with $B_{r}^{\circ}, B_{r}^{\circ,\circ}, B_{r}^{\circ,\circ,\circ}$ of type 1,2,3,4 with ends $a, b$ and Jordan cells $J(r), J = (\lambda(J), k(J)), \lambda(J) \in \mathbb{R}, k(J) \in \mathbb{Z}_{>0}$. The ends $a, b$ are the first a critical angle $\theta_i$ the second of the form $\theta_j + 2\pi k$, $\theta_j$ a critical angle $k$ a non negative integer.

In both cases (real and angle valued maps) it will be convenient to record the bar codes $B_{r}^{\circ}(f) \cup B_{r}^{\circ}(f)$ as a configuration $C_r(f)$ of points in the plane $\mathbb{R}^2$ resp. the cylinder $\mathbb{T}$ defined by $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}$ or equivalently $\mathbb{C} \setminus 0$, see picture below. Precisely $\mathbb{T}$ is the quotient space of $\mathbb{R}^2$ the Euclidean plane, by the additive group of integers $\mathbb{Z}$, w.r. to the action $\mu : \mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by $\mu(n; (x, y)) = (x + 2\pi n, y + 2\pi n)$. The identification of $\mathbb{T}$ to $\mathbb{C} \setminus 0$ is done via the map $(x, y) \to e^{ix}$. 

One denotes by $\Delta \subset \mathbb{R}^2$ resp. $\Delta \subset \mathbb{T}$ the diagonal of $\mathbb{R}^2$ resp. the quotient of the diagonal of $\mathbb{R}^2$ by the group $\mathbb{Z}$. The points above or on diagonal, $(x, y), x \leq y$, will
be used to record closed bar codes \([x, y] \in B_\mathbb{R}(f)\) and the points below the diagonal, \((x, y), x > y\) to record open bar codes \((y, x) \in B_{\mathbb{R}^{-1}}\). When \(\mathbb{T}\) is identified to \(\mathbb{C} \setminus 0\) the diagonal \(\Delta\) corresponds to the unit circle, the points above or on the diagonal to the the points outside or on the unit circle and those below diagonal to points inside unit circle.

If we identify a point in \((x, y) \in \mathbb{R}^2\) with \(z = x + iy\), hence \(\mathbb{R}^2\) to \(\mathbb{C}\), and \(\mathbb{T}\) to \(\mathbb{C} \setminus 0\) it is convenient to regard \(C_r(f)\) as the monic polynomial \(P_r^f(z)\) whose roots are the elements of \(C_r(f)\). In the second case \(P_r^f(z)\) is a monic polynomial with nonzero free coefficient since the roots are all nonzero.

Figure 1. Configurations

The first picture is the configuration \(C_{r_r}(f)\) for a tame real valued map the second for an angle valued map. The points in blue (above or on the diagonal resp. outside or on the unit circle) represent closed \(r\)– bar codes, the ones in red (below diagonal resp. inside the unit circle) open \((r - 1)\)– bar codes.
5. Graph representation

To describe $\mathcal{B}_r(f)$ and $\mathcal{J}_r(f)$ we will use two graphs $\mathcal{Z}$ for real valued maps, and $G_{2m}$ for angle valued maps. The graph $\mathcal{Z}$ has vertices $x_i$, $i \in \mathbb{Z}$, and edges $a_i$ from $x_{2i-1}$ to $x_{2i}$ and $b_i$ from $x_{2i+1}$ to $x_{2i}$.

\[ \cdots \xleftarrow{b_{i-1}} x_{2i-1} \xrightarrow{a_i} x_{2i} \xleftarrow{b_i} x_{2i+1} \xrightarrow{a_{i+1}} x_{2i+2} \xleftarrow{b_{i+1}} \cdots \]

The graph $\mathcal{Z}$

and $\Gamma = G_{2m}$ has vertices $x_1, x_2, \ldots, x_{2m}$ and edges $a_i$, $1 \leq i \leq m$, and $b_i$, $1 \leq i \leq m-1$, as above and $b_m : x_1 \to x_{2m}$.

Let $\kappa$ be a fixed field. A $\Gamma$-representation $\rho$ is an assignment which to each vertex $x$ of $\Gamma$ assigns a finite dimensional vector space $V_x$ and to each oriented arrow from the vertex $x$ to the vertex $y$ a linear map $V_x \to V_y$. The concepts of morphism, isomorphism= equivalence, sum, direct summand, zero and nontrivial representations are obvious.

A $\mathcal{Z}$-representation is given by the collection

\[ \rho := \left\{ V_r, \quad \alpha_i : V_{2i-1} \to V_{2i}, \quad \beta_i : V_{2i+1} \to V_{2i} \right\}, \quad r, i \in \mathbb{Z}, \]

while a $G_{2m}$ representation by the collection

\[ \rho := \left\{ V_r, \quad \alpha_i : V_{2i-1} \to V_{2i}, \quad \beta_i : V_{2i+1} \to V_{2i} \right\}, \quad 1 \leq r \leq 2m, \quad 1 \leq i \leq m, \quad V_{2m+1} = V_1. \]

Both will be abbreviated by $\rho = \{ V_r, \alpha_i, \beta_i \}$.

A finitely supported $\mathcal{Z}$-representation\(^1\), resp. an arbitrary $G_{2m}$-representation can be uniquely decomposed as a sum of indecomposable representations. In the case of the graph $\mathcal{Z}$ the indecomposable representations are indexed by one of the four types of intervals (bar codes) with ends $i, j \in \mathbb{Z}$, $i \leq j$ for type (1) and $i < j$

\(^1\)i.e. all but finitely many vector spaces $V_x$ have dimension zero
for types (2), (3) and (4). For reasons which will be understandable later on we regard the ends $i, j$ as associated to the vertices $x_{2i}, x_{2j}$. We refer to both the indecomposable representation and the interval as bar code.

Here is the description of all bar codes (for the graph $\mathcal{Z}$).

1. $\rho([i, j]), i \leq j$ has $V_r = \kappa$ for $r = \{2i, 2i+1, \cdots, 2j\}$ and $V_r = 0$ if $r \not\in [2i, 2j]$, 
2. $\rho([i, j]), i < j$ has $V_r = \kappa$ for $r = \{2i, 2i+1, \cdots, 2j\}$ and $V_r = 0$ if $r \not\in [2i, 2j+1]$, 
3. $\rho([i, j]), i < j$ has $V_r = \kappa$ for $r = \{2i, 2i+1, \cdots, 2j\}$ and $V_r = 0$ if $r \not\in [2i+1, 2j]$, 
4. $\rho([i, j]), i < j$ has $V_r = \kappa$ for $r = \{2i, 2i+1, \cdots, 2j\}$ and $V_r = 0$ if $r \not\in [2i+1, 2j-1]$, 

with all $\alpha_i$ and $\beta_i$ the identity provided that the source and the target are both non zero. The above description is implicit in [9].

In the case of the graph $G_{2m}$ for simplicity we consider the field $\kappa$ algebraically closed. The indecomposable representations are indexed by similar intervals (bar codes) with ends $i, j + mk, 1 \leq i, j \leq m, k \in \mathbb{Z}_{\leq 0}$, $i \leq j$ with $1 \leq i \leq m$ and by Jordan cells. Again $i, j$ are associated to vertices $x_{2i}, x_{2j}$. We refer to both the indecomposable representation and the interval resp. the Jordan cell as bar code resp. Jordan cell.

**Type I: (bar codes for the graph $G_{2m}$)** For any triple of integers $\{i, j, k\}, 1 \leq i, j \leq m, k \geq 0$, we have the representations denoted by

1. $\rho^I([i, j]; k) \equiv \rho^I([i, j + mk]), \quad 1 \leq i, j \leq m, k \geq 0$
2. $\rho^I([i, j]; k) \equiv \rho^I([i, j + mk]), \quad 1 \leq i, j \leq m, k \geq 0$
3. $\rho^I([i, j]; k) \equiv \rho^I([i, j + mk]), \quad 1 \leq i, j \leq m, k \geq 0$
4. $\rho^I([i, j]; k) \equiv \rho^I([i, j + mk]), \quad 1 \leq i, j \leq m, k \geq 0$

described as follows.

Suppose the vertices of $G_{2m}$ are located counter-clockwise on the unit circle with evenly indexed vertices $\{x_2, x_4, \cdots x_{2m}\}$ corresponding to the angles $0 < s_1 < s_2 < \cdots < s_m < 2\pi$. Draw the spiral curve for $a = s_i$ and $b = s_j + 2\pi k$ with the ends a black or an empty circle if the end of the bar code is closed or open (see picture below for $k = 2$).

![Figure 2. The spiral for $[i, j + 2m]$.](image)

Denote by $V_i$ the vector space generated by the intersection points of the spiral with the radius corresponding to the vertex $x_i$. Let $\alpha_i$ resp. $\beta_i$ be defined as follows:
a generator $e$ of $V_{2i+1}$ is sent to the generator $e'$ of $V_{2i}$ if connected by a piece of spiral and to 0 otherwise.

**Type II:** $\rho^{II}(\lambda, k)$ defined by

$$\rho^{II}(\lambda, k) = \{V'_i = \lambda^k, \alpha'_i = T(\lambda, k), \alpha'_i = Id \ i \neq 1, \beta'_i = Id\}.$$ (2)

For a $Z$-representation or a $G_{2m}$-representation $\rho$ one denotes by $B(\rho)$ the set of all bar codes and write $B(\rho)$ as $B(\rho) = B^c(\rho) \sqcup B^o(\rho) \sqcup B^{o,c}(\rho) \sqcup B^{c,o}(\rho)$ where $B^c(\rho)$, $B^o(\rho)$, $B^{o,c}(\rho)$ and $B^{c,o}(\rho)$ are the subsets of closed, open, left open right closed, and right open left closed bar codes.

For a $G_{2m}$ representation $\rho$ one denotes by $J(\rho)$ the set of all Jordan cells resp. Jordan cells.

6. **The invariants associated to a tame map $f$.**

Given a tame map $f: X \to \mathbb{R}$ resp. $f: X \to S^1$ consider the critical values resp. the critical angles $\theta_1 < \theta_2 < \cdots < \theta_m$. In the second case we have $0 < \theta_1 < \cdots < \theta_m \leq 2\pi$. Choose $t_i$, $i = 1, 2, \ldots, m$, with $\theta_1 < t_1 < \theta_2 < \cdots < t_{m-1} < \theta_m < t_m$. In the second case choose $t_m$ s.t. $2\pi < t_m < \theta_1 + 2\pi$.

The tameness of of the map when $f$ is a real valued map induces the diagram

$$\cdots \xrightarrow{b_{i-1}} X_{t_{i-1}} \xrightarrow{a_i} X_{\theta_i} \xleftarrow{b_i} X_{t_i} \xrightarrow{a_{i+1}} X_{\theta_{i+1}} \xleftarrow{b_{i+1}} \cdots$$

and when $f$ is angle valued map the diagram

Here $X_t = f^{-1}(t)$ resp. $X_\theta = f^{-1}(\theta)$. Different choices of $t_i$ lead to different diagrams but all homotopy equivalent.

For any $r \leq \dim X$ let $\rho_r = \rho(f)$ be the $Z$- resp. $G_{2m}$-representation associated to the tame map $f$ defined by

$$V_{2i} = H_r(X_{\theta_i}), V_{2i+1} = H_r(X_{t_i}), \quad \alpha_i : V_{2i-1} \to V_{2i}, \quad \beta_i : V_{2i+1} \to V_{2i}$$

for brevity in writing denote the critical values of both, real and angle valued maps by $\theta_i$. 

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\[for\ \text{brevity in writing denote the critical values of both, real and angle valued maps by } \theta_i.\]
with $\alpha_i$ and $\beta_i$ the linear maps induced by the continuous maps $a_i$ and $b_i$ in the diagrams above. Here and below $H_r(Y)$ denotes the singular homology in dimension $r$ with coefficients in a fixed chosen field $\kappa$.

In order to relate the indecomposable components of $\rho_r$ to the critical values of $f$, for a real valued map one converts the intervals $\{i, j\}$ into $\{\theta_i, \theta_j\}$ and for an angle valued map the intervals $\{i, j + km\}$, $1 \leq i, j \leq m$, into the intervals $\{\theta_i, \theta_j + 2\pi k\}$.

**Definition:** The sets $B_r(f) := B(\rho_r)$, $r = 1, 2, \cdots \dim X$, with the intervals $I$ converted into intervals with ends $\theta_i$’s and $(\theta_i + 2\pi k)$’s and $J_r(f) := J(\rho_r)$ are the r-invariants of the map $f$.

For a real valued map one has only bar codes, while for an angle valued map one has bar codes and Jordan cells.

We denote by $(V_r(f), T_r(f))$ the pair

$$(V_r(f), T_r(f)) = \bigoplus_{(\lambda, k) \in J_r(f)} (\mathbb{R}^k, T(\lambda, k))$$

and refer to it as as the $r$-monodromy of the angle valued map $f$.

7. The main results

Recall that for $f : X \to \mathbb{R}$ a continuous map denote by: $X_t = f^{-1}(t)$ and $X_{[t_1, t_2]} = f^{-1}[t_1, t_2]$. For $f : X \to S^1$ a continuous map denote by: $X_\theta = f^{-1}(\theta)$, $\theta$ angle, $\xi_f \in H^1(X; \mathbb{Z})$ the cohomology class represented by $f$ and $\tilde{f} : \tilde{X} \to \mathbb{R}$ the lift of $f$ to the infinite cyclic cover of $X \to X$ defined by $\xi_f$. The covering $\tilde{X} \to X$ is the pull back of the infinite cyclic cover $\mathbb{R} \to S^1$ by $f$ or any other map in the class $\xi_f$. For $I$ an interval $\subset \mathbb{R}$ denote by $n_\theta(I) = \sharp\{k \in \mathbb{Z} \mid \theta + 2\pi k \in I\}$ and for $J$ a Jordan cell write $J = (\lambda(J), k(J))$.

**Theorem 7.1.**

1. If $f : X \to \mathbb{R}$ is a tame map then:

$$\beta_r(X_t) = \sharp\{I \in B_r(f) \mid t \in I\}$$

$\dim \text{im}(H_r(X_t) \to H_r(X)) = \sharp\{I \in B_r^c(f) \mid t \in I\}$

$\beta_r(X) = \sharp\{B_r^c(f) + \sharp B_r^{c-1}(f)\}.$

2. If $f : X \to S^1$ is a tame map then:

$$\beta_r(X_\theta) = \sum_{I \in B_r(f)} n_\theta(I) + \sum_{J \in J_r(f)} k(J)$$

$\dim \text{im}(H_r(X_\theta) \to H_r(X)) = \sharp\{I \in B_r^c(f) \mid \theta \in I\} + \sharp\{(\lambda, k) \in J_r(f) \mid \lambda(J) = 1\}$

$$\beta_r(X) = \begin{cases} \sharp\{B_r^c(f) + \sharp B_r^{c-1}(f)\} + \\ \sharp\{(\lambda, k) \in J_r(f) \mid \lambda(J) = 1\} + \\ \sharp\{(\lambda, k) \in J_{r-1}(f) \mid \lambda(J) = 1\} \end{cases}$$

$$\beta N_r(X; \xi_f) = \sharp\{B_r^c(f) + \sharp B_r^{c-1}(f)\}.$$\footnote{we use the symbol “{” for both “(” and “[ ” and “)” for both “)” and “]”.}
Theorem 7.2. Let \( f : X \rightarrow S^1 \) be a tame map and \( \tilde{B}(f) := \{ I' = I + 2\pi k \mid k \in \mathbb{Z}, I \in B(f) \} \). Then:

1.

\[
\beta_r(\tilde{X}_{[a,b]}) = \begin{cases} 
\sharp\{ I' \in \tilde{B}_r(f), I' \cap [a,b] \text{closed and } \neq 0 \} + \\
\sum_{J \in \mathcal{J}_r(f)} k(J) 
\end{cases}
\]

\[
\dim \text{im}(H_r(\tilde{X}_{[a,b]}) \rightarrow H_r(\tilde{X})) = \begin{cases} 
\sharp\{ I \in \tilde{B}_r(f), I \cap [a,b] \neq 0 \} + \\
\sum_{J \in \mathcal{J}_r(f)} k(J) 
\end{cases}
\]

\[
\dim \text{im}(H_r(\tilde{X}_{[a,b]}) \rightarrow H_r(X)) = \begin{cases} 
\sharp\{ I \in \tilde{B}_r([a,b] \cap (I + 2\pi k) \neq 0 \} + \\
\sharp\{ I \in B_{r-1}(f), I + 2\pi k \subset [a,b] \} + \\
\sharp\{ J \in \mathcal{J}_r(f) \lambda(J) = 1 \} + \\
\sharp\{ J \in \mathcal{J}_{r-1}(f) \lambda(J) = 1 \}. 
\end{cases}
\]

2. \( V_r(\xi_f) := \ker(H_r(\tilde{X}) \rightarrow H_r^N(X;\xi_f)) \) is a finite dimensional \( \kappa \)-vector space and \( (V_r(\xi_f) \otimes \pi, T_r(\xi_f) \otimes \pi) = (V_r(f), T_r(f)) \).

3. \( H_r(X) = \kappa[t^{-1}, t]^N \oplus V_r(\xi_f) \) as \( \kappa[t^{-1}, t] \)-modules with \( N = \beta N_r(f) = z B_r^O(f) + z B_{r-1}^O(f) \).

Theorems 7.1 and 7.2 imply that for \( f \) real valued the number \( z B_r^O + z B_{r-1}^O \) is a homotopy invariant and for \( f \) angle valued the number \( \beta B_r^O + \beta B_{r-1}^O \) and the collection \( J_r(f) \) are homotopy invariants. Therefore \( C_r(f) \) can be regarded as points in the symmetric product \( S^N(\xi_f)(\mathbb{R}) \) resp. \( S^N_\Sigma(\xi_f)(\mathbb{T}) \) which are nice stratified spaces. Recall that \( S^N(M) = (M \times M \times \cdots M)/\Sigma_N \) where the product contains \( N \) terms and \( \Sigma_N \) denotes the \( N \)-symmetric group.

Let \( C^0_{\text{tame}}(M;\mathbb{R}) \) resp. \( C^0_{\text{tame}}(M;\mathbb{S}^1) \) denote the set of tame maps with the topology induced from \( C^0(M;\mathbb{R}) \) resp. \( C^0(M;\mathbb{S}^1) \) equipped with the compact open topology. This set is dense in the space of all continuous maps.

Theorem 7.3. The assignments \( f \mapsto C_r(f) \) is a continuous map on \( C^0_{\text{tame}}(M;\mathbb{R}) \) resp. \( C^0_{\text{tame}}(M;\mathbb{S}^1) \) hence has a continuous extension to the entire \( C^0(M;\mathbb{R}) \) resp. \( C^0(M;\mathbb{S}^1) \).

As a consequence the configuration \( C_r(f) \), hence the closed \( r \) bar codes and the open \( (r-1) \) bar codes, as well as the collection of Jordan cells can be defined for any continuous maps. Consequently the monic polynomials \( P_r(f)(z) \) are well defined and the assignment \( f \mapsto P_r(f)(z) \) continuous. Note that the collection \( J_r(f) \) remains constant on a connected component of \( C^0(M;\mathbb{S}^1) \). Consequently, for \( f : X \rightarrow \mathbb{C} \setminus 0 \) a continuous map one has the monic polynomials \( P_r([f])(z) \) and \( Pr([f])(z) \) which can be regarded as refinements of \( \beta_r(f) \) and \( \beta_1(f) \) with respect to \( f \).

The above results show that for \( f : X \rightarrow S^1 \) a tame map only the bar codes in \( B^O_r(f) \), \( B^\cdot_r(f) \) and the Jordan cells \( J_r(f) \) are relevant for the topology of \( X \). The bar codes in \( B^O_r \) and in \( B^\cdot_r \) are related only with the specifics of the map \( f \) and have no contribution to the topology of \( X \). More about will be discussed in [3].
8. The meaning of the bar codes

For \( f : X \to \mathbb{R} \) the following concepts are fundamental to describe the meaning of the invariants we have considered.

- The element \( x \in H_r(X_t) \) is dead (to the right) at \( t' > t \) resp. dead (to the left) at \( t'' < t \) if its image by \( H_r(X_t) \to H_r(X_{[t,t']}) \) resp. by \( H_r(X_t) \to H_r(X_{[t',t]}) \) vanishes.
- The element \( x \in H_r(X_t) \) is observable at \( t' \neq t \) if its image by \( H_r(X_t) \to H_r(X_{[t,t']}) \) is contained in the image of \( H_r(X_t') \to H_r(X_{[t',t]}) \).

**Definition 8.1.**

(1) For \( x \in H_r(X_t) \) define \( \tau^+(x) \in \mathbb{R}_+ \cup \infty \) resp. \( \tau^+(x) \in \mathbb{R}_+ \cup \infty \) by the following property: \( x \) is dead (to the right) at \( t + \tau^+(x) \) resp. (to the left) at \( t - \tau^- \) but not before, i.e. for \( t' \) with \( t < t' < t + \tau^+(x) \) resp. \( t > t' > t - \tau^- \).

(2) For \( x \in H_r(X_t) \) define \( o^+(x) \in \mathbb{R}_+ \cup \infty \) resp. \( o^-(x) \in \mathbb{R}_+ \cup \infty \) by the following property: \( x \) is observable at \( t + o^+(x) \) resp. \( t - o^-(x) \) but not at \( t + o^+(x) + \epsilon \) resp. \( t - o^-(x) - \epsilon \) for \( \epsilon > 0 \).

**Definition 8.2.** For \( f : X \to \mathbb{R} \), with critical values \( s_i \) denote by:

(1) \( N_r(s_i, s_j) \) \((s_i \leq s_j)\) the maximal number of linearly independent elements \( x \in H_r(X_t) \) with \( t + \tau^+(x) = s_j, t - \tau^-(x) = s_i \) for any \( t \) in the open interval \((s_i, s_j]\),

(2) \( N_r[s_i, s_j) \) \((s_i < s_j)\) the maximal number of linearly independent elements \( x \in H_r(X_t) \) with \( t + o^+(x) = s_j, t - o^-(x) = s_i \) for any \( t \) in the open interval \((s_i, s_j]\),

(3) \( N_r(s_i, s_j] \) \((s_i < s_j)\) the maximal number of linearly independent elements \( x \in H_r(X_t) \) with \( t + o^+(x) = s_j, t - o^-(x) = s_i \) for any \( t \) in the open interval \((s_i, s_j]\),

(4) \( N_r[\infty, s_j) \) \((s_i < s_j)\) the maximal number of linearly independent elements \( x \in H_r(X_t) \) with \( t + \tau^+(x) = s_j, t - o^-(x) = s_i \) for any \( t \) in the open interval \((s_i, s_j]\).

For \( f \) real values the number \( N_r(s_i, s_j] \) represents the multiplicity of the bar code \( \{s_i, s_j]\) with the convention that non existence of such bar codes means multiplicity zero.

For \( f \) angle valued with critical values \( \theta_i \) the number \( N_r(\theta_i, \theta_j + 2\pi k) \) represents the multiplicity of the bar code \( \{\theta_i, \theta_j + 2\pi k\} \) which is the same as the multiplicity of \( \{\theta_i, \theta_j + 2\pi k\} \) for the real valued \( f : \tilde{X} \to \mathbb{R} \) of \( f \).

Note that for \( f : X \to S^1 \) and \( \theta \in (0, 2\pi) \) one can have elements \( x \in H_r(X_\theta) = H_r(\tilde{X}_\theta) \) which never die and remain observable for ever. The existence of such elements is guaranteed by the presence of Jordan cells. The Jordan cells provide rather complete information on the maximal number such elements which remain observable and linearly independent for any \( \theta' \) as well as about how they return in \( H_r(\tilde{X}_{\theta + 2\pi}) = H_r(\tilde{X}_\theta) \) when \( \theta' \) goes from \( \theta \) to \( \theta + 2\pi \), equivalently how do they change when observed in \( H_r(X_{\theta + 2\pi}) \).

---

4if \( X \) is compact in particular if \( f \) is tame as defined above \( o^+(x) \in \mathbb{R}_+ \), so can not be infinite
5It suffices to happen for one \( t \) and then it happens for any other \( t \)
9. About the proof (The canonical long exact sequence)

Since a tame real valued map can be regarded as a tame angle valued map (by identifying \( \mathbb{R} \) to an open subset of \( S^1 \)), we will consider only the case of tame angle valued maps.

Let \( f: X \to S^1 \) be a tame map with \( m \) critical angles \( \theta_1, \theta_2, \ldots, \theta_m \) and regular angles \( t_1, t_2, \ldots, t_m \). First observe that, up to homotopy, the space \( X \) and the map \( f: X \to S^1 \) can be regarded as the iterated mapping torus \( T \) and the map \( f^T: T \to [0, m]/\sim \) described below. Consider the collection of spaces and continuous maps:

\[
\begin{align*}
X_{\theta_1} & \xrightarrow{a_1} X_{t_m} \\
X_{t_1} & \xleftarrow{b_1} X_{t_m} \\
X_{\theta_2} & \xleftarrow{a_2} X_{t_m} \\
X_{t_2} & \xrightarrow{b_2} X_{t_m} \\
X_{\theta_3} & \xrightarrow{a_3} X_{t_m} \\
X_{t_3} & \xleftarrow{b_3} X_{t_m} \\
\cdots & \cdots \\
X_{\theta_m} & \xrightarrow{a_m} X_{t_m} \\
X_{t_m} & \xleftarrow{b_m} X_{t_m}
\end{align*}
\]

with \( R_i := X_{t_i} \) and \( X_i := X_{\theta_i} \). Denote by \( T = T(\alpha_1 \cdots \alpha_m; \beta_1 \cdots \beta_m) \) the space obtained from the disjoint union

\[
\left( \bigsqcup_{1 \leq i \leq m} R_i \times [0, 1] \right) \sqcup \left( \bigsqcup_{1 \leq i \leq m} X_i \right)
\]

by identifying \( R_i \times \{1\} \) to \( X_i \) by \( \alpha_i \) and \( R_i \times \{0\} \) to \( X_{i-1} \) by \( \beta_{i-1} \). Denote by \( f^T: T \to [0, m]/\sim = S^1 \) the map given by \( f^T: R_i \times [0, 1] \to [i-1, i] \) is the projection on \([0, 1]\) followed by the translation of \([0, 1]\) to \([i-1, i]\) and \([0, m]/\sim\) the space obtained from the segment \([0, m]\) by identifying the ends. The map \( f^T: T \to [0, m]/\sim \) is a homotopical reconstruction of \( f: X \to S^1 \) provided that, with the choice of angles \( t_i, \theta_i \), the maps \( a_i, b_i \) are those described in section 6 for \( X_i := f^{-1}(\theta_i) \) and \( R_i := f^{-1}(t_i) \).

Let \( P' \) denote the space obtained from the disjoint union

\[
\left( \bigsqcup_{1 \leq i \leq m} R_i \times (\epsilon, 1] \right) \sqcup \left( \bigsqcup_{1 \leq i \leq m} X_i \right)
\]

by identifying \( R_i \times \{\epsilon\} \) to \( X_i \) by \( \alpha_i \), and \( P'' \) denote the space obtained from the disjoint union

\[
\left( \bigsqcup_{1 \leq i \leq m} R_i \times [0, 1-\epsilon] \right) \sqcup \left( \bigsqcup_{1 \leq i \leq m} X_i \right)
\]

by identifying \( R_i \times \{0\} \) to \( X_{i-1} \) by \( \beta_{i-1} \).

Let \( \mathcal{R} = \bigsqcup_{1 \leq i \leq m} R_i \) and \( \mathcal{X} = \bigsqcup_{1 \leq i \leq m} X_i \). Then, one has:

\[
(1) \quad T = P' \cup P''
\]
(2) \( \mathcal{P}' \cap \mathcal{P}'' = \left( \bigsqcup_{1 \leq i \leq m} R_i \times (\epsilon, 1 - \epsilon) \right) \sqcup \mathcal{X} \), and

(3) the inclusions \( \left( \bigsqcup_{1 \leq i \leq m} R_i \times \{1/2\} \right) \sqcup \mathcal{X} \subset \mathcal{P}' \cap \mathcal{P}'' \) as well as the obvious

inclusions \( \mathcal{X} \subset \mathcal{P}' \) and \( \mathcal{X} \subset \mathcal{P}'' \) are homotopy equivalences.

The Mayer–Vietoris long exact sequence applied to \( T = \mathcal{P}' \cup \mathcal{P}'' \) leads to the
diagram:

\[
\begin{array}{cccccccc}
H_r(\mathcal{R}) & \xrightarrow{M(\rho_r)} & H_r(\mathcal{X}) \\
\downarrow{pr_1} & & \downarrow{(id, -id)} \\
\cdots & \xrightarrow{\partial_{r+1}} & H_r(\mathcal{T}) & \xrightarrow{N} & H_r(\mathcal{X}) \oplus H_r(\mathcal{X}) & \xrightarrow{(\iota_r, -\iota_r)} & H_r(T) \\
\downarrow{in_2} & & \uparrow{\Delta} \\
H_r(\mathcal{X}) & \xrightarrow{id} & H_r(\mathcal{X})
\end{array}
\]

Diagram 2

Here \( \Delta \) denotes the diagonal, \( in_2 \) the inclusion on the second component, \( pr_1 \)
the projection on the first component, \( \iota_r \) the linear map induced in homology by
the inclusion \( \mathcal{X} \subset \mathcal{T} \). The matrix \( M(\rho_r) \) is defined by

\[
M(\rho_r) = \begin{pmatrix}
\alpha_1^r & -\beta_1^r & 0 & \cdots & 0 \\
0 & \alpha_2^r & -\beta_2^r & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \alpha_{m-1}^r & -\beta_{m-1}^r \\
-\beta_m^r & 0 & \cdots & 0 & \alpha_m^r
\end{pmatrix}
\]

with \( \alpha_i^r : H_r(R_i) \to H_r(X_i) \) and \( \beta_i^r : H_r(R_{i+1}) \to H_r(X_i) \) induced by the maps \( \alpha_i \)
and \( \beta_i \) and the matrix \( N \) is defined by

\[
\begin{pmatrix}
\alpha^r & Id \\
-\beta^r & Id
\end{pmatrix}
\]

where \( \alpha^r \) and \( \beta^r \) are the matrices

\[
\begin{pmatrix}
\alpha_1^r & 0 & \cdots & 0 \\
0 & \alpha_2^r & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \alpha_{m-1}^r
\end{pmatrix}
\] \quad \text{and} \quad
\begin{pmatrix}
0 & \beta_1^r & 0 & \cdots & 0 \\
0 & 0 & \beta_2^r & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \beta_{m-1}^r
\end{pmatrix}
\]

The long exact sequence

\[
\cdots \to H_r(\mathcal{R}) \xrightarrow{M(\rho_r)} H_r(\mathcal{X}) \to H_r(T) \to H_{r-1}(\mathcal{T}) \xrightarrow{M(\rho_{r-1})} H_{r-1}(\mathcal{X}) \to \cdots
\]

derived from Diagram 2 is referred to as the canonical sequence associated with a
tame.

This long exact sequence implies the short exact sequence

\[
0 \to \text{coker} M(\rho_r) \to H_r(T) \to \ker M(\rho_{r-1}) \to 0
\]
and then the noncanonical isomorphism

$$H_r(T) = \text{coker}M(\rho) \oplus \ker M(\rho_{r-1}).$$

(5)

Any splitting $s: \ker M(\rho_{r-1}) \to H_r(T)$ in the short exact sequence (4) provides an isomorphism (5). The calculation of $\ker M(\rho)$ and $\text{coker}M(\rho)$ for $\rho = \rho_r$ is reduced to the case $\rho$ is indecomposable hence to bar codes and Jordan cells cf [1] calculation provided in [1] Proposition 5.3.

Note that the long exact sequence (3) holds also for homology with local coefficients (i.e. homology with coefficients in a representation of the fundamental group). Such sequence can be derived from a similar diagram as Diagram 2, where instead of homology with coefficients in $\kappa$ one uses homology with local coefficients. Of particular interest is the case the local coefficients system is $u\xi_f$, the representation described by the composition $H_1(M; \mathbb{Z}) \xrightarrow{\xi_f} \mathbb{Z} \xrightarrow{\hat{u}} \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. In this case the vector spaces $H_r(\mathbb{R})$ and $H_r(X)$ are independent on $u\xi_f$ and represent always the cohomology with coefficients in the trivial representation (corresponding to $u = 1$) hence with coefficients in the field $\kappa$. Manipulations of these sequences cf [1], [2] lead to the proof of Theorems 7.1, 7.2. The proof of Theorem 7.3 requires refinements of Theorems 7.1 and 7.2 and will be contained in [3].

Note:
1. The canonical long exact sequence contains more information than used in the present discussion. In case $\kappa$ is a field of characteristic zero $H_r(\mathbb{R})$ and $H_r(X)$ contain inside the lattice of integral homology. This can be used for calculating a more subtle invariant "torsion".
2. Theorems 7.1, 7.2 imply that up to an isomorphism (of vector spaces) the canonical long exact sequence it is completely determined by the bar codes and the Jordan cells.

10. About computability of the bar codes and the Jordan cells

For $f: X \to \mathbb{R}$ or $f: X \to S^1$ simplicial maps algorithms of relative low complexity to calculate the bar codes and Jordan cells are described in [1]. Here simplicial means that $X$ is a simplicial complex and, when the target of $f$ is $\mathbb{R}$, the restriction of $f$ to any simplex $\sigma$ of $X$ is linear, and when the target is $S^1$, any lift $f: \sigma \to \mathbb{R}$ of $f|_{\sigma} \to S^1$ ($\pi \cdot f = f|_{\sigma}$ with $\pi: \mathbb{R} \to S^1$ the universal cover) is linear. Note that any simplicial map $f$ is tame and its critical values are among the values of $f$ on the vertices of $X$.

The algorithms we proposed consist of two steps. In the first step one inputs the simplicial complex and the values of $f$ on the vertices of $X$ and derive which of these values are critical and then by choosing regular values (for example midle between two consecutive critical values) the representations $\rho_r$ as collections of matrices $\{\alpha^r, \beta^r\}$. A summary presentation of this part is provided in [1]. The second part inputs the representations $\rho_r$ and outputs the bar codes and the Jordan cells. Details are provided in the Appendix to [1].

As long as the first input is concerned, to record the simplicial complex $X$ we choose a total order of the set of vertices, $\{v_1, v_2, \cdots\}$, extend this order to a total order of the set of all simplices of $X$ such that the following two conditions hold:

C1: If $\tau$ is a face of $\sigma$ then $\tau < \sigma$,
C2: If \( \dim \tau < \dim \sigma \) then \( \tau < \sigma \). It is obvious that such total orders exists. Note that the ordering of the vertices provide an orientation on each simplex.

One records the simplicial complex \( X \) as an \( N \times N \) upper triangular matrix \( M(X) \) with zero on diagonal and entries \( 0, +1, -1 \) where \( N \) the cardinality of the set of all simplices. Precisely the entry corresponding to the pair \( (\tau, \sigma) \) is zero if \( \tau \) is not a face of \( \sigma \) and equal to \( +1 \) if it is. In this case is \( +1 \) if the orientation of \( \sigma \) induces the orientation of \( \tau \) and \( -1 \) otherwise.

To the matrix called \( M(X) \) one add the values of \( f \) on vertices. In the first step one determines which values of \( f \) are critical and then using sub matrices of \( M(X) \) (possibly enhanced) one recover the matrices \( M(\rho_r) \) equivalently the representations \( \rho_r \) as indicated in [1].

In the second step a new algorithm whose input is the matrix \( M(\rho_r) \) describing the representations \( \rho_r \) and output is the barcodes and the Jordan cells finalize the calculations. More details about this algorithm can be found in [1].

### 11. Examples

The Picture below describes a tame real valued map \( p : Y \to [0, 2\pi] \subset \mathbb{R} \) and an angle valued map \( f : X \to S^1 \) whose bar codes and Jordan cells are given in the tables below. The space \( X \) is obtained from \( Y \) by identifying its right end \( Y_1 \) (a union of three circles) to the left end \( Y_0 \) (a union of three circles) following the map \( \phi : Y_1 \to Y_0 \) described as follows:
- circle 1 goes 3 times around circle 1
- circle 2 go 2 time around circle 2
- circle 3 goes1 time around 2 counter clockwise and 2 times around circle 3.

The map \( p : Y \to \mathbb{R} \) is the projection on \( [0, 2\pi] \) and the map \( f : X \to S^1 \) is induced by the projection of \( p : Y \to [0, 2\pi] \) by passing to the quotient spaces \( X = Y/\phi \) and \([0, 2\pi]/\sim\). Note that \( H_1(Y_1) = H_1(Y_0) = \kappa \oplus \kappa \oplus \kappa \) and \( \phi \) induces a linear map in

![Figure 3. The tames maps p and f.](image-url)
$H_1$-homology represented by the matrix

\[
\begin{pmatrix}
3 & 0 & 0 \\
1 & 2 & -1 \\
0 & 0 & 2 \\
\end{pmatrix}
\]

The bar codes of the map $p$ are given in the Table 1. There are no bar codes in dimension 2 since each fiber of $f$ is one-dimensional.

<table>
<thead>
<tr>
<th>dimension</th>
<th>bar codes</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$(0, 2\pi)$</td>
</tr>
<tr>
<td>1</td>
<td>$(\theta_4, \theta_5)$, $(\theta_6, 2\pi)$, $(\theta_2, \theta_3)$, $(0, 2\pi)$</td>
</tr>
</tbody>
</table>

Table 1:

For the angle valued map $f : X \to S^1$ there are no bar codes or Jordan cells in dimension 2 since each fiber of $f$ is one-dimensional and, as all fibers are connected in dimension zero we have only one Jordan cell $\rho^{II}(1; 1)$. The bar codes and the Jordan cells in dimensions 0 and 1 are described Table 2. In More details on their calculation are presented in [1] and [2].

<table>
<thead>
<tr>
<th>dimension</th>
<th>bar codes</th>
<th>Jordan cells</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$(\theta_6, \theta_1 + 2\pi)$</td>
<td>$(1,1)$</td>
</tr>
<tr>
<td>1</td>
<td>$(\theta_4, \theta_5)$, $(\theta_2, \theta_3)$</td>
<td>$(2,2)$</td>
</tr>
</tbody>
</table>

Table 2.

REFERENCES

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