

Cyclic theory for commutative differential graded algebras and s-cohomology.

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Abstract. In this paper one considers three homotopy functors on the category of manifolds, hH^* , cH^* , sH^* , and parallel them with other three homotopy functors on the category of connected commutative differential graded algebras, HH^* , CH^* , SH^* . If P is a smooth 1-connected manifold and the algebra is the de-Rham algebra of P the two pairs of functors agree but in general do not. The functors HH^* and CH^* can be also derived as Hochschild resp. cyclic homology of commutative differential graded algebra, but this is not the way they are introduced here. The third SH^* , although inspired from negative cyclic homology, can not be identified with any sort of cyclic homology of any algebra. The functor sH^* might play some role in topology. Important tools in the construction of the functors HH^* , CH^* and SH^* , in addition to the linear algebra suggested by cyclic theory, are Sullivan minimal model theorem and the "free loop" construction described in this paper.

(dedicated to A. Connes for his 60-th birthday)

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1. Introduction

This paper deals with commutative graded algebras and the results are of significance in “commutative” geometry/topology. However they were inspired largely by the linear algebra underlying Connes’ cyclic theory. The topological results formulated here, Theorem 2 and Theorem 3, were first established as a consequence of the identification of the cohomology resp. S^1 –equivariant cohomology of the free loop spaces of a 1-connected smooth manifold with the Hochschild resp. cyclic homology of its de-Rham algebra, cf. [J], [BFG], [B2]. In this paper this identification is circumvented. Still, the results illustrate the powerful influence of Connes’ mathematics in areas outside the non commutative geometry.

In this paper, inspired by the relationship between Hochschild, cyclic and negative cyclic homology of a unital algebra, one considers two systems of graded vector space valued homotopy functors hH^* , cH^* , sH^* and HH^* , CH^* , SH^* and investigate their relationship. The first three functors are defined on the category of smooth manifolds and smooth maps via the free loop space P^{S^1} of a smooth manifold P , which is a smooth S^1 –manifold of infinite dimension. The next three functors are defined on the category of connected commutative differential graded algebras via an algebraic analogue of “free loop” construction and via Sullivan minimal model theorem, Theorem 1. The relationship between them is suggested by the general nonsense- diagram Fig 2 in section 2.

When applied to the De–Rham algebra of a 1-connected smooth manifold the last three functors take the same values as the first three. This is not the case when the smooth manifold is not 1-connected; the exact relationship will be addressed in a future work.

The first three functors are based on a formalism (manipulation with differential forms) which can be considered for any smooth (finite or infinite dimensional) manifold M and any smooth vector field X on M . However it seems to be of relevance when the vector field X comes from a smooth S^1 –action on M . This is of mild interest if the manifold is of finite dimension but more interesting when the manifold is of infinite dimension. In particular, it is quite interesting when $M = P^{S^1}$, the free loop space of P , and the action is the canonical S^1 –action on P^{S^1} . Manipulation with differential forms on P^{S^1} leads to the graded vector spaces $hH^*(P)$, $cH^*(P)$, $sH^*(P)$ with the first two being the cohomology resp. the S^1 –equivariant cohomology of P^{S^1} but sH^* being a new homotopy functor, referred here as s –cohomology.

This functor was first introduced in [B1], [B2] but so far not seriously investigated. The functor sH^* relates, at least in the case of a 1-connected manifold P , the Waldhausen algebraic K –theory of P and the Atiyah–Hirtzebruch complex K –theory (based on complex vector bundles) of P . It has a rather simple description in terms of infinite sequence of smooth invariant differential forms on P^{S^1} .

The additional structures on P^{S^1} , the power maps ψ_k $k = 1, 2, \dots$, and the involution $\tau = \psi_{-1}$, provide endomorphisms of $hH^*(P)$, $cH^*(P)$, $sH^*(P)$

whose eigenvalues and eigenspaces are interesting issues. They are clarified only when P is 1-connected. This is done in view of the relationship with the functors HH^*, CH^*, SH^* .

It might be only a coincidence but still a appealing observation that the symmetric resp. antisymmetric part of $sH^*(P)$ w.r. to the canonical involution τ calculates, for a 1-connected manifold P and in the stability range, the vector space $Hom(\pi_*(H/Diff(P), \kappa), \kappa)$, $\kappa = \mathbb{R}, \mathbb{C}$; the symmetric part when $\dim P$ is even the antisymmetric part when $\dim P$ is odd, cf. [Bu], [B1]. Here $H/Diff(P)$ denotes the (homotopy) quotient of the space of homotopy equivalences of P by the group of diffeomorphisms with the C^∞ -topology.

The functors HH^*, CH^*, SH^* are the algebraic version of hH^*, cH^*, sH^* and are defined on the category of (homologically connected) commutative differential graded algebras. Their definition uses the “free loop” construction, an algebraic analogue of the free loop space, described in this paper only for free connected commutative differential graded algebras $(\Lambda[V], d_V)$. A priori these functors are defined only for free connected commutative differential graded algebras. Since they are homotopy functors they extend to all connected commutative differential graded algebras via Sullivan minimal model theorem, Theorem 1.

Using the definition presented here one can take full advantage of the simple form the algebraic analogue of the power maps take on the free loop construction. As a consequence one obtains a simple description of the eigenvalues and eigenspaces of the endomorphisms induced from the algebraic power maps on HH^* and CH^* and implicitly understand their additional structure.

The extension of the results of Sullivan–Vigué, cf. [VS], to incorporate S^1 -action and the power maps in the minimal model of P^{S^1} summarized in Section 7 leads finally to results about hH^*, cH^*, sH^* when P is 1-connected, cf. Theorem 3.

In addition to the algebraic definition of HH^*, CH^*, SH^* this paper contains the proof of the homotopy invariance of sH^* .

2. Mixed complexes, a formalism inspired from Connes' cyclic theory

A mixed complex (C^*, δ^*, β_*) consists of a graded vector space C^* ($*$ a non negative integer) and linear maps, $\delta^* : C^* \rightarrow C^{*+1}$, $\beta_{*+1} : C^{*+1} \rightarrow C^*$ which satisfy:

$$\begin{aligned} \delta^{*+1} \cdot \delta^* &= 0 \\ \beta_* \cdot \beta_{*+1} &= 0 \\ \beta_{*+1} \cdot \delta^* + \delta^{*-1} \cdot \beta_* &= 0. \end{aligned}$$

When no risk of confusion the index $*$ will be deleted and we write (C, δ, β) instead of (C^*, δ^*, β_*) . Using the terminology of [B2], [BV2] a mixed complex can be viewed either as a cochain complex (C^*, d^*) equipped with an S^1 -action β_* , or as a chain complex (C^*, β_*) equipped with an algebraic S^1 -action δ^* .

To a mixed complex (C^*, δ^*, β_*) one associates a number of cochain, chain and 2–periodic cochain complexes, and then their cohomologies, homologies and 2–periodic cohomologies¹, as follows.

First denote by

$$\begin{aligned} {}^+C^r &= \prod_{k \geq 0} C^{r-2k} & -C^r &:= \prod_{k \geq 0} C^{r+2k} \\ \mathbb{P}C^{2r+1} &= \prod_{k \geq 0} C^{2k+1} & \mathbb{P}C^{2r} &= \prod_{k \geq 0} C^{2k} \text{ for any } r \\ PC^{2r+1} &= \bigoplus_{k \geq 0} C^{2k+1} & PC^{2r} &= \bigoplus_{k \geq 0} C^{2k} \text{ for any } r. \end{aligned} \quad (1)$$

Since our vector spaces are $\mathbb{Z}_{\geq 0}$ -graded the direct product ${}^+C^r$ involves only finitely many factors.

Next introduce

$$\begin{aligned} {}^+D_\beta^r(w_r, w_{r-2}, \dots) &:= (\delta w_r, (\delta w_{r-2} + \beta w_r), \dots) \\ {}^+D_r^\delta(w_r, w_{r-2}, \dots) &:= ((\beta w_r + \delta w_{r-2}), (\beta w_{r-2} + \delta w_{r-4}), \dots) \\ -D_\beta^r(\dots, w_{r+2}, w_r) &:= (\dots, (\beta w_{r+4} + \delta w_{r+2}), (\beta w_{r+2} + \delta w_r)) \\ -D_r^\delta(\dots, w_{r+2}, w_r) &:= (\dots, (\delta w_r + \beta w_{r+2}), \beta w_r) \\ D^{2r}(\dots, w_{2r+2}, w_{2r}, \dots, w_0) &= (\dots, (\delta w_{2r} + \beta w_{2r+2}), \dots) \\ D^{2r+1}(\dots, w_{2r+3}, w_{2r+1}, \dots, w_1) &= (\dots, (\delta w_{2k+1} + \beta w_{2k+3}), \dots). \end{aligned} \quad (2)$$

Finally consider the cochain complexes

$$\mathcal{C} := (C^*, \delta^*), \quad {}^+\mathcal{C}_\beta := ({}^+C^*, {}^+D_\beta^*), \quad -\mathcal{C}_\beta := (-C^*, -D_\beta^*),$$

the chain complexes

$$\mathcal{H} := (C_*, \beta_*), \quad {}^+\mathcal{H}^\delta := ({}^+C^*, {}^+D_*^\delta), \quad -\mathcal{H}^\delta := (-C_*, -D_*^\delta)$$

and the 2–periodic cochain complexes²

$$PC := (PC^*, D^*), \quad \mathbb{P}C := (\mathbb{P}C^*, D^*)$$

whose cohomology, homology, 2–periodic cohomology are denoted by

$$\begin{aligned} H^* &:= H^*(C, \delta), & {}^+H_\beta^* &:= {}^+H_\beta^*(C, \delta, \beta), & -H_\beta^* &:= -H_\beta^*(C, \delta, \beta), \\ H_* &:= H_*(C, \beta), & {}^+H_*^\delta &:= {}^+H_*^\delta(C, \delta, \beta), & -H_*^\delta &:= -H_*^\delta(C, \delta, \beta), \\ PH^* &:= PH^*(C, \delta, \beta), & & & \mathbb{P}H^* &:= \mathbb{P}H^*(C, \delta, \beta). \end{aligned}$$

¹We will use the word "homology" for a functor derived from a chain complex and "cohomology" for one derived from a cochain complex. The 2–periodic chain and cochain complexes can be identified.

²here $(\mathbb{P}C^*, D^*)$ is regarded as a cochain complex with D^* obtained from the degree +1 derivation δ perturbed by the degree –1 derivation β ; the same complex can be regarded chain complex with D^* obtained from the degree –1 derivation β perturbed by the degree +1 derivation δ ; the cohomology for the first is the same as homology for the second

In this paper the chain complexes $\mathcal{H}, {}^\pm \mathcal{H}^\delta$, will only be used to derive conclusions about the cochain complexes $\mathcal{C}, {}^\pm \mathcal{C}_\beta, \mathbb{P}\mathcal{C}$.

The obvious inclusions and projections lead to the following commutative diagrams of short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & -\mathcal{H}_*^\delta & \longrightarrow & \mathbb{P}\mathcal{C}^* & \longrightarrow & +\mathcal{H}_{*-2}^\delta \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow id \\
 0 & \longrightarrow & \mathcal{H}_* & \longrightarrow & +\mathcal{H}_*^\delta & \longrightarrow & +\mathcal{H}_{*-2}^\delta \longrightarrow 0 \\
 \\
 0 & \longrightarrow & +\mathcal{C}_\beta^{*-2} & \xrightarrow{i^{*-2}} & +\mathcal{C}_\beta^* & \longrightarrow & \mathcal{C}^* \longrightarrow 0 \\
 & & \downarrow id & & \downarrow I^* & & \downarrow \\
 0 & \longrightarrow & +\mathcal{C}_\beta^{*-2} & \xrightarrow{I^{*-2}} & \mathbb{P}\mathcal{C}^* & \longrightarrow & -\mathcal{C}_\beta^* \longrightarrow 0.
 \end{array}$$

They give rise to the following commutative diagram of long exact sequences

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & -H_r^\delta & \longrightarrow & \mathbb{P}H(r) & \longrightarrow & +H_{r-2}^\delta \longrightarrow -H_{r-1}^\delta \longrightarrow \dots \\
 & & \downarrow h_r & & \downarrow & & \downarrow id \\
 \dots & \longrightarrow & H_r & \xrightarrow{J_r} & +H_r^\delta & \xrightarrow{S_r} & +H_{r-2}^\delta \xrightarrow{B_{r-2}} H_{r-1} \longrightarrow \dots \\
 & & & & \downarrow & & \downarrow h_{r-1}
 \end{array}$$

Fig 1.

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & +H_\beta^{r-2} & \xrightarrow{S^{r-2}} & +H_\beta^r & \longrightarrow & H^r \xrightarrow{B^r} +H_\beta^{r-1} \longrightarrow \dots \\
 & & \downarrow id & & \downarrow \mathbb{I}^* & & \downarrow i^r \\
 \dots & \longrightarrow & +H_\beta^{r-2} & \xrightarrow{\mathbb{I}^{r-2}} & \mathbb{P}H^r & \xrightarrow{J^r} & -H_\beta^r \xrightarrow{B^r} +H_\beta^{r-1} \longrightarrow \dots \\
 & & & & \downarrow & & \downarrow id
 \end{array}$$

Fig 2.

and

$$\begin{array}{ccc}
 +H_\beta^{r-2} & \xrightarrow{S^{r-2}} & +H_\beta^r \\
 \downarrow \mathbb{I}^{r-2} & \swarrow \mathbb{I}^r & \downarrow \\
 \mathbb{P}H^r = \mathbb{P}H^{r+2} & \longleftarrow & \lim_{\substack{\longrightarrow \\ S}} +H_\beta^{r+2k}
 \end{array}$$

The diagram (Fig1) is the one familiar in the homological algebra of Hochschild versus cyclic homologies, cf [Lo]. The diagram Fig 2 is the one we will use in this paper.

Note that Hochschild, cyclic, periodic cyclic, negative cyclic homology of an associative unital algebra A as defined in [Lo], is $H_*^+, H_*^\delta, \mathbb{P}H^*, -H_*^\beta$ of the Hochschild mixed complex with $C^r := A^{\otimes(r+1)}$, β the Hochschild boundary, and $\delta^r = (1 - \tau_{r+1}) \cdot s_r \cdot (1 + \tau_r + \cdots + \tau_r^r)$ where $\tau_r(a_0 \otimes a_1 \otimes \cdots \otimes a_r) = (a_r \otimes a_0 \otimes \cdots \otimes a_{r-1})$ and $s_r(a_0 \otimes a_1 \otimes \cdots \otimes a_r) = (1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_r)$.

A morphism $f : (C_1^*, \delta_1^*, \beta_1^1) \rightarrow (C_2^*, \delta_2^*, \beta_2^2)$ is a degree preserving linear map which intertwines δ 's and β 's. It induces degree preserving linear maps between any of the homologies /cohomologies defined above. The following elementary observations will be used below.

Proposition 1. *Let (C, δ, β) be a mixed cochain complex.*

1. $\mathbb{P}H^r = \varinjlim^+ H_\beta^{r+2k}$, where $S^{k+2r} : H_\beta^{k+2r} \rightarrow H_\beta^{k+2r+2}$ is induced by the inclusion ${}^+C_\beta^* \rightarrow {}^+C_\beta^{*+2}$.

2. The following is an exact sequence

$$0 \longrightarrow \varinjlim^+ H_{r-1+2k}^\delta \longrightarrow \mathbb{P}H^r \longrightarrow \varinjlim^+ H_{r+2k}^\delta \longrightarrow 0,$$

with $S_{k+2r} : H_{k+2r}^\delta \rightarrow H_{k+2r-2}^\delta$ induced by the projection ${}^+\mathcal{H}_*^\delta \rightarrow {}^+\mathcal{H}_{*-2}^\delta$.

Let $f^* : (C_1^*, \delta_1^*, \beta_1^1) \rightarrow (C_2^*, \delta_2^*, \beta_2^2)$ be a morphism of mixed complexes.

3. If $H^*(f)$ is an isomorphism then so is ${}^+H_\beta^*(f)$ and $\mathbb{P}H^*(f)$.

4. If $H_*(f)$ is an isomorphism then so is ${}^+H_*^\delta(f)$ and $\mathbb{P}H^*(f)$.

5. If $H^*(f)$ and $H_*(f)$ are both isomorphisms then, in addition to the conclusions in (3) and (4), $-H_\beta^*(f)$ is an isomorphism.

Proof. (1): Recall that a direct sequence of cochain complexes

$$\mathcal{C}_0^* \xrightarrow{i_0} \mathcal{C}_1^* \xrightarrow{i_1} \mathcal{C}_2^* \xrightarrow{i_2} \cdots$$

induces, by passing to cohomology, the direct sequence

$$H^*(\mathcal{C}_0^*) \xrightarrow{H(i_0)} H^*(\mathcal{C}_1^*) \xrightarrow{H(i_1)} H^*(\mathcal{C}_2^*) \xrightarrow{i_2} \cdots$$

and that $H^j(\varinjlim \mathcal{C}_i^*) = \varinjlim H^j(\mathcal{C}_i^*)$ for any j .

(2): Recall that an inverse sequence of chain complexes

$$\mathcal{H}_*^0 \xleftarrow{p_0} \mathcal{H}_*^1 \xleftarrow{p_1} \mathcal{H}_*^2 \xleftarrow{p_2} \cdots$$

induces, by passing to homology, the sequence

$$H_*(\mathcal{H}_*^0) \xleftarrow{H(p_0)} H_*(\mathcal{H}_*^1) \xleftarrow{H(p_1)} H_*(\mathcal{H}_*^2) \xleftarrow{p_2} \cdots$$

and the following short exact sequence cf.[Lo] 5.1.9.

$$0 \longrightarrow \varinjlim^+ H_{j-1}(\mathcal{H}_*^i) \longrightarrow H_j(\varinjlim \mathcal{H}_*^i) \longrightarrow \varinjlim H_j(\mathcal{H}_*^i) \longrightarrow 0$$

for any j .

Item (3) follows by induction on degree from the naturality of the first exact sequence in the diagram Fig 2 and (1).

Item (4) follows by induction from the naturality of the second exact sequence of the diagram Fig 1 and from (2).

Item (5) follows from the naturality of the second exact sequence in diagram Fig 2 and from (3) and (4). \square

The mixed complex (C^*, δ^*, β_*) is called β -acyclic if β_1 is surjective and $\ker(\beta_r) = \text{im}(\beta_{r+1})$. If so consider the diagram whose rows are short exact sequences of cochain complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & (\text{Im}(\beta)^*, \delta^*) & \xrightarrow{i} & (C^*, \delta^*) & \xrightarrow{\beta} & ((\text{Im}(\beta))^{*-1}, \delta^{*-1}) \longrightarrow 0 \\ & & \searrow j & & \searrow id & & \\ 0 & \longrightarrow & (+C_{\beta}^{*-2}, +D_{\beta}^{*-2}) & \xrightarrow{i^{*-2}} & (+C_{\beta}^*, +D_{\beta}^*) & \longrightarrow & (C^*, \delta^*) \longrightarrow 0 \end{array}$$

Each row induces the long exact sequence in the diagram below and a simple inspection of boundary map in these long exact sequences permits to construct linear maps θ^r and to verify that the diagram below is commutative.

$$\begin{array}{ccccccc} H^{r-2}(\text{Im}(\beta), \delta) & \longrightarrow & H^r(\text{Im}(\beta), \delta) & \longrightarrow & H^r(C, \delta) & \longrightarrow & H^{r-1}(\text{Im}(\beta), \delta) \\ \downarrow \theta^{r-2} & & \downarrow \theta^r & & \downarrow id & & \downarrow \theta^{r-1} \\ +H_{\beta}^{r-2}(C, \delta, \beta) & \longrightarrow & +H_{\beta}^r(C, \delta, \beta) & \longrightarrow & H^r(C, \delta) & \longrightarrow & +H_{\beta}^{r-1}(C, \delta, \beta) \end{array}$$

As a consequence one verifies by induction on degree that the inclusion $j : (\text{Im}\beta, \delta) \rightarrow (+C, +D_{\beta})$ induces an isomorphism $H^*(\text{Im}(\beta), \delta) \rightarrow +H_{\beta}^*(C, \delta, \beta)$.

Mixed complex with power maps and involution

A collection of degree zero (degree preserving) linear maps $\Psi_k, k = 1, 2, \dots, \tau := \Psi_{-1}$ which satisfy

- (i) $\Psi_k \circ \delta = \delta \circ \Psi_k$,
- (ii) $\Psi_k \circ \beta = k\beta \circ \Psi_k$,
- (iii) $\Psi_k \circ \Psi_{\tau} = \Psi_{\tau} \circ \Psi_k = \Psi_{kr}, \Psi_1 = id$

will be referred to as “power maps and involution”, or simpler ”power maps”, $\Psi_k, k = -1, 1, 2, \dots$ ³. They provide the morphisms of cochain complexes

$$\begin{aligned} \Psi_k &: \mathcal{C} \rightarrow \mathcal{C}, \\ \pm \Psi_k &: \pm \mathcal{C}_{\beta} \rightarrow \pm \mathcal{C}_{\beta} \\ \mathbb{P}\Psi_k &: \mathbb{P}\mathcal{C} \rightarrow \mathbb{P}\mathcal{C} \end{aligned}$$

defined as follows

$$\begin{aligned} +\Psi_k^r(w_r, w_{r-2}, \dots) &= (\Psi^r(\omega_r), \frac{1}{k}\Psi^{r-2}(\omega_{r-2}), \dots) \\ -\Psi_k^r(\dots, w_{r+2}, w_r) &:= (\dots, k\Psi_k^{2r+2}(\omega_{r+2}), \Psi_k^r(\omega_r)) \end{aligned}$$

and

³We use the notation τ for Ψ_{-1} to emphasize that is an involution and to suggest consistency with other familiar involutions in homological algebras and topology.

$$\begin{aligned}
& \mathbb{P}\Psi_k^{2r}(\cdots, \omega_{2r+2}, \omega_{2r}, \omega_{r-2}, \cdots, \omega_0) = \\
& = (\cdots, k\Psi_k^{2r+2}(\omega_{r+2}), \Psi_k^{2r}(\omega_{2r}), \frac{1}{k}\Psi_k^{2r-2}(\omega_{2r-2}), \cdots, \frac{1}{k^r}\Psi_k^0(\omega_0)) \\
& \quad \mathbb{P}\Psi_k^{2r+1}(\cdots, \omega_{2r+3}, \omega_{2r+1}, \omega_{2r-1}, \cdots, \omega_1) = \\
& = (\cdots, k\Psi_k^{2r+3}(\omega_{2r+3}), \Psi_k^{2r+1}(\omega_{2r+1}), \frac{1}{k}\Psi_k^{2r-1}(\omega_{2r-1}), \cdots, \frac{1}{k^r}\Psi_k^1(\omega_1))
\end{aligned}$$

Consequently they induce the endomorphisms,

$$\begin{aligned}
\Psi_k^* &: H^* \rightarrow H^* \\
\pm \underline{\Psi}_k^* &: \pm H_\beta^* \rightarrow \pm H_\beta^* \\
\underline{\mathbb{P}\Psi}_k^* &: \mathbb{P}H^* \rightarrow \mathbb{P}H^*
\end{aligned}$$

Note that in diagram (Fig2):

\mathbb{J}^*, J^* and the vertical arrows intertwine the endomorphisms induced by Ψ_k ,
 \mathbb{B}^* resp. B^* intertwine $k(-\underline{\Psi}_k)$ resp. $k\underline{\Psi}_k$ with $+\underline{\Psi}_k$,
 \mathbb{I}^{*-2} resp. S^{*-2} intertwine $+\underline{\Psi}_k$ with $k\underline{\mathbb{P}\Psi}_k$ resp. $k(+\underline{\Psi}_k)$,

The above elementary linear algebra will be applied to CDGA's in the next sections.

3. Mixed commutative differential graded algebras

Let κ be a field of characteristic zero (for example $\mathbb{Q}, \mathbb{R}, \mathbb{C}$).

Definition 1. (i) A commutative graded algebra abbreviated CGA, is an associative unital augmentable graded algebra \mathcal{A}^* , (the augmentation is not part of the data) which is commutative in the graded sense, i.e.

$$a_1 \cdot a_2 = (-1)^{r_1 r_2} a_2 \cdot a_1, \quad a_i \in \mathcal{A}^{r_i}, i = 1, 2.$$

(ii) An exterior differential $d_{\mathcal{A}}^* : \mathcal{A}^* \rightarrow \mathcal{A}^{*+1}$, is a degree +1-linear map which satisfies

$$d(a_1 \cdot a_2) = d(a_1) \cdot a_2 + (-1)^{r_1} a_1 \cdot d(a_2), a_1 \in \mathcal{A}^{r_1}, \quad d_{\mathcal{A}}^{*+1} d_{\mathcal{A}}^* = 0.$$

(iii) An interior differential $\beta_{\mathcal{A}}^* : \mathcal{A}^* \rightarrow \mathcal{A}^{*-1}$ is a degree -1 linear map which satisfies

$$\beta(a_1 \cdot a_2) = \beta(a_1) \cdot a_2 + (-1)^{r_1} a_1 \cdot \beta(a_2), a_1 \in \mathcal{A}^{r_1}, \quad \beta_{\mathcal{A}^{*-1}}^* \beta_{\mathcal{A}^*}^* = 0.$$

(iv) The exterior and interior differentials d^* and β_* are compatible if

$$d^{*-1} \cdot \beta_* + \beta_{*+1} \cdot d^* = 0.$$

(v) A pair (\mathcal{A}^*, d^*) , \mathcal{A}^* a CGA and d^* exterior differential, is called CDGA and a triple $(\mathcal{A}^*, d^*, \beta_*)$, \mathcal{A}^* a CGA, d^* exterior differential and β_* interior differential, with d^* and β_* compatible, is called a mixed CDGA.

A mixed CDGA is a mixed cochain complex .

A degree preserving linear map $f^* : \mathcal{A}^* \rightarrow \mathcal{B}^*$ is a morphism of CGA's, resp. CDGA's, resp. mixed CDGA's if is a unit preserving graded algebra homomorphism and intertwines d 's and β 's when the case.

We will consider the categories of CGA's, CDGA's and mixed CDGA's. In all these three categories there is a canonical tensor product and in the category of CDGA's a well defined concept of homotopy between two morphisms (cf. [Lo], [Ha])⁴. The category of mixed CDGA's is a subcategory of mixed cochain complexes and all definitions and considerations in section 2 can be applied.

For a (commutative) differential graded algebra $(\mathcal{A}^*, d_{\mathcal{A}}^*)$, the graded vector space $H^*(\mathcal{A}^*, d_{\mathcal{A}}^*) = Ker(d^*)/Im(d^{*-1})$ is a commutative graded algebra whose multiplication is induced by the multiplication in \mathcal{A}^* . A morphism $f = f^* : (\mathcal{A}^*, d_{\mathcal{A}}^*) \rightarrow (\mathcal{B}^*, d_{\mathcal{B}}^*)$ induces a degree preserving linear map, $H^*(f) : H^*(\mathcal{A}^*, d_{\mathcal{A}}^*) \rightarrow H^*(\mathcal{B}^*, d_{\mathcal{B}}^*)$, which is an algebra homomorphism.

Definition 2. A morphism of CDGA's f , with $H^k(f)$ isomorphism for any k , is called a quasi isomorphism.

The CDGA $(\mathcal{A}, d_{\mathcal{A}})$ is called homologically connected if $H^0(\mathcal{A}, d_{\mathcal{A}}) = \kappa$ and homologically 1-connected if homologically connected and $H^1(\mathcal{A}, d_{\mathcal{A}}) = 0$.

The full subcategory of homologically connected CDGA's will be denoted by c-CDGA. For all practical purposes (related to geometry and topology) it suffices to consider only c-CDGA' s.

Definition 3. 1. The CDGA (\mathcal{A}, d) is called free if $\mathcal{A} = \Lambda[V]$, where $V = \sum_{i \geq 0} V^i$ is a graded vector space and $\Lambda[V]$ denotes the free commutative graded algebra generated by V . If in addition $V^0 = 0$ is called free connected commutative differential graded algebra, abbreviated fc-CDGA.

2. The CDGA (\mathcal{A}, d) is called minimal if it is a fc-CDGA and in addition

i. $d(V) \subset \Lambda^+[V] \cdot \Lambda^+[V]$, with $\Lambda^+[V]$ the ideal generated by V ,

ii. $V^1 = \oplus_{\alpha \in I} V_{\alpha}$ with I a well ordered set and $d(V_{\beta}) \subset \Lambda[\oplus_{\alpha < \beta} V_{\alpha}]$ (the set I and its order are not part of the data)

Observation 1. If $(\Lambda[V], d_V)$ is minimal and 1-connected, then $V^1 = 0$ and, for $v \in V^i$, $d_V(v)$ is a linear combination of products of elements $v_j \in V^j$ with $j < i$. In particular for $v \in V^2$ one has $dv = 0$.

The interest of minimal algebras comes from the following result [L], [Ha].

⁴let $k(t, dt)$ be the free commutative graded algebra generated by the symbol t of degree zero and dt of degree one, equipped with the differential $d(t) = dt$. A morphism $F : (\mathcal{A}, d_{\mathcal{A}}) \rightarrow (\mathcal{B}, d_{\mathcal{B}}) \otimes_k (k(t, dt), d)$, is called elementary homotopy from f to g , $f, g : (\mathcal{A}, d_{\mathcal{A}}) \rightarrow (\mathcal{B}, d_{\mathcal{B}})$, if $\rho_0 \cdot F = f$, and $\rho_1 \cdot F = g$ where

$$\begin{aligned} \rho_0(a \otimes p(t)) &= p(0)a, \quad \rho_0(a \otimes p(t)dt) = 0, \\ \rho_1(a \otimes p(t)) &= p(1)a, \quad \rho_1(a \otimes p(t)dt) = 0, \end{aligned}$$

The homotopy is the equivalence relation generated by elementary homotopy.

Theorem 1. 1 (D. Sullivan)

1. A quasi isomorphism between two minimal CDGA's is an isomorphism.
2. For any homologically connected CDGA, $(\mathcal{A}, d_{\mathcal{A}})$, there exists quasi isomorphisms $\theta : (\Lambda[V], d_V) \rightarrow (\mathcal{A}, d_{\mathcal{A}})$ with $(\Lambda[V], d_V)$ minimal. Such θ will be called minimal model of $(\mathcal{A}, d_{\mathcal{A}})$.
3. Given a morphism $f : (\mathcal{A}, d_{\mathcal{A}}) \rightarrow (\mathcal{B}, d_{\mathcal{B}})$ and the minimal models $\theta_A : (\Lambda[V_A], d_{V_A}) \rightarrow (\mathcal{A}, d_{\mathcal{A}})$ and $\theta_B : (\Lambda[V_B], d_{V_B}) \rightarrow (\mathcal{B}, d_{\mathcal{B}})$, there exists morphisms $f' : (\Lambda[V_A], d_{V_A}) \rightarrow (\Lambda[V_B], d_{V_B})$ such that $f \cdot \theta_A$ and $\theta_B \cdot f'$ are homotopic; moreover any two such (f') 's are homotopic.

We can therefore consider the homotopy category of c-CDGA's, whose the morphisms are homotopy classes of morphisms of CDGA's. By the above theorem the full subcategory of fc-CDGA is a skeleton, and therefore any homotopy functor a priori defined on fc-CDGA's admits extensions to homotopy functors defined on the full homotopy category of c-CDGA's and all these extensions are isomorphic as functors. In particular any statement about a homotopy functor on the category c-CDGA suffices to be verified for fc-CDGA.

Precisely for any c-CDGA, $(\mathcal{A}, d_{\mathcal{A}})$, choose a minimal model $\theta_A : (\Lambda[V_A], d_{V_A}) \rightarrow (\mathcal{A}, d_{\mathcal{A}})$ and for any $f : (\mathcal{A}, d_{\mathcal{A}}) \rightarrow (\mathcal{B}, d_{\mathcal{B}})$ choose a morphism $f' : (\Lambda[V_A], d_{V_A}) \rightarrow (\Lambda[V_B], d_{V_B})$ so that $\theta_B \cdot f'$ and $f \cdot \theta_A$ are homotopic. Define the value of the functor on $(\mathcal{A}, d_{\mathcal{A}})$ to be the value on $(\Lambda[V_A], d_{V_A})$ and the value on a morphism $f : (\mathcal{A}, d_{\mathcal{A}}) \rightarrow (\mathcal{B}, d_{\mathcal{B}})$ to be the value on the morphism $f' : (\Lambda[V_A], d_{V_A}) \rightarrow (\Lambda[V_B], d_{V_B})$.

There are two natural examples of mixed CDGA's; one is provided by a smooth manifold equipped with a smooth vector field, the other by a construction referred to as "the free loop", considered first by Sullivan-Vigué. The free loop construction applies directly only to a fc-CDGA but in view of Theorem 1 can be indirectly used for any c-CDGA.

The first will lead to (the de-Rham version of) a new homotopy functor defined on the category of possibly infinite dimensional manifolds (hence on the homotopy category of all countable CW complexes), the **s-cohomology**, and its relationship with other familiar homotopy functors⁵, cf section 4 below. The second leads to simple definitions of three homotopy functors defined on the full category of c-CDGA's (via the minimal model theorem) with values in the graded vector spaces endowed with weight decomposition, cf section 5 below. Their properties lead to interesting results about cohomology of the free loop space of 1-connected spaces.

⁵this functor was called in [B2] and [B4] string cohomology for its unifying role explained below, cf. Observation 2. The name "string homology" was afterwards used by Sullivan and his school to designate the homology and equivariant homology of the free loop space of a closed manifold when endowed with additional structures induced by intersection theory and the Pontrjagin product in the chains of based pointed loops cf. [CS].

4. De-Rham Theory in the presence of a smooth vector field

Let M be a smooth manifold, possibly of infinite dimension. In the last case the manifold is modeled on a good Frechet space ⁶, for which the differential calculus can be performed as expected.

Consider the CDGA of differential forms $\Omega^*(M)$ with exterior differential $d^* : \Omega^*(M) \rightarrow \Omega^{*+1}(M)$ and interior differential $i_*^X : \Omega^*(M) \rightarrow \Omega^{*-1}(M)$, the contraction along the vector field X . They are not compatible. However we can consider the Lie derivative $L_X := d \cdot i^X + i^X \cdot d$ and define $\Omega_X(M) := \{\omega \in \Omega(M) | L_X \omega = 0\}$; $\Omega_X(M)$ consists of the smooth forms invariant by the flow induced by X . The graded vector space $\Omega_X^*(M)$ is a commutative graded algebra, a sub algebra of $\Omega^*(M)$, and the restriction of d^* and of i_*^X leave invariant $\Omega_X^*(M)$ and are compatible. Consequently $(\Omega_X^*(M), d^*, i_*^X)$ is a mixed CDGA.

Denote by

- (i) $H_X^*(M) := H^*(\Omega_X^*, d^*)$,
- (ii) ${}^\pm H_X^*(M) := {}^\pm H^*(\Omega_X^*, d^*, i_*^X)$,
- (iii) $PH_X^*(M) := PH^*(\Omega_X^*, d^*, i_*^X)$,
- (iv) $\mathbb{P}H_X^*(M) := \mathbb{P}H^*(\Omega_X^*, d^*, i_*^X)$.

The diagram Fig 2 becomes

$$\begin{array}{ccccccc}
 \longrightarrow & +H_X^r(M) & \xrightarrow{S^r} & +H_X^{r+2}(M) & \xrightarrow{J^{r+2}} & H^{r+2}(M) & \xrightarrow{B^{r+2}} & +H_X^{r+1}(M) & \longrightarrow & \dots \\
 & \downarrow id & & \downarrow \mathbb{I}^{r+2} & & \downarrow & & \downarrow & & \\
 \longrightarrow & +H_X^r(M) & \xrightarrow{\mathbb{I}^r} & \mathbb{P}H^{r+2}(M) & \xrightarrow{\mathbb{J}^{r+2}} & -H_X^{r+2}(M) & \xrightarrow{\mathbb{B}^{r+2}} & +H_X^{r+1}(M) & \longrightarrow & \dots
 \end{array}$$

Fig 3

The above diagram becomes more interesting if the vector field X is induced from an S^1 action $\mu : S^1 \times M \rightarrow M$ (i.e. if $x \in M$ then $X(x)$ is the tangent to the orbits through x). In this section we will explore particular cases of this diagram.

Observe that since μ is a smooth action, the subset F of fixed points is a smooth sub manifold. For any $x \in F$ denote by $\rho_x : S^1 \times T_x(M) \rightarrow T_x(M)$ the linearization of the action at x which is a linear representation. The inclusion $F \subset M$ induces the morphism $r^* : (\Omega_X^*(M), d^*, i_*^X) \rightarrow (\Omega^*(F), d^*, 0)$.

For a linear representation $\rho : S^1 \times V \rightarrow V$ on a good Frechet space denote by V^f the fixed point set and by X the vector field associated to ρ when regarded as a smooth action.

Definition 4. A linear representation $\rho : S^1 \times V \rightarrow V$ on the good Frechet space is good if the following conditions hold:

- a. V^f , the fixed point set, is a good Frechet space,
- b. The map $r^* : \Omega^*(V) \rightarrow \Omega^*(V^f)$ induced by the inclusion is surjective,

⁶this is a Frechet space with countable base which admits a smooth partition of unity; Note that if a Frechet space V is good then the space of smooth maps $C^\infty(S^1, V)$ equipped with the C^∞ -topology is also good.

c. $(\Omega_X^*(V, V^f), i_*^X)$ with $(\Omega_X^*(V, V^f) = \ker r^*$ is acyclic.

We have:

Proposition 2. 1. Any representation on a finite dimensional vector space is good.

2. If V is a good Frechet space then the regular representation $\rho : S^1 \times C^\infty(S^1, V) \rightarrow C^\infty(S^1, V)$, with $C^\infty(S^1, V)$, the Frechet space of smooth functions, is good.

For a proof consult Appendix [B1]. The proof is based on an explicit formula for i^X in the case of irreducible S^1 -representation and on the writing of the elements of $C^\infty(S^1, V)$ as Fourier series.

Definition 5. A smooth action $\mu : S^1 \times M \rightarrow M$ is good if its linearization at any fixed point is a good representation.

Then a smooth action on any finite dimensional manifold is good and so is the canonical smooth action of S^1 on P^{S^1} , the smooth manifold of smooth maps from S^1 to P where P is any smooth Frechet manifold (in particular a finite dimensional manifold). In view of the definitions above observe the following.

Proposition 3. If $\tilde{M} = (M, \mu)$ is a smooth S^1 -manifold and X is the associated vector field, then:

1. $H_X^*(M) = H^*(M)$,
2. ${}^+H_X^*(M) = H_{S^1}^*(\tilde{M})$ and $S : H^*(M) \rightarrow H^{*+2}(M)$ identifies to the multiplication with $u \in H_{S^1}^2(pt)$, the generator of the equivariant cohomology of one point space,
3. $PH_X^*(M) = \lim_{\substack{\rightarrow \\ S}} H_{S^1}^{*+2k}(\tilde{M})$.

If the action is good then:

4. $\mathbb{P}H_X^*(M) = K^*(F)$ where

$$K^r(F) = \prod_k H^{2k}(F) \text{ if } r \text{ even}$$

$$K^r(F) = \prod_k H^{2k+1}(F) \text{ if } r \text{ odd.}$$

If M is a closed of n -dimension manifold then :

5. ${}^-H_X^*(M) = H_{n-1-*}^{S^1}(\tilde{M}, \mathcal{O}_M)$ with $H_*^{S^1}(\tilde{M}, \mathcal{O}_M)$ the equivariant homology with coefficients in the orientation bundle ⁷ of M .

⁷Recall that $H_*^{S^1}(M, \mathcal{O}_M) = H_*(M//S^1, \mathcal{O}_M)$ where $M//S^1$ is the homotopy quotient of this action. This equivariant homology can be derived from invariant currents in the same way as equivariant cohomology from invariant forms, cf. [AB86]. The complex of invariant currents (with coefficients in orientation bundle) contains the complex $(\Omega_X^{n-*}(M, \mathcal{O}_M), \partial_{n-*})$ as a quasi isomorphic sub complex.

Proof. 1. The verification is standard since S^1 is compact and connected; one construct $av^* : (\Omega^*(M), d^*) \rightarrow (\Omega_X^*, d^*)$ by S^1 -averaging using the compacity of S^1 . The homomorphism induced in cohomology by av^* is obviously surjective. To check it is injective one has to show that any closed k differential form ω which becomes exact after applying av is already exact, precisely $\int_c \omega = 0$ for any smooth k -cycle c . Indeed, since the connectivity of S^1 implies $\int_c \omega = \int_{\mu(-\theta, c)} \omega$, $\theta \in S^1$ one has:

$$\begin{aligned} \int_c \omega &= 1/2\pi \int_{S^1} \left(\int_c \omega \right) d\theta = \\ 1/2\pi \int_{S^1} \left(\int_{\mu(-\theta, c)} \omega \right) d\theta &= 1/2\pi \int_{S^1} \left(\int_c \mu_\theta^*(\omega) \right) d\theta = \\ \int_c (1/2\pi \int_{S^1} \mu_\theta^*(\omega) d\theta) &= \int_c av^*(\omega) = 0. \end{aligned}$$

Here μ_θ denotes the diffeomorphism $\mu(\theta, \cdot) : M \rightarrow M$.

2. Looking at the definition in section 2 one recognizes one of the most familiar definition of equivariant cohomology using invariant differential forms cf. [AB86].

3. The proof is a straightforward consequence of Proposition 1 in section 2 and (2) above.

4. Let F be the smooth sub manifold of the fixed points of μ . Clearly $r^* : (\Omega_X^*(M), d^*, i_*^X) \rightarrow (\Omega^*(F), d_F^*, 0)$ is a morphism of mixed CDGA, hence of mixed complexes. If the smooth action is good then the above morphism induces an isomorphism in homology $H_*(\Omega_X^*, i_*^X) \rightarrow H_*(\Omega^*(F), 0)$. To check this we have to show that $(\ker r^*, i_*^X)$, with $\ker r^* := \{\omega \in \Omega_X^*(M) | \omega|_F = 0\}$, is acyclic. This follows (by S^1 -average) from the acyclicity of the chain complex $(\Omega^*(M, F), i_*^X)$ which in turn can be derived using the linearity w.r. to functions of i^X . Indeed, using a "partition of unity" argument it suffices to verify this acyclicity locally. For points outside F the acyclicity follows from the acyclicity of the complex

$\dots \Lambda^{*-1}(V) \xrightarrow{i^e} \Lambda^*(V) \xrightarrow{i^e} \Lambda^{*-1}(V) \rightarrow \dots$, where V is a Frechet space, $\Lambda^k(F)$ the space of skew symmetric k -linear maps from V to $\kappa = \mathbb{R}, \mathbb{C}$ and $e \in V \setminus 0$. For points $x \in F$ this follows from the fact that the linearization of the action at x is a good representation, as stated in Proposition 2.

5. If \tilde{M} is a finite dimensional smooth S^1 -manifold we can equip M with an invariant Riemannian metric g and consider $\star : \Omega^*(M) \rightarrow \Omega^{n-*}(M; \mathcal{O}_M)$ the Hodge star operator. Denote by $\omega \in \Omega^1(M)$ the 1-form corresponding to X w.r. to the metric g , by $e_\omega : \Omega^*(M; \mathcal{O}_M) \rightarrow \Omega^{*+1}(M; \mathcal{O}_M)$ the exterior multiplication with ω and by $\partial_* : \Omega^*(M; \mathcal{O}_M) \rightarrow \Omega^{*-1}(M; \mathcal{O}_M)$ the formal adjoint of d^{*-1} w.r. to g i.e. $\partial_* = \pm \star \cdot d^{n-*} \cdot \star^{-1}$. Note that $e_\omega = \pm \star \cdot i^X \cdot \star^{-1}$. All these operators leave Ω_X invariant since g is invariant. Clearly $(\Omega_X^*(M; \mathcal{O}_M), e_\omega^*, \partial_*)$ is a mixed cochain complex and we have

$${}^-H_{i^X}^*(\Omega_X(M), d, i^X) = {}^+H_{n-*}^\partial(\Omega_X(M; \mathcal{O}_M), e^\omega, \partial).$$

The equivariant homology of \tilde{M} with coefficients in the orientation bundle can be calculated from the complex of invariant currents which, if M closed, contains the complex $(\Omega_X^{n-*}(M, \mathcal{O}_M), \partial_{n-*})$ as a quasi isomorphic sub complex. As a consequence we have

$$\begin{aligned} H_{n-*}(\Omega_X(M; \mathcal{O}_M), \partial) &= H_*(M; \mathcal{O}_M) \\ {}^+H_{n-*}^{e_\omega}(\Omega_X(M; \mathcal{O}_M), e_\omega, \partial) &= H_*^{S^1}(M; \mathcal{O}_M) \end{aligned}$$

(cf. section 2 for notations). \square

As a consequence of Proposition 3 (1)-(4) for any smooth S^1 manifold with good S^1 -action the second long exact sequence in diagram Fig 3 becomes

$$\begin{array}{ccccccc} \cdots \rightarrow H_{S^1}^{r-2}(\tilde{M}) & \xrightarrow{\mathbb{I}^{r-2}} & K^r(F) & \xrightarrow{\mathbb{J}^r} & -H_{S^1}^r(\tilde{M}) & \xrightarrow{\mathbb{B}^r} & H_{S^1}^{r-1}(\tilde{M}) \rightarrow \cdots \\ & \searrow & \uparrow & & & & \\ & & \lim_{\substack{\rightarrow \\ S}} H_{S^1}^{r+2k}(\tilde{M}) & & & & \end{array}$$

Fig 4

The sequence above is obviously natural in the sense that $f : \tilde{M} \rightarrow \tilde{N}$, an S^1 -equivariant smooth map, induces a commutative diagram whose rows are the above exact sequence Fig 4 for \tilde{M} and \tilde{N} . Then if f and its restriction to the fixed point set induce isomorphisms in cohomology it induces isomorphisms in $H_{S^1}^*$ and K^* and then all other types of equivariant cohomologies $-H_{S^1}^*$, $PH_{S^1}^*$, $\mathbb{P}H_{S^1}^*$.

If \tilde{M} is a compact smooth S^1 -manifold in view of Proposition 3 (5) one identifies $-H_{S^1}^r(M)$ to $H_{n-r}^{S^1}(M; \mathcal{O}_M)$ and in view of this identification write Pd_{n-r} instead of \mathbb{B}^r . The long exact sequence becomes

$$\cdots \longrightarrow K^r(F) \longrightarrow H_{n-r}^{S^1}(\tilde{M}; \mathcal{O}_M) \xrightarrow{Pd_{n-r}} H_{S^1}^{r-1}(\tilde{M}) \longrightarrow K^{r+1}(F) \longrightarrow \cdots$$

Fig 5

In case that the fixed point set is empty we conclude that

$$Pd_{n-r} : H_{n-r}^{S^1}(\tilde{M}, \mathcal{O}_M) \rightarrow H_{S^1}^{r-1}(\tilde{M})$$

is an isomorphism. In this case the orbit space M/S^1 is a \mathbb{Q} -homological manifold of dimension $(n-1)$, hence

$$\begin{aligned} H_{n-r}^{S^1}(\tilde{M}; \mathcal{O}_M) &= H_{n-r}(M/S^1; \mathcal{O}_{M/S^1}) \\ H_{S^1}^{r-1}(\tilde{M}; \mathcal{O}_M) &= H^{r-1}(M/S^1; \mathcal{O}_{M/S^1}) \end{aligned}$$

and Pd_* is nothing but the Poincaré duality isomorphism for \mathbb{Q} -homology manifolds. In general the long exact sequence Fig 5 measures the failure of the Poincaré duality map, Pd_* , to be an isomorphism.

5. The free loop space and s-cohomology

A more interesting example is provided by the S^- -manifold $P^{\tilde{S}^1} := (P^{S^1}, \mu)$. Here P^{S^1} denotes the smooth manifold of smooth maps from S^1 to P modeled by the Frechet space $C^\infty(S^1, V)$ where V is the model for P (finite or infinite dimensional Frechet space) cf [B1]. This smooth manifold is equipped with the canonical smooth S^1 -action $\mu : S^1 \times P^{S^1} \rightarrow P^{S^1}$ defined by

$$\mu(\theta, \alpha)(\theta') = \alpha(\theta + \theta'), \quad \alpha : S^1 \rightarrow P, \quad \theta, \theta' \in S^1 = \mathbb{R}/2\pi.$$

The fixed points set of the action μ consists of the constant maps hence identifies with P . This action is the restriction of the canonical action of $O(2)$, the group of isometries of S^1 , to the subgroup of orientation preserving isometries identified to S^1 itself. For any $x \in P$ viewed as a fixed point of μ the linearization representation is the regular representation of S^1 on $V = T_x(P)$. In view of Proposition 2 the action μ is good. The space P^{S^1} is also equipped with the natural maps ψ_k , $k = 1, 2, \dots$, the geometric power maps and with the involution τ , defined by

$$\begin{aligned} \psi_k(\alpha)(\theta) &= \alpha(k\theta) \\ \tau(\alpha)(\theta) &= \alpha(-\theta) \end{aligned}$$

with $\alpha \in P^{S^1}$, and $\theta \in S^1$.

The involution τ is the restriction of the action of $O(2)$ to the reflexion $\theta \rightarrow -\theta$ in S^1 . Then $(\Omega_X^*(P^{S^1}), d^*, i_*^X)$ is a mixed CDGA, hence a mixed complex with power maps Ψ_k, τ and involution τ induced from ψ_k and τ .

Suppose $f : P_1 \rightarrow P_2$ is a smooth map. It induces a smooth equivariant map $f^{S^1} : P_1^{S^1} \rightarrow P_2^{S^1}$ whose restriction to the fixed points set is exactly f . If f is a homotopy equivalence then so is f^{S^1} .

Introduce the notation

$$\begin{aligned} hH^*(P) &:= H^*(P^{S^1}), \\ cH^*(P) &:= H_{S^1}^*(P^{\tilde{S}^1}), \\ sH^*(P) &:= H_{S^1}^*(P^{\tilde{S}^1}). \end{aligned}$$

The assignments $P \rightsquigarrow hH^*(P)$, $P \rightsquigarrow cH^*(P)$, $P \rightsquigarrow sH^*(P)$ are functors ⁸ with the property that $hH^*(f), cH^*(f), sH^*(f)$ are isomorphism if f is a homotopy equivalence, hence they are all homotopy functors. They are related by the commutative diagram below. This diagram is the same as diagram (Fig 3) applied to $\tilde{M} = P^{\tilde{S}^1}$ with the specifications provided by Proposition 3.

⁸the notations hH^*, cH^* are motivated by the Hochschild resp. cyclic homology interpretation of these functors, while sH^* is abbreviation from string cohomology.

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & cH^{r-2}(P) & \xrightarrow{S^{r-2}} & cH^r(P) & \xrightarrow{J^r} & hH^r(P) & \xrightarrow{B^r} & cH^{r-1}(P) & \longrightarrow & \cdots \\
& & \downarrow id & & \downarrow \mathbb{I}^r & & \downarrow & & \downarrow id & & \\
\cdots & \longrightarrow & cH^{r-2}(P) & \xrightarrow{\mathbb{I}^{r-2}} & K^r(P) & \xrightarrow{J^r} & sH^r(P) & \xrightarrow{\mathbb{B}^r} & cH^{r-1}(P) & \longrightarrow & \cdots \\
& & \downarrow S & \nearrow \mathbb{I}^r & \uparrow & & & & & & \\
& & cH^r(P) & \longrightarrow & \varinjlim cH^{r+2k}(P) & & & & & &
\end{array}$$

Fig 6

where $\varinjlim cH^{r+2k}(P) = \varinjlim \{ \cdots \rightarrow cH^{r+2k}(P) \xrightarrow{S} cH^{r+2k+2}(P) \rightarrow \cdots \}$.

The linear map

$$cH^*(P) = H_{S^1}^*(P^{\tilde{S}^1}) \xrightarrow{\mathbb{I}^r} K^*$$

factors through $\varinjlim cH^{r+2k}(P)$ which depends only on the fundamental group of P .

Indeed, it is shown in [B3] that if $P(1)$ ⁹ is a smooth manifold (possibly of infinite dimension) which has the homotopy type of $K(\pi, 1)$ and $p(1) : P \rightarrow P(1)$ is smooth map inducing an isomorphism for the fundamental group then $\varinjlim H_{S^1}^{r+2k}(P^{\tilde{S}^1}) \rightarrow \varinjlim H_{S^1}^{r+2k}(P(1)^{S^1})$ is an isomorphism. cf [B3]. Then if one denotes by $\overline{cH}^*(M) := \text{coker}(cH^*(M) \rightarrow cH^*(pt))$ and $\overline{K}^*(M) := \text{coker}(K^*(M) \rightarrow K^*(pt))$ ¹⁰ one obtains

Theorem 2. *If P is a 1-connected smooth manifold then we have the following short exact sequence:*

$$0 \rightarrow \overline{K}^r(P) \otimes \kappa \xrightarrow{J^r} sH^r(P) \xrightarrow{\mathbb{B}^r} \overline{cH}^{r-1}(P) \rightarrow 0$$

where $\kappa = \mathbb{R}$ or \mathbb{C} .

Observation 2. *The vector space $K^*(P)$ can be identified via the Chern character to the Atiyah–Hirzebruch (complex) K -theory tensored with the field $\kappa = \mathbb{R}$ or \mathbb{C} , depending of what sort of differential forms one consider (real or complex valued). When P is 1-connected $\overline{cH}^*(P)$ identifies to $\text{Hom}(\tilde{A}_*(P), \kappa)$ where $\tilde{A}_*(P)$ denotes the reduced Waldhausen algebraic K -theory¹¹, cf [B2]. From this perspective sH^* unifies topological (Atiyah–Hirzebruch) K -theory and Waldhausen algebraic K -theory.*

Observation 3. *In view of the definition of ${}^-H_\beta^*(C^*, \delta^*, \beta_*)$, cf. section 2, observe that $sH^*(P)$ is represented by infinite sequences¹², rather than eventually finite sequences of invariant differential forms on P^{S^1} . If instead of “infinite sequences” we would have considered “eventually finite sequences” the outcome would have*

⁹the notation for the first stage Postnikov term of P .

¹⁰clearly $K^r(pt) = H_{S^1}^r(pt) = \kappa$ resp. 0 if r is even resp. odd

¹¹often referred to as A -theory.

¹² $sH^*(P)$ is the cohomology of the cochain complex $({}^-C^*, {}^-D^*)$ with ${}^-C^r = \prod_{k \geq 0} \Omega_{inv}^{r+2k}(P^{S^1})$ and ${}^-D^r(\cdots, \omega_{r+2}, \omega_r) = (\cdots, (i^X \omega_{r+2} + d\omega_r))$, cf. section 2.

been different for infinite dimensional manifolds. The difference between "infinite sequences" and "eventually finite sequences" exists only for infinite dimensional manifolds which P^{S^1} always is.

The power maps ψ_k induce the endomorphisms $h\Psi_k$, $c\Psi_k$, $s\Psi_k$ and $K\Psi_k$ on hH^* , cH^* , sH^* , and K^* .

In general only $K\Psi_k$ are easy to describe. Precisely if r is even then $K^r = \prod_{i \geq 0} H^{2i}(P)$ and if r is odd then $K^r = \prod_{i \geq 0} H^{2i+1}(P)$, and in both cases $K\Psi_k = \prod_{i \geq 0} k^{i-r} Id$.

The symmetric part with respect to the involution $c\Psi_{-1}$, i.e. the eigenspaces corresponding to the eigenvalues $+1$ identifies with $H_{O(2)}^*(P^{S^1})$, the equivariant cohomology for the canonical $O(2)$ -action.

However, if P is 1-connected, in view of the section 6, one can describe both the eigenvalues and the eigenspaces of the power maps $h\Psi_k$ and $c\Psi_k$ and then of $s\Psi_k$. We have:

Theorem 3. *Let P be a 1-connected manifold.*

1. *All eigenvalues of the endomorphisms $h\Psi_k$ and $c\Psi_k$ are k^r , $r = 0, 1, 2, \dots$, and the eigenspaces corresponding to k^r are independent of k provided $k \geq 2$.*

2. *Denotes these eigenspaces by $hH^*(M)(r)$ and $cH^*(M)(r)$. Then $hH^*(0) = H^*(X; \kappa)$, $cH^*(0) = H^{*+1}(X; \kappa)$, and*

$hH^r(p) = cH^r(p) = 0$, $p \geq r + 1$.

3. *If $\sum_i \dim \pi_i(P) \otimes \kappa < \infty$, κ the field of real or complex numbers and $\sum_i \dim(H^i(P)) < \infty$ then for any $r \geq 0$ one has*

$$\sum_i \dim hH^i(P)(r) < \infty, \quad \sum_i \dim cH^i(P)(r) < \infty.$$

If P is "formal" in the sense of rational homotopy theory ¹³, (a projective complex algebraic variety, or more general a closed Kaehler manifold is formal, cf [DGMS]) then the Euler Poincaré characteristic

$$\chi^h(\lambda) := \sum_{i,r} \dim hH^i(r) \lambda^r$$

and

$$\chi^c(\lambda) := \sum_{i,r} \dim cH^i(r) \lambda^r$$

can be explicitly calculated in terms of the numbers $\dim H^i(P)$, cf [B]. The explicit formulae are quite complicated. They require the results of P.Hanlon [H] about the eigenspaces of Adams operations in Hochschild and cyclic homology as well as the identification of $hH^*(P)$ resp. $cH^*(P)$ with the Hochschild resp. cyclic homology

¹³i.e. for each connected component of P De-Rham algebra and the cohomology algebra equipped with the differential 0 are homotopy equivalent, cf section 6.

of the graded algebra $H^*(P)$. These are not discussed in this paper but the reader can consult [BFG] and [B4] for precise statements.

The functor $\overline{sH}^r(P)$ is of particular interest in geometric topology. In the case P is 1-connected it calculates in some ranges the homotopy groups of the (homotopy) quotient space of homotopy equivalences by the group of diffeomorphisms [B1], [B4].

6. The free loop construction for CDGA

The "free loop" construction associates to a free connected CDGA, $(\Lambda[V], d_V)$ a mixed CDGA, $(\Lambda[V \oplus \overline{V}], \delta_V, i^V)$, endowed with power maps Ψ_k and involution τ defined as follows.

- (i) Let $\overline{V} = \bigoplus_{i \geq 0} \overline{V}^i$ with $\overline{V}^i := V^{i+1}$ and let $\Lambda[V \oplus \overline{V}]$ be the commutative graded algebra generated by $V \oplus \overline{V}$.
- (ii) Let $i^V : \Lambda[V \oplus \overline{V}] \rightarrow \Lambda[V \oplus \overline{V}]$ be the unique internal differential (of degree -1) which extends $i^V(v) = \overline{v}$ and $i^V(\overline{v}) = 0$.
- (iii) Let $\delta_V : \Lambda[V \oplus \overline{V}] \rightarrow \Lambda[V \oplus \overline{V}]$ be the unique external differential (of degree $+1$) which extends $\delta_V(v) = d(v)$ and $\delta_V(\overline{v}) = -i^V(d(v))$.
- (iv) Let $\Psi_k : (\Lambda[V \oplus \overline{V}], \delta_V) \rightarrow (\Lambda[V \oplus \overline{V}], \delta_V)$, $k = -1, 1, 2, \dots$ be the unique morphisms of CDGA which extends $\Psi_k(v) = v$, $\Psi_k(\overline{v}) = k\overline{v}$. We put $\tau := \Psi_{-1}$. The maps Ψ_k $k \geq 1$ are called the power maps and τ the canonical involution. One has

$$\begin{aligned}\Psi_k \cdot \Psi_r &= \Psi_{kr} \\ \Psi_k \cdot i_V &= k i_V \cdot \Psi_k\end{aligned}$$

- (v) Let $\Lambda^+[V \oplus \overline{V}]$ be the ideal of $\Lambda[V \oplus \overline{V}]$ generated by $V \oplus \overline{V}$ or the kernel of the augmentation which vanishes on $V \oplus \overline{V}$.

Note that :

Observation 4. 1. $(\text{Im}(i^V), \delta_V, 0)$ is a mixed sub complex of $(\Lambda^+[V \oplus \overline{V}], \delta_V, i^V) \subset (\Lambda[V \oplus \overline{V}], \delta_V, i^V)$

2. Ψ_k , $k = -1, 1, 2, \dots$ leave $(\Lambda^+[V \oplus \overline{V}], \delta_V, i^V)$ and $(\text{Im}(i^V), \delta)$ invariant and have k^r $r = 0, 1, 2, \dots$ as eigenvalues. These are all eigenvalues.

For $k \geq 2$ the eigenspace of $\Psi_r : \Lambda[V \oplus \overline{V}] \rightarrow \Lambda[V \oplus \overline{V}]$ corresponding to the eigenvalue k^r is exactly $\Lambda[V] \otimes \overline{V}^{\otimes r}$, resp. $\Lambda^+[V \oplus \overline{V}] \cap \Lambda[V] \otimes \overline{V}^{\otimes r}$, resp. $\text{Im}(i^V)(r) = \text{Im}(i^V) \cap \Lambda[V] \otimes \overline{V}^{\otimes r}$, hence independent of k . Each such eigenspace is δ_V -invariant.

3. The mixed complex $(\Lambda^+[V \oplus \overline{V}], \delta_V, i^V)$ is i^V -acyclic,

4. We have the decomposition

$$(\Lambda[V \oplus \overline{V}], \delta_V) = \bigoplus_{r \geq 1} (\Lambda[V] \otimes \overline{V}^{\otimes r}, \delta_V)$$

and the analogous decomposition for $(\Lambda^+[V \oplus \overline{V}], \delta_V)$ and $(\text{Im}(i^V), \delta_V)$ referred from now on as the weight decompositions.

Consider the complex $(\Lambda[V] \otimes \bar{V}^{\otimes r}, \delta_V)$ and the filtration provided by $\Lambda[V] \otimes F_p(\bar{V}^{\otimes r})$ with $F_p(\bar{V}^{\otimes r})$ the span of elements in $\bar{V}^{\otimes r}$ of total degree $\leq p$.

For a graded vector space $W = \oplus_i W^i$ denote by $\dim W = \sum \dim W^i$.

Observation 5. 1. $(\Lambda[V] \otimes F_p(\bar{V}^{\otimes r}), \delta_V)$ is a sub complex of $(\Lambda[V] \otimes \bar{V}^{\otimes r}, \delta_V)$.

2. If $(\Lambda[V], d_V)$ is minimal and one connected then, by Observation 1, $\delta(F_p(\bar{V}^{\otimes r})) \subset \Lambda[V] \otimes F_{p-1}(\bar{V}^{\otimes r})$ and then

$$(\Lambda[V] \otimes F_p(\bar{V}^{\otimes r}) / \Lambda[V] \otimes F_{p-1}(\bar{V}^{\otimes r}), \delta_V) = (\Lambda[V], d_V) \otimes F_p(\bar{V}^{\otimes r}) / F_{p-1}(\bar{V}^{\otimes r}).$$

$$2. \sum_p \dim(F_p(\bar{V}^{\otimes r}) / F_{p-1}(\bar{V}^{\otimes r})) = \dim(\bar{V}^{\otimes r}) = (\dim V)^r.$$

If $f : (\Lambda[V], d_V) \rightarrow \Lambda[W, d_W]$ is a morphism of CDGAs then it induces $\tilde{f} : (\Lambda[V \oplus \bar{V}], \delta_V, i^V) \rightarrow (\Lambda[W \oplus \bar{W}], \delta_W, i^W)$ which intertwines Ψ'_k 's and then preserves the weight decompositions.

We introduce the the notation $HH^*, CH^* PH^*$

$$HH^*(\Lambda[V], d_V) := H^*(\Lambda[V \oplus \bar{V}], \delta_V)$$

$$CH^*(\Lambda[V], d_V) := {}^+H_{i^V}^*(\Lambda[V \oplus \bar{V}], \delta_V, i^V)$$

$$PH^*(\Lambda[V], d_V) := PH_{i^V}^*(\Lambda[V \oplus \bar{V}], \delta_V, i^V)$$

and for a morphism f denote by $HH(f), CH(f), PH(f)$ the linear maps induced by \tilde{f} . The assignments HH^*, CH^*, PH^* provide functors from the category of fc-CDGA's to graded vector spaces. They come equipped with the operations $H\Psi_k, C\Psi_k$ etc. induced from Ψ_k . Since for f quasi isomorphisms $HH^*(f), CH^*(f), PH^*(f)$ are isomorphisms these functors, as shown in section 3 extend to the category of c-CDGA's. We have the following result.

Theorem 4. Let $(\mathcal{A}, d_{\mathcal{A}})$ be a connected CDGA.

1. All eigenvalues of the endomorphisms $H\Psi_k$ and $C\Psi_k$ are $k^r, r = 0, 1, 2 \dots$, and their eigenspaces are independent of k provided $k \geq 2$. One denotes them by $HH(\mathcal{A}, d_{\mathcal{A}})(r)$, and $CH(\mathcal{A}, d_{\mathcal{A}})(r)$.

2.

$$HH^*(\mathcal{A}, d_{\mathcal{A}})(0) = H^*(\mathcal{A}, d_{\mathcal{A}}),$$

$$CH^*(\mathcal{A}, d_{\mathcal{A}})(0) = H^{*+1}(\mathcal{A}, d_{\mathcal{A}})$$

$$HH^r(\mathcal{A}, d_{\mathcal{A}})(p) = CH^r(\mathcal{A}, d_{\mathcal{A}})(p) = 0, \quad p \geq r + 1$$

3. Suppose $(\mathcal{A}, d_{\mathcal{A}})$ is 1-connected with minimal model $(\Lambda[V], d_V)$. If $\sum_i \dim V^i < \infty$ and $\sum_i \dim H^i(\mathcal{A}, d_{\mathcal{A}}) < \infty$ then for any $r \geq 0$ one has

$$\sum_i \dim HH^i(\mathcal{A}, d_{\mathcal{A}})(r) < \infty, \quad \sum_i \dim CH^i(\mathcal{A}, d_{\mathcal{A}})(r) < \infty.$$

Proof. It suffices to check the statements for $(\mathcal{A}, d_{\mathcal{A}}) = (\Lambda[V], d_V)$ minimal. Items 1) and 2) are immediate consequences of Observation 4.

Item 3) follows from Observation 5. Indeed for a fixed r one has

$$\begin{aligned} & \sum_i \dim H^i(\Lambda[V] \otimes \bar{V}^{\otimes r}, \delta_V) \leq \\ \sum_{i,p} \dim H^i(\Lambda[V] \otimes F_p(\bar{V}^{\otimes r}) / \Lambda[V] \otimes F_{p-1}(\bar{V}^{\otimes r}), \delta_V) = \\ & (\dim V)^r \cdot \sum_i \dim H^i(\Lambda[V], d_V) \end{aligned}$$

□

In addition to $\chi(\mathcal{A}, d_{\mathcal{A}}) := \sum (-1)^i \dim H^i(\mathcal{A}, d_{\mathcal{A}})$ one can consider

$$\begin{aligned} \chi^H(\mathcal{A}, d_{\mathcal{A}})(r) &:= \sum (-1)^i \dim HH^i(\mathcal{A}, d_{\mathcal{A}})(r) \text{ and} \\ \chi^C(\mathcal{A}, d_{\mathcal{A}})(r) &:= \sum (-1)^i \dim CH^i(\mathcal{A}, d_{\mathcal{A}})(r), \end{aligned}$$

and then the power series in λ ,

$$\chi^H(\mathcal{A}, d_{\mathcal{A}})(\lambda) := \sum \chi^H(\mathcal{A}, d_{\mathcal{A}})(r) \lambda^r, \quad \chi^C(\mathcal{A}, d_{\mathcal{A}})(\lambda) := \sum \chi^C(\mathcal{A}, d_{\mathcal{A}})(r) \lambda^r.$$

Theorem 4 (3) implies that for $(\mathcal{A}, d_{\mathcal{A}})$ 1-connected with $\sum_i \dim V^i < \infty$ and $\sum_i \dim H^i(\mathcal{A}, d_{\mathcal{A}}) < \infty$ the partial Euler–Poincaré characteristics $\chi^H(\mathcal{A}, d_{\mathcal{A}})(r)$ and $\chi^C(\mathcal{A}, d_{\mathcal{A}})(r)$ and therefore the power series $\chi^H(\mathcal{A}, d_{\mathcal{A}})(\lambda)$ and $\chi^C(\mathcal{A}, d_{\mathcal{A}})(\lambda)$ are well defined. The results of Hanlon [H] permit to calculate explicitly $\chi^H(\lambda)$ and $\chi^C(\lambda)$ in terms of $\dim H^i(\mathcal{A}, d_{\mathcal{A}})$ if $(\mathcal{A}, d_{\mathcal{A}})$ is 1-connected and formal, i.e. there exists a quasi isomorphisms $(\Lambda[V], d) \rightarrow (H^*(\Lambda[V], d), 0)$, $(\Lambda[V], d)$ a minimal model of $(\mathcal{A}, d_{\mathcal{A}})$.

We want to define an algebraic analogue of the functor sH^* on the category of cCDGA's. Recall that for a morphism $f^* : (C_1^*, d_1^*) \rightarrow (C_2^*, d_2^*)$ the “mapping cone” $Cone(f^*)$ is the cochain complex with components $C_f^* = C_2^* \oplus C_1^{*+1}$ and with

$$d_f^* = \begin{pmatrix} d_2^* & f^{*+1} \\ 0 & -d_1^{*+1} \end{pmatrix}.$$

Notice that, when f^* is injective, the morphism $Cone(f^*) \rightarrow C_2^*/f^*(C_1^*)$ defined by the composition $C_2^* \oplus C_1^{*+1} \rightarrow C_2^* \rightarrow C_2^*/f^*(C_1^*)$ is a quasi isomorphism.

We will consider the composition

$$\underline{I}^{*-2} :+ C_{iV}^{*-2}(\Lambda[V \oplus \bar{V}], \delta_V, i^V) \xrightarrow{I^{*-2}} \mathbb{P}C^*(\Lambda[V \oplus \bar{V}], \delta_V, i^V) \xrightarrow{\mathbb{P}^*(p)} \mathbb{P}C^r(\Lambda[V], d_V, 0)$$

with the first arrow provided by the natural transformation $I^{*-2} :+ C_{\beta}^{*-2} \rightarrow \mathbb{P}C^*$ described in section 2 applied to the mixed complex $(\Lambda[V \oplus \bar{V}], \delta_V, i^V)$ and the second induced by the projection on the zero weight component of $(\Lambda[V \oplus \bar{V}], \delta_V, i^V)$.

The mapping cone $Cone(\underline{I}^{*-2})$, is functorial when regarded on the category of fc-CDGA's. Define

$$SH^*(\Lambda[V], d_V) := H^*(Cone(\underline{I}^*)).$$

The assignment $(\Lambda[V], d_V) \rightsquigarrow SH^*(\Lambda[V], d_V)$ is an homotopy functor.

Consider the commutative diagrams

$$\begin{array}{ccccc}
+C_{i^V}^{*-2}(\Lambda[V \oplus \bar{V}], \delta_V, i^V) & \xrightarrow{i^{*-2}} & +C_{i^V}^*(\Lambda[V \oplus \bar{V}], \delta_V, i^V) & \longrightarrow & Cone(i^{*-2}) \\
\downarrow id & & \downarrow \underline{I}^* & & \downarrow \\
+C_{i^V}^{*-2}(\Lambda[V \oplus \bar{V}], \delta_V, i^V) & \xrightarrow{\underline{I}^{*-2}} & \mathbb{P}C^*(\Lambda[V], d_V, 0) & \longrightarrow & Cone(\underline{I}^{*-2}).
\end{array}$$

and

$$\begin{array}{ccccc}
+C_{i^V}^{*-2}(\Lambda[V \oplus \bar{V}], \delta_V, i^V) & \xrightarrow{i^{*-2}} & +C_{i^V}^*(\Lambda[V \oplus \bar{V}], \delta, i^V) & \longrightarrow & Cone(i^{*-2}) \\
\downarrow id & & \downarrow id & & \downarrow \\
+C_{i^V}^{*-2}(\Lambda[V \oplus \bar{V}], \delta_V, i^V) & \xrightarrow{i^{*-2}} & +C_{i^V}^*(\Lambda[V \oplus \bar{V}], \delta_V, i^V) & \longrightarrow & C^*(\Lambda[V \oplus \bar{V}], \delta_V)
\end{array}$$

with the last vertical arrow in the second diagram a quasi isomorphism as noticed above.

The long exact sequence induced by passing to cohomology in the first diagram combined with the identifications implied by the second diagram lead to

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & CH^{r-2} & \xrightarrow{S^{r-2}} & CH^r & \xrightarrow{J^r} & HH^r & \xrightarrow{B^r} & CH^{r-1} & \longrightarrow & \cdots \\
& & \downarrow id & & \downarrow T^r & & \downarrow & & \downarrow id & & \\
\cdots & \longrightarrow & CH^{r-2} & \xrightarrow{\mathbb{I}^{r-2}} & K^r & \xrightarrow{\mathbb{J}^r} & SH^{r+2} & \xrightarrow{\mathbb{B}^r} & CH^{r-1} & \longrightarrow & \cdots \\
& & \downarrow S^{r-2}\mathbb{I}^r & \nearrow & \uparrow & & & & & & \\
& & CH^r & \longrightarrow & PH^r & & & & & &
\end{array}$$

Fig 7

with $PH^r = \varinjlim CH^{r+2k}$ and $K^r := K^r(\Lambda[V], d_V)$ given by $= \prod_k H^{2k}(\Lambda[V], d_V)$ resp. $\prod_k H^{2k+1}(\Lambda[V], d_V)$ if r is even resp. odd. It is immediate that Theorem 2 remains true for sH^*, cH^* replaced by SH^*, CH^* as follows easily from diagram Fig 7. The diagram Fig 7 should be compared with diagram Fig 6. This explains why $SH^*(\Lambda[V], d^V)$ will be regarded as the algebraic analogue of $sH^*(P)$.

It is natural to ask if the functors HH^*, CH^*, SH^* applied to $(\Omega^*(P), d^*)$ calculate hH^*, cH^*, sH^* applied to P and the diagram Fig 7 identifies to the diagram Fig 6. The answer is in general no, but is yes if P 1-connected .

The minimal model theory, discussed in the next section, permits to identify, $HH^*(\Omega^*(P), d^*), CH^*(\Omega^*(P), d^*)$ to $hH^*(P), CH^*(P)$ and then $SH^*(\Omega^*(P), d^*)$ to $SH^*(P)$ and actually diagram Fig 6 to diagram Fig 7 when P is 1-connected.

7. Minimal models and the proof of Theorem 3

Observe that if $(\mathcal{A}^*, d^*, \beta_*)$ is a mixed CDGA equipped with the power maps and involution $\Psi_k, k = -1, 1, 2, \dots$, then the diagram

$$\begin{array}{ccc}
{}^+H_{\beta}^*(\mathcal{A}^*, d^*, \beta_*) & \longrightarrow & H^*(\mathcal{A}^*, d^*) \\
\downarrow {}^+\Psi_k & & \downarrow {}^+\Psi_k \\
{}^+H_{\beta}^*(\mathcal{A}^*, d^*, \beta_*) & \longrightarrow & H^*(\mathcal{A}^*, d^*)
\end{array}$$

can be derived by passing to cohomology in the commutative diagram of CDGA's.

$$\begin{array}{ccc}
(\mathcal{A}^* \otimes \Lambda[u], \mathcal{D}[u]) & \longrightarrow & (\mathcal{A}^*, d^*) \\
\downarrow \Psi_k[u] & & \downarrow \Psi_k \\
(\mathcal{A}^* \otimes \Lambda[u], \mathcal{D}[u]) & \longrightarrow & (\mathcal{A}^*, d^*)
\end{array}$$

where $\Lambda[u]$ is the free commutative graded algebra generated by the symbol u of degree 2, $\mathcal{D}[u](a \otimes u^r) = d(a) \otimes u^r + \beta(a) \otimes u^{r+1}$ and $\Psi[u](a \otimes u^r) = 1/k^r \Psi(a) \otimes u^r$.

For P 1-connected and $(\Lambda([V], d_V)$ a minimal model of $(\Omega^*(P), d^*)$ we want to establish the existence of the homotopy commutative diagram

$$\begin{array}{ccccc}
& & A & \xrightarrow{\quad} & B \\
& \nearrow \tilde{\theta} & \downarrow & \nearrow \theta & \downarrow \\
C & \xrightarrow{\quad} & D & & \Psi_k^P \downarrow B \\
& \searrow \Psi_k^P[u] & \downarrow & \searrow \Psi_k & \downarrow \\
& & A & \xrightarrow{\quad} & B \\
\Psi_k[u] \downarrow C & \nearrow \tilde{\theta} & \downarrow & \nearrow \theta & \downarrow \\
C & \xrightarrow{\quad} & D & & \downarrow
\end{array}$$

where

$$\begin{aligned}
A &= (\Omega_X(P^{S^1}) \otimes \Lambda[u], \mathcal{D}[u]) \\
B &= (\Omega_X(P^{S^1}), d) \\
C &= (\Lambda[V \oplus \bar{V}] \otimes \Lambda[u], \delta[u]) \\
D &= (\Lambda[V \oplus \bar{V}], \delta)
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{D}[u](\omega \otimes u^r) &= d(\omega) \otimes u^r + i^X(\omega) \otimes u^{r+1} \\
\delta[u](a \otimes u^r) &= \delta(a) \otimes u^r + i^V(\omega) \otimes u^{r+1}.
\end{aligned}$$

The existence of the quasi isomorphism θ was established in in [SV]. The existence of the quasi isomorphism $\tilde{\theta}$ and the homotopy commutativity of the top square was established in [BV] and the homotopy commutativity of the side squares was verified in [BFG]. The right side square resp. left side square in this diagram provide identifications of $HH^*(\Lambda[V], d_V)$ with $hH^*(P)$ resp. of $CH^*(\Lambda[V], d_V)$

with $cH^*(P)$. These identifications are compatible with all natural transformations defined above and with the endomorphisms induced by the algebraic resp. geometric power maps. In particular one derive Theorem 3 from Theorem 4. It is tedious but straightforward to derive, under the hypothesis of 1- connectivity for P , the identification of the diagram Fig 6 for P and the diagram Fig 7 for $(\Omega^*(P), d^*)$.

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