Results on infinite dimensional topology and applications to the structure of the critical set of nonlinear Sturm-Liouville operators

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Abstract

We consider the nonlinear Sturm-Liouville differential operator $F(u) = -u'' + f(u)$ for $u \in H^2_D([0,\pi])$, a Sobolev space of functions satisfying Dirichlet boundary conditions. For a generic nonlinearity $f : \mathbb{R} \to \mathbb{R}$ we show that there is a diffeomorphism in the domain of F converting the critical set C of F into a union of isolated parallel hyperplanes. For the proof, we show that the homotopy groups of connected components of C are trivial and prove results which permit to replace homotopy equivalences of systems of infinite dimensional Hilbert manifolds by diffeomorphisms.

1 Introduction

Consider the nonlinear Sturm-Liouville problem

$$-u''(t) + f(u(t)) = g(t), \qquad u(0) = u(\pi) = 0$$

and for any smooth nonlinearity $f : \mathbb{R} \to \mathbb{R}$ denote by F the differential operator

$$\begin{aligned} F : H^2_D([0,\pi]) &\to L^2([0,\pi]). \\ u &\mapsto -u'' + f(u) \end{aligned}$$

Here $H^2_D([0,\pi])$ is the Sobolev space of functions $u(t)$ with square integrable second derivatives which satisfy Dirichlet boundary conditions $u(0) = u(\pi) = 0$

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and $L^2([0,\pi])$ is the usual Hilbert space of square integrable functions in $[0,\pi]$. We are interested in the critical set C of F,

$$C = \{u \in H^2_D([0,\pi]) \mid DF(u) \text{ has } 0 \text{ as an eigenvalue}\}.$$

Here $DF(u) : H^2_D([0,\pi]) \subset L^2([0,\pi]) \to L^2([0,\pi])$ is the Fredholm linear operator $DF(u)v = -v'' + f'(u)v$ of index 0. Let $\Sigma = \{m \in \mathbb{Z} \mid m > 0, -m^2 \in f'(\mathbb{R})\}$. Our main result about C is the following.

Theorem 1 *For tame nonlinearities f (to be defined below), C is the disjoint union of connected components C_m, $m \in \Sigma$, where each C_m is a smooth contractible hypersurface in $H^2_D([0,\pi])$. Furthermore, there is a diffeomorphism of $H^2_D([0,\pi])$ to itself taking C to a union of parallel hyperplanes.*

In the proof of this theorem we shall make use of two results concerning the topology of infinite dimensional manifolds, Theorem 2 (proved in Section 2) and Theorem 3 (in Section 3). The topological theorems are described in a generality which is greater than needed in this paper, since we believe that the results are of independent interest.

Theorem 2 *Let X and Y be separable Banach spaces. Suppose $i : Y \to X$ is a bounded, injective linear map with dense image and $M \subset X$ a smooth, closed submanifold of finite codimension. Then $N = i^{-1}(M)$ is a smooth closed submanifold of Y, and the restrictions $i : Y \setminus N \to X \setminus M$ and $i : (Y,N) \to (X,M)$ are homotopy equivalences.*

Our second topological result concerns H-manifolds, i.e., manifolds modeled on the separable infinite dimensional Hilbert space H.

Theorem 3 *Suppose $f : (V_1, \partial V_1) \to (V_2, \partial V_2)$ is a smooth homotopy equivalence of H-manifolds with boundary, $K_2 \subset V_2 \setminus \partial V_2$ a closed submanifold of finite codimension and $K_1 = f^{-1}(K_2)$. Suppose also that f is transversal to K_2 and the maps $f : K_1 \to K_2$ and $f : V_1 \setminus K_1 \to V_2 \setminus K_2$ are homotopy equivalences. Then there exists a diffeomorphism $h : (V_1; \partial V_1, K_1) \to (V_2; \partial V_2, K_2)$, which is homotopic to f as maps of triples.*

This theorem remains true for H replaced by a large class of Banach spaces to be described in Section 3. This amplification in turn allows us to extend Theorem 1 to other domains for F such as $C^r_D([0,\pi])$, $r \geq 2$ (i.e., functions of class C^r with Dirichlet boundary conditions) and higher order Sobolev spaces.

We now sketch the main steps in the proof of Theorem 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function: we call f *appropriate* if *either* $f''(0) \neq 0$ *or* the two conditions below hold:

(a) the roots of f'' are isolated,

(b) $f'(0)$ is not of the form $-m^2$, $m \in \mathbb{Z}$.

We call an appropriate function $f : \mathbb{R} \to \mathbb{R}$ *tame* iff $f''(x) \neq 0$ whenever $f'(x)$ is of the form $-m^2$ for some integer m. Notice that tame functions are generic. In section 4 we introduce a smooth functional $\omega : H^2_D \to \mathbb{R}$; for tame f, Σ consists of regular values of ω and C is the union of the non-empty smooth H-manifolds $C_m = \omega^{-1}(\{m\})$. Theorem 1 follows from Theorem 3 once we prove that each C_m is contractible. As any H-manifold is an ANR, this fact is asserted by the following technical result, which is the core of section 4.

Proposition 1.1 *If f is tame and C_m is non-empty, then C_m is path-connected and its homotopy groups $\pi_k(C_m)$ are all trivial.*

To prove this proposition, we observe that the functional ω smoothly extends to $\tilde{\omega} : C^0_D([0,\pi]) \to \mathbb{R}$ (Lemma 4.1). Theorem 2 then shows that the inclusion $\iota : H^2_D \to C^0_D$ induces a homotopy equivalence between the levels $C_m \subset H^2_D$ and the levels $\tilde{C}_m = \tilde{\omega}^{-1}(m\pi) \subset C^0_D, m \in \Sigma$, and we are left with showing that the manifolds $\tilde{C}_m$ are path connected and have trivial homotopy groups: this is simpler than the similar task for $C_m \subset H^2_D$, since we only have to control the continuity of the homotopy of spheres to a point with respect to the weaker C^0 norm.

We refer to the geometric and topological study of the set of solutions of $F(u) = g$ (for varying g) as the *geometric approach.* A pioneering example of the geometric approach applied to PDEs is the work of Ambrosetti and Prodi on the Laplacian on a bounded open set $\Omega \subset \mathbb{R}^n$ with Dirichlet conditions ([1]),

$$F_{AP}(u) = -\Delta u + f(u), u|_{\partial\Omega} = 0.$$

In the Ambrosetti-Prodi scenario, the hypotheses on the nonlinearity are such that the critical set is diffeomorphic to a hyperplane. Subsequent work ([4]) then established that F_{AP} is a global fold. Theorem 1 above is the $n = 1$ case of Ambrosetti-Prodi but now we consider more general nonlinearities f. For convex f, Theorem 1 was proved ([12], [5]) by showing that each connected component of C is the graph of a continuous function from $H_{\sin}$ to $\mathbb{R}$, where $H_{\sin}$ is the hyperplane of functions in $H^2_D([0,\pi])$ orthogonal to $\sin(t)$. This result, applied to the nonlinearity $f(u) = u^2/2$, yields the following rather standard (but not trivial) fact in spectral theory: the set of potentials $u \in H^2_D([0,\pi])$ for which the operator

$$v \in H^2([0,\pi]) \mapsto -v'' + uv \in L^2([0,\pi])$$

has 0 as its n-th eigenvalue is a topological hyperplane. For more general nonlinearities, however, we were not able to make spectral theory work for us. Our hypotheses do not demand that the nonlinearity f have a prescribed asymptotic behavior at infinity.

A more elementary version of this approach has been exploited in [11] to show that the critical set of the operator

$$\begin{aligned} F_1 : H^1(\mathbb{S}^1) &\to L^2(\mathbb{S}^1) \\ u &\mapsto -u' + f(u) \end{aligned}$$

is either empty or a hyperplane. In this case, the critical set is the zero level of a Nemytskii operator,

$$\begin{aligned} \varphi : H^1(\mathbb{S}^1) &\to \mathbb{R} \\ u &\mapsto \int_{\mathbb{S}^1} f'(u) \end{aligned}$$

whose contractibility was shown ([10]) by means of ergodic-like arguments, robust enough to admit extensions to functionals from spaces of functions acting in domains in higher dimensions and taking values on $\mathbb{R}^n$. To show contractibility of the connected components C_m of F, however, we recur constantly to Sturm oscillation, and this is the main reason why our proof does not appear to extend to operators on functions in many variables. Notice that Lemma 5.3 in [11] is a corollary of our Theorem 3.

In order to provide global geometric information about the operator F_1, the authors of [11] considered the stratification of the critical set into Morin singularities of different types. Generically, the singularities of F are also of Morin type, but we do not explore the matter further in this paper.

2 Homotopy equivalence

The aim of this section is to prove Theorem 2. As in the statement of the theorem, X and Y are separable Banach spaces, $i : Y \to X$ is an injective bounded linear map with dense image and $M \subset X$ is a closed smooth submanifold of finite codimension k. Let P be a compact smooth manifold with boundary of dimension $r + k$; P has a fixed but arbitrary Riemannian metric.

Definition 2.1 *The smooth function $f : P \to X$ is called a* smooth M-proper embedding *if f is a smooth embedding, $f(\partial P) \cap M = \emptyset$ and f is transversal to M.*

The set $Q = f^{-1}(M)$ is a smooth compact manifold (with no boundary) of dimension r.

Lemma 2.2 *If $f : P \to X$ is a smooth M-proper embedding then there exists $\epsilon > 0$ such that any smooth map $g : P \to X$ with $\|f - g\|_{C^1} < \epsilon$ is a smooth M-proper embedding. For this ϵ, if $\|f-g\|_{C^1} < \epsilon$ then $Q_g = g^{-1}(M)$ is diffeomorphic to Q and there exists a smooth embedding $\theta : Q \to P$ homotopic to the inclusion $Q \subset P$ and such $\theta(Q) = Q_g$. Moreover, if $S \subset P$ is a compact submanifold and $f|_S = g|_S$ then θ can be chosen to be the identity on $S \cap Q$ and the homotopy to be relative to $S \cap Q$.*

Notice that the lemma also holds if $Q = \emptyset$.

Proof: The existence of ϵ follows from the fact that all of the following properties are open in f in the C^1 topology: being an embedding, $f(\partial P) \cap M = \emptyset$ and f is transversal to M. Let $g_t : P \to X$ be defined by $g_t(p) = (1-t)f(p)+tg(p)$; clearly $g_0 = f$ and $g_1 = g$ and by the previous remark g_t is a smooth M-proper embedding for all $t \in [0, 1]$. Let $G : P \times [0, 1] \to X \times [0, 1]$ be $G(p, t) = (g_t(p), t)$ and let $\tilde{Q} = G^{-1}(M \times [0, 1])$: $\tilde{Q}$ is a compact manifold with two boundary components, $Q \times \{0\}$ and $Q_g \times \{1\}$. The function $\pi : \tilde{Q} \to [0, 1]$ defined by $\pi(p, t) = t$ is a submersion. Notice that $(S \cap Q) \times [0, 1] \subseteq \tilde{Q}$. Construct on $\tilde{Q}$ a tangent vector field α such that $\alpha(p, t) = (0, 1)$ for $p \in S \cap Q$ and $D\pi(p, t) \cdot \alpha(p, t) = 1$ for all $(p, t) \in \tilde{Q}$ (it is easy to construct such a vector field α in a neighborhood of a point (p, t); use partitions of unity to define it on all $\tilde{Q}$). Integrating this vector field yields θ and the desired homotopy. ∎

Let X, Y and $i : Y \to X$ be as above and let P be a compact manifold with boundary. Let $C^1(P, X)$ (resp. $C^1(P, Y)$) be the metric space of C^1 functions from P to X (resp. Y) with the C^1 metric. Similarly, let $C^1_c(\mathbb{R}^k, X)$ (resp. $C^1_c(\mathbb{R}^k, Y)$) be the normed vector spaces of C^1 functions from $\mathbb{R}^k$ to X (resp. Y) with compact support with the C^1 norm. Define $i^* : C^1(P, Y) \to C^1(P, X)$ and $i^* : C^1_c(\mathbb{R}^k, Y) \to C^1_c(\mathbb{R}^k, X)$ by composition.

Lemma 2.3 *The images of $i^* : C^1(P, Y) \to C^1(P, X)$ and $i^* : C^1_c(\mathbb{R}^k, Y) \to C^1_c(\mathbb{R}^k, X)$ are dense in $C^1(P, X)$ and $C^1_c(\mathbb{R}^k, X)$, respectively.*

Proof: We first prove the lemma for $\mathbb{R}^k$ by induction on k (the case $k = 0$ is trivial). Let $f : \mathbb{R}^{k+1} \to X$ be a C^1 function with compact support and $\epsilon > 0$ a real number. We may assume without loss of generality that f is smooth (take a convolution with a smooth bump) and that the support of f is contained in $(0, 1)^{k+1}$. We want to construct $\tilde{f} : \mathbb{R}^{k+1} \to Y$ such that $d_{C^1}(f, i \circ \tilde{f}) < \epsilon$ and such that the support of $\tilde{f}$ is also contained in $(0, 1)^{k+1}$. Take $\delta = 1/N > 0$ such that if $v, v' \in \mathbb{R}^{k+1}$, $d(v, v') < \delta$, then f and all partial derivatives of f of order 1 or 2 differ by at most $\epsilon/16$ between v and v'. Assume furthermore that the support of f is contained in $(\delta, 1 - \delta)^{k+1}$. Let $g = \partial f/\partial x_{k+1}$ and consider the functions $g_0, g_1, \ldots, g_N : \mathbb{R}^k \to X$ defined by

$$g_j(x_1, x_2, \ldots, x_k) = g(x_1, x_2, \ldots, x_k, j/N);$$

notice that $g_0 = g_1 = g_{N-1} = g_N = 0$. By induction hypothesis, we may pick $\tilde{g}_0, \ldots, \tilde{g}_N : \mathbb{R}^k \to Y$ with $d_{C^1}(g_j, i \circ \tilde{g}_j) < \epsilon/16$ and with supports contained in $(0,1)^k$; take $\tilde{g}_0 = \tilde{g}_1 = \tilde{g}_{N-1} = \tilde{g}_N = 0$. Now define $\tilde{g} : \mathbb{R}^{k+1} \to Y$ by

$$\tilde{g}(x_1, x_2, \ldots, x_k, j/N) = \tilde{g}_j(x_1, x_2, \ldots, x_k)$$

and by affine interpolation for other values of x_{k+1}. Clearly, the distances $d_{C^0}(g, i \circ \tilde{g})$, $d_{C^0}(\partial g/\partial x_1, i \circ \partial\tilde{g}/\partial x_1)$, $\ldots$, $d_{C^0}(\partial g/\partial x_k, i \circ \partial\tilde{g}/\partial x_k)$ are all smaller than $\epsilon/4$. Therefore, the function $\tilde{h} : \mathbb{R}^n \to Y$ defined by $\tilde{h}(x_1, \ldots, x_n) = \int_0^1 \tilde{g}(x_1, \ldots, x_k, t)dt$ satisfies $d_{C^1}(h, 0) < \epsilon/4$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth non-negative function with support contained in $(0,1)$, integral equal to 1 and $d_{C^1}(\phi, 0) < 3$. Then the function

$$\tilde{f}(x_1, \ldots, x_k, x_{k+1}) = \int_0^{x_{k+1}} (\tilde{g}(x_1, \ldots, x_k, t) - \phi(t)\tilde{h}(x_1, \ldots, x_k))dt$$

satisfies all the requirements.

We now prove the lemma for a compact manifold P. Take a finite open cover of P by disks and a corresponding smooth partition of unity. In order to approximate f it suffices to approximate each product of f by a function in the partition of unity, provided the support of the approximation is still contained in the corresponding open set. But that is precisely what we did in the previous case. ∎

Let M be a submanifold of finite codimension k of a separable Banach space X. A closed tubular neighborhood of M is a 0-codimensional smooth embedding $\phi : D(\xi) \to X$, where $D(\xi)$ is the closed unit disk bundle of a smooth $\mathbb{R}^k$-bundle over M so that

(1) ϕ restricted to the 0-section is the inclusion $M \hookrightarrow X$,

(2) $\phi(D^0(\xi))$ is an open subset of X,

(3) $\phi(D(\xi))$ is a closed subset of X.

Here, $D^0(\xi)$ is the open unit disk bundle. Clearly, $\phi(\partial D(\xi))$ is a codimension 1 smooth submanifold of X. It is a well known fact ([9] or [7]) that finite codimensional Banach submanifolds of a separable Banach space admit closed tubular neighborhoods.

Proof of Theorem 2: It suffices to prove the two following facts. Let S and Q be compact manifolds with no boundary. If $i_0 : S \to Q$, $i_1 : S \to Y \setminus N$ (resp. $i_1 : S \to N$) and $i_2 : Q \to X \setminus M$ (resp. $i_2 : Q \to M$) are embeddings with $i \circ i_1 = i_2 \circ i_0$ then there exists $u : Q \to Y \setminus N$ (resp. $u : Q \to N$) such that $u \circ i_0$ is homotopic to i_1 and i_2 is homotopic to $i \circ u$.

In the first case we use Lemma 2.2 and Lemma 2.3 to obtain $\tilde{i}_2 : P \to X \setminus M$ near i_2 and of the from $\tilde{i}_2 = i \circ u$, for $u : P \to Y \setminus N$, proving the first claim.

Consider a closed tubular neighborhood of M in X. If $i_2 : Q \to M$ is an embedding then the pull-back of the tubular neighborhood is a bundle $P \to Q$. We have a smooth M-proper embedding also called $i_2 : P \to X$ where P is a compact manifold with boundary; the dimension of P is $r + k$ where k is the codimension of M and r is the dimension of Q. Use this construction in the second case to define P, again use both lemmas to obtain $\tilde{i}_2 : P \to X$ near i_2, a smooth M-proper embedding of the form $\tilde{i}_2 = i \circ u_1$, where $u_1 : P \to Y$ a smooth N-proper embedding. The homotopy and the function θ constructed in Lemma 2.2 now obtain $u : Q \to Y$ and the desired homotopies. ■

3 Global changes of variable

In this section we prove Theorem 3. We will write $\mathbb{H}$ for the infinite dimensional separable real Hilbert space and we call a $\mathbb{H}$-manifold (resp. $\mathbb{H}$-manifold with boundary) a Hausdorff paracompact smooth manifold with local model $\mathbb{H}$ (resp. $\mathbb{H} \times [0, +\infty)$).

Proposition 3.1 *Suppose $f : (V, \partial V) \to (W, \partial W)$ is a homotopy equivalence between two smooth $\mathbb{H}$-manifolds with boundary*

1. *There exists a diffeomorphism $h : (V, \partial V) \to (W, \partial W)$ so that h and f are homotopic maps of pairs.*
2. *If $h^\partial : \partial V \to \partial W$ is a diffeomorphism homotopic (resp. equal) to $f^\partial = f|_{\partial V} : \partial V \to \partial W$, then one can extend h^∂ to a diffeomorphism $h : (V, \partial V) \to (W, \partial W)$ homotopic (resp. relative homotopic) to f.*

Here, an $\mathbb{H}$-manifold is modeled on the separable infinite dimensional real Hilbert space. In view of [7], it is not too hard to see that both our proofs and our theorems remain correct for $\mathbb{H}$ replaced by a Banach space $\mathbb{B}$ provided $\mathbb{B}$ has the following properties.

1. The group $GL(\mathbb{B})$ is contractible with the norm topology.
2. The direct sum $\mathbb{B} \oplus \mathbb{B}$ and any finite codimension subspace of $\mathbb{B}$ are all isomorphic to $\mathbb{B}$ (as Banach spaces).
3. The norm of $\mathbb{B}$ is a smooth map away from the origin.

The proof of Proposition 3.1 is based on the following known results on Hilbert manifolds.

Fact 1 *[3], [2] If $f : M \to N$ is a homotopy equivalence between two $\mathbb{H}$-manifolds, there exists $h : M \to N$, a diffeomorphism which is homotopic to f.*

An isotopy between diffeomorphisms $h_0, h_1 : M \to N$ is a diffeomorphism $h : \mathbb{R} \times M \to \mathbb{R} \times N$ taking (t, m) to a point of the form $h(t, m) = (t, h_t(m))$ so that $h_t = h_0$ if $t \leq 0$ and $h_t = h_1$ if $t \geq 1$.

Fact 2 *[2] Homotopic diffeomorphisms $h_0, h_1 : M \to N$ between $\mathbb{H}$-manifolds are isotopic.*

Fact 3 *[3], [7] Given two homotopic closed embeddings of infinite codimension $\ell_i : V \to M$, $i = 0, 1$, with V and M $\mathbb{H}$-manifolds, there exists an isotopy of diffeomorphisms $h : \mathbb{R} \times M \to \mathbb{R} \times M$ such that h_t is the identity for $t \leq 0$, h_t is constant for $t \geq 1$ and $h_1 \circ \ell_0 = \ell_1$.*

Let M be an $\mathbb{H}$-manifold and $V \subset M$ be a closed $\mathbb{H}$-submanifold. As in Section 2 define a closed tubular neighborhood of V to be a 0-codimensional smooth embedding $\phi : D(\xi) \to M$, where $D(\xi)$ is the closed unit disk bundle of a smooth vector bundle with the same properties (1), (2) and (3) as before. Now, the fiber of ξ is isomorphic to $\mathbb{R}^k$ if the codimension of V in M is k and $\mathbb{H}$ if the codimension is infinite. If the fiber is isomorphic to $\mathbb{H}$ then $D(\xi)$ is necessarily a trivial bundle since the linear group of invertible bounded operators on $\mathbb{H}$ is contractible ([8]).

Fact 4 *[3] Let $V \subset M$ as above and $\phi_i : D(\xi) \to M$, $i = 0, 1$, be two closed tubular neighborhoods. Then there exists an isotopy of diffeomorphisms $h : \mathbb{R} \times M \to \mathbb{R} \times M$ such that h_t is the identity for $t \leq 0$, h_t is constant for $t \geq 1$, $h_1 \circ \phi_0 = \phi_1 \circ \theta$, $\theta : D(\xi) \to D(\xi)$ a (vector) bundle isomorphism which can be taken to be identity when the fiber of ξ is $\mathbb{H}$ and $h_1|_V$ is the identity.*

The hypothesis of finite codimension implies K_1 and K_2 of infinite dimension. A similar conclusion remains true for K_1 and K_2 of infinite dimension and infinite codimension, and the proof in this case is a straightforward consequence of results in [3] and [2].

Proof of Proposition 3.1 Let $V_0 = V$ and $V_1 = W$. Set K_i, $i = 0, 1$, to be $\mathbb{H}$-manifolds diffeomorphic to ∂V_i via diffeomorphisms $\psi_i : \partial V_i \to K_i$. Let $B_i = K_i \times D^\infty$ and $K_i^0 = K_i \times \{0\} \subset B_i$. By Fact 1, there exists diffeomorphisms $\theta_i : \partial V_i \to K_i \times \mathbb{S}^\infty$, such that θ_i is homotopic to the map $\tilde{\psi}_i$ taking $v \in \partial V_i$ to $(\psi_i(v), x) \in K_i \times \mathbb{S}^\infty$, where x is an arbitrary but fixed element in $\mathbb{S}^\infty$. Also, define the smooth $\mathbb{H}$-manifolds $\tilde{V}_i = V_i \cup_{\theta_i} B_i$.

The inclusions $(\tilde{V}_i, K_i^0) \hookrightarrow (\tilde{V}_i, B_i)$ and $(V_i, \partial V_i) \hookrightarrow (\tilde{V}_i, B_i)$ are homotopy equivalences of pairs. Also, by hypothesis, $f : (V_0, \partial V_0) \to (V_1, \partial V_1)$ is a homotopy equivalence of pairs. Hence, there exists $\tilde{f} : (\tilde{V}_0, K_0^0) \to (\tilde{V}_1, K_1^0)$ which makes the diagram below homotopy commutative. From Fact 1, let $\tilde{f}' : \tilde{V}_0 \to \tilde{V}_1$ be a diffeomorphism which is homotopic to $\tilde{f}$. Notice that the behavior of $\tilde{f}'$ at K_0^0 or ∂V_0 is not controlled. Let $\ell : K_0 = K_0^0 \to K_1 = K_1^0$ be a diffeomorphism homotopic to $\tilde{f}|_{K_0^0}$. Notice that $f|_{\partial V_0} : \partial V_0 \to \partial V_1$ is homotopic to both $\psi_1^{-1} \circ \ell \circ \psi_0$ and to $\theta_1^{-1} \circ (\ell \times \mathbf{id}) \circ \theta_0$.

Let $\ell_0 : K_0 \to \tilde{V}_1$ be the composition of ℓ with the inclusion $K_1 = {K_1}^0 \subset \tilde{V}_1$. Also, let $\ell_1 : K_0 = K_0^0 \to \tilde{V}_1$ be the restriction of $\tilde{f}'$ to K_0^0. Notice that both maps embed K_0 in $\tilde{V}_1$ as an infinite codimensional submanifold. By Fact 3, there exists an isotopy $h' : \mathbb{R} \times \tilde{V}_1 \to \mathbb{R} \times \tilde{V}_1$ with $h'_t = \mathbf{id}$ for $t \leq 0$, which is independent of t for $t \geq 1$ and such that $h'_1 \circ \ell_0 = \ell_1$. The diffeomorphism $\tilde{f}'' = (h'_1)^{-1} \circ \tilde{f}' : \tilde{V}_0 \to \tilde{V}_1$ is clearly homotopic to $\tilde{f}$. Also, $\tilde{f}''$ takes K_0^0 to ${K_1}^0$, and $\tilde{f}''$ is then homotopic to $\tilde{f}$ as a map of pairs, from $(\tilde{V}_0, K_0^0)$ to $(\tilde{V}_1, {K_1}^0)$.

Let $\phi_0, \phi_1 : {K_1}^0 \times D^\infty \to \tilde{V}_1$ be tubular neighborhoods given by $\phi_0 = \tilde{f}'' \circ \mathbf{incl} \circ (\ell^{-1} \times \mathbf{id})$, where $\mathbf{incl} : K_1 \times D^\infty \to \tilde{V}_1$ is the inclusion, and ϕ_1 is the inclusion of B_1 in $\tilde{V}_1$. By Fact 4, there is an isotopy $h'' : \mathbb{R} \times \tilde{V}_1 \to \mathbb{R} \times \tilde{V}_1$ with $h''_t = \mathbf{id}$, for $t \leq 0$, which is independent of t for $t \geq 1$ and such that $h''_1 \circ \phi_0 = \phi_1$. Finally, let $\tilde{f}''' = h''_1 \circ \tilde{f}''$. Clearly, $\tilde{f}'''$ restricts to a diffeomorphism of pairs from $(V_0, \partial V_0)$ to $(V_1, \partial V_1)$, homotopic to f. This finishes item (a).

Item (b) is a straightforward consequence of item (a), Fact 2 and the existence of collar neighborhoods for the boundary of an $\mathbb{H}$-manifold. ■

Before we proceed with the proof of Theorem 3 we describe a construction which is sometimes refered to as *cutting a manifold along a submanifold.*

Let $(V, \partial V)$ be an $\mathbb{H}$-manifold with boundary. A proper finite codimension submanifold is a closed subset $K \subset V$ such that

1. $K \subset V \setminus \partial V$,
2. K is a closed, finite codimensional smooth submanifold of $V \setminus \partial V$.

The *normal bundle* ν_K is the quotient $T(V)|_K/T(K)$. Denote by $E(\nu_K)$ the total space of ν_K and by $S(\nu_K)$ the corresponding sphere bundle $(E(\nu_K)\setminus K)/\mathbb{R}^+$, where $\mathbb{R}^+$ acts on $E(\nu_K) \setminus K$ by multiplication. Also, denote by $D(\nu_K)$ the fiberwise compactification of $E(\nu_K)$ obtained by adding a point at infinity for each half-line from the origin. Clearly, $D(\nu_K)$ and $S(\nu_K)$ are isomorphic to the closed unit disk bundle and to the unit sphere bundle of ν_K provided ν_K is equipped with a smooth fiberwise scalar product, and we may therefore consider $S(\nu_K)$ as the boundary of $D(\nu_K)$.

Let $\psi_D : D(\nu_K) \to V$ be a closed tubular neighborhood of K. Let $\psi_S : S(\nu_K) \times [0, 1) \to V$ be defined by $\psi_S(n, t) = \psi_D(tn)$. The *cutting* of $(V, \partial V)$ *along* K is an $\mathbb{H}$-manifold with boundary (actually a bordism) $(\tilde{V}, \partial_+\tilde{V}, \partial_-\tilde{V})$, together with a canonical smooth map $p : (\tilde{V}, \partial_+\tilde{V}, \partial_-\tilde{V}) \to (V, \partial V)$. The manifold $\tilde{V}$ as a set is the disjoint union of $V \setminus K$ and $S(\nu_K)$; more precisely, $\tilde{V} = (V \setminus K) \cup_{\psi_S} (S(\nu_K) \times [0, 1))$. Set p to be the identity on $V \setminus K$ and equal to the bundle projection $S(\nu_K) \to K$ on K. From Fact 4, the smooth structure thus defined on $\tilde{V}$ actually does not depend on the choice of tubular neighborhood θ. Now take $\partial_+\tilde{V} = \partial V$ and $\partial_-\tilde{V} = S(\nu_K)$.

Consider $\check{V} = \tilde{V} \cup D(\nu_K)$, obtained by identifying $\partial_-\tilde{V}$ and $\partial D(\nu_K)$, both equal to $S(\nu_K)$. The pair $(\check{V}, \partial\check{V} = \partial_+\tilde{V})$ is a smooth $\mathbb{H}$-manifold with boundary and $K \subset D(\nu_K) \subset \check{V}$ is a proper smooth embedding of finite codimension. There exists a well defined class of *thickening diffeomorphisms* (all isotopic) $\check{\theta} : (\check{V}, \partial_+\check{V}, K) \to (V, \partial V, K)$ so that the restriction of $\check{\theta}$ to K is the identity.

The constructions above are functorial in the following sense. Let

$$f : (V_1, \partial V_1, K_1) \to (V_2, \partial V_2, K_2)$$

be a smooth map so that f is transversal to K_2 and $K_1 = f^{-1}(K_2)$. Such a map induces a map $\tilde{f} : (\tilde{V}_1, \partial_+\tilde{V}_1, \partial_-\tilde{V}_1) \to (\tilde{V}_2, \partial_+\tilde{V}_2, \partial_-\tilde{V}_2)$ and bundle maps $D(\nu_{K_1}) \to D(\nu_{K_2})$ sending $S(\nu_{K_1})$ into $S(\nu_{K_2})$. The restriction of $\tilde{f}$ to $\partial_-\tilde{V}_1$ is the induced bundle map from $S(\nu_{K_1})$ to $S(\nu_{K_2})$; on each fiber, this map is projective (i.e., $\tilde{f}$ restricted to $\partial_-\tilde{V}_1$ comes from a vector bundle map $E(\nu_{K_1}) \to E(\nu_{K_2})$). Such maps f will be called *morphisms*. We consider morphisms $f : (V_1, \partial V_1, K_1) \to (V_2, \partial V_2, K_2)$ with the additional property that the maps f :

$V_1 \to V_2$, $f : \partial V_1 \to \partial V_2$, $f : K_1 \to K_2$, $f : V_1 \setminus K_1 \to V_2 \setminus K_2$ are all homotopy equivalences. This clearly implies that $\tilde{f} : (\tilde{V}_1, \partial_+ \tilde{V}_1, \partial_- \tilde{V}_1) \to (\tilde{V}_2, \partial_+ \tilde{V}_2, \partial_- \tilde{V}_2)$ is a homotopy equivalence of triples.

Proof of Theorem 3: Start with $f : (V_1, \partial V_1, K_1) \to (V_2, \partial V_2, K_2)$ which induces the homotopy equivalence $\tilde{f} : (\tilde{V}_1, \partial_+ \tilde{V}_1, \partial_- \tilde{V}_1) \to (\tilde{V}_2, \partial_+ \tilde{V}_2, \partial_- \tilde{V}_2)$ and the bundle map $D(f) : D(\nu_{K_1}) \to D(\nu_{K_2})$. Notice that the restriction $f^K = f|_{K_1}$ is a homotopy equivalence, and therefore so is $D(f)$.

By Fact 1, f^K is homotopic to a diffeomorphism $h^K : K_1 \to K_2$ by a homotopy f^K_t, where $f^K_0 = f^K$ and $f^K_1 = h^K$. Since f^K comes from a bundle map $D(f)$, the homotopy f^K_t lifts to a homotopy of bundle maps $D(f^K_t)$, and since $D(f^K_1)$ induces on the base a diffeomorphism h^K, $D(f^K_1)$ itself is a diffeomorphism.

This shows that $\tilde{f}$ is a homotopy equivalence between $\partial_- \tilde{V}_1$ and $\partial_- \tilde{V}_2$, and therefore a homotopy equivalence between the boundaries of $\tilde{V}_1$ and $\tilde{V}_2$. We may then apply Proposition 3.1 to obtain a homotopic diffeomorphism $\tilde{h}$ between the triples. Also, the restriction of $D(f^K_1)$ to $S(\nu_{K_1}) = \partial_- \tilde{V}_1$ is a diffeomorphism homotopic to the restriction of $\tilde{h}$. By Fact 2, there exists an isotopy between these two diffeomorphisms, and we may therefore glue them in order to obtain the desired diffeomorphism. ∎

Corollary 3.2 *Let X and Y be a separable Hilbert spaces and $i : Y \to X$ an injective bounded linear map with dense image. If M is a finite codimensional closed submanifold of X then $N = i^{-1}(M)$ is a finite codimensional closed submanifold of Y and there exists a diffeomorphism $h : (Y, N) \to (X, M)$ homotopic to $i : (Y, N) \to (X, M)$.*

Proof: From Theorem 2, $i : Y \setminus N \to X \setminus M$ and $i : (Y, N) \to (X, M)$ are homotopy equivalences. We now apply Theorem 3 with $V_1 = Y$ and $V_2 = X$, $\partial V_1 = \partial V_2 = \emptyset$, $K_2 = M$, $K_1 = N$ and $f = i$. ∎

4 Constructing the homotopy

Let $X = C^0_D([0, \pi])$, $Y = H^2_0([0, \pi])$ and $Z = L^2([0, \pi])$. Consider the operator

$$\begin{aligned} F : Y &\to Z \\ u &\mapsto -u'' + f(u) \end{aligned}$$

with derivative at u given by

$$\begin{aligned} DF(u) : Y &\to Z. \\ w &\mapsto -w'' + f'(u)w \end{aligned}$$

We recall some facts from the theory of second order differential equations ([6]). Given $u \in Y$, let $v(u,\cdot)$ be the solution in $[0,\pi]$ of

$$-v''(u,t) + f'(u(t))v(u,t) = 0, \qquad v(u,0) = 0, \quad v'(u,0) = 1.$$

The characterization of critical points of F follows from standard Fredholm theory of Sturm-Liouville operators: u is in C if and only if the kernel of $DF(u)$ is non-trivial. Thus, a point u belongs to the critical set C of F if and only if $v(u,\pi) = 0$; in this case, by the simplicity of the spectrum of $DF(u)$, $\ker DF(u)$ is spanned by $v(u,\cdot)$.

Let $\omega : Y \times [0,\pi] \to \mathbb{R}$ be the continuously defined argument of $(v'(u,t), v(u,t))$, with $\omega(u,0) = 0$. One can show that u is critical if and only if $\omega(u,\pi) = m\pi$, for $m \in \mathbb{Z}$; moreover, m has to be positive since $\omega(u,t) > 0$ for all $t > 0$. Let $C_m = \{u \in Y \mid \omega(u,\pi) = m\pi\}$: the critical set C of F is the (disjoint) union of C_m, $m \in \mathbb{N}^*$. A standard computation shows that, for any given t, $\omega(u,t)$ is smooth as a function of $u \in Y$ and we have

$$\frac{\partial}{\partial u}\omega(u,t) \cdot \varphi = -\frac{1}{(v(u,t))^2 + (v'(u,t))^2} \int_0^t f''(u(s))\varphi(s)(v(u,s))^2 ds.$$

where $\frac{\partial}{\partial u}\omega(u,t)$ denotes the differential of $\omega(\cdot,t) : Y \to \mathbb{R}$ and $\frac{\partial}{\partial u}\omega(u,t) \cdot \varphi$ the value of this differential on the element $\varphi \in Y$. Let f be appropriate: using the formula above one can see that $m\pi$ is a regular value for the real valued function $\omega(\cdot,\pi)$, and therefore the sets C_m are either empty or smooth manifolds of codimension 1.

When considering a given C_m we use the more convenient *m-argument* at u: $\omega_m : Y \times [0,\pi] \to \mathbb{R}$ is the argument of (v', mv), i.e., it satisfies

$$v'(u,t)\tan(\omega_m(u,t)) = mv(u,t), \quad \omega_m(u,0) = 0.$$

It is easy to see that $m\pi$ is a regular value of ω_m, as it is of ω. The advantage of this definition is that if $f'(u(t)) = -m^2$ for t in some interval then $\omega_m(u,\cdot)$ is a linear map of slope m in this interval.

We will later consider *local m-arguments*. More precisely, given u we solve the differential equation

$$-\hat{v}'' + f'(u)\hat{v} = 0, \qquad \hat{v}(t_0) = a_0, \quad \hat{v}'(t_0) = b_0$$

and set $\hat{\omega}_m(t)$ to be the argument of $(\hat{v}', m\hat{v})$. The notation $\hat{\omega}_m(u,t)$ leaves unspecified the values of t_0, a_0 and b_0: these values will always be specified in the context.

We list some properties relating u and ω_m, most of which are simple or standard. The obvious adaptations to $\hat{\omega}_m$ will be frequently left to the reader.

Lemma 4.1 *The m-argument ω_m satisfies*

$$\omega_m'(u,t) = m - \frac{m^2 + f'(u(t))}{m}\sin^2(\omega_m(u,t)), \qquad \omega_m(u,0) = 0 \qquad (*)$$

(here $\omega_m'(u,t) = \frac{d}{dt}\omega_m(u,t)$);

(a) $\omega_m'(u,t) = m$ if and only if either $\omega_m(u,t) = j\pi$, $j \in \mathbb{Z}$, or $f'(u(t)) = -m^2$;

(b) if $f'(u(t)) < -m^2$ (resp., $f'(u(t)) > -m^2$) then $\omega_m'(u,t) \geq m$ (resp., $\omega_m'(u,t) \leq m$);

(c) the differential equation $(*)$ defines $\omega_m(\cdot,\pi)$ as a smooth function on $C^0([0,\pi])$ which is L^1-continuous on bounded sets;

(d) for any given t, the differential equation $(*)$ implies that

$$\frac{\partial}{\partial u}\omega_m(u,t)\cdot\varphi = -\frac{m}{(mv(u,t))^2 + (v'(u,t))^2}\int_0^t f''(u(s))\varphi(s)(v(u,s))^2 ds;$$

(e) if u is smooth and not flat at t_0, $f''(u(t_0)) \neq 0$ and $\omega_m'(u,t_0) = m$ then $\omega_m'(u,\cdot)$ is not flat at t_0.

Item (c) of this result entitles us to define the extension $\tilde{\omega}_m : X \to \mathbb{R}$ and implies that the maps ω and ω_m can be extended to smooth maps on X and one can consider the codimension one smooth manifolds $\tilde{C}_m = \{u \in X \mid \omega(u,\pi) = m\pi\}$. From Corollary 3.2, the inclusion $i : (Y, C_m) \to (X, \tilde{C}_m)$ is homotopic to a homeomorphism. From now on we drop the tilde both on $\tilde{C}_m$ and $\tilde{\omega}_m$: thus, our basic functional space will be $X = C^0_D([0,\pi])$.

Recall that a smooth function $\zeta : \mathbb{R} \to \mathbb{R}$ is *flat* at t_0 if its Taylor expansion at t_0 is constant.

Proof of Lemma 4.1: We only prove the last item. Assume by contradiction that $\omega_m'(u,\cdot)$ is flat at t_0. Since $f''(u(t_0)) \neq 0$, the function $f' \circ u$ is not flat at t_0. In the differential equation

$$\omega_m'(u,t) - m = -\frac{m^2 + f'(u(t))}{m}\sin^2(\omega_m(u,t))$$

the left hand side is zero and flat at t_0 and the right hand side is a product of two non-flat functions. Thus, the formal Taylor series around t_0 of the left hand side is identically zero, whereas the corresponding series for the right hand side is non-zero. ■

Lemma 4.1 shows how to obtain $\omega_m(u,\cdot)$ directly from u, without referring to $v(u,\cdot)$. More generally, we define the local m-argument $\hat{\omega}_m$ by the equation

$$\hat{\omega}'_m(u,t) = m - \frac{m^2 + f'(u(t))}{m} \sin^2(\hat{\omega}_m(u,t)), \qquad \hat{\omega}_m(u,t_0) = \theta_0$$

where $\tan(\theta_0) = ma_0/b_0$.

Proposition 4.2 *Let f be appropriate; then $C_m \neq \emptyset$ if and only if the number $-m^2$ belongs to the interior of the image of f'.*

Notice that the proposition makes no claim concerning connectedness or any other topological property of C_m.

Proof: Assume that $-m^2 \in \text{int}(f'(\mathbb{R}))$: for some $\epsilon > 0$ there exist $x_-, x_+ \in \mathbb{R}$ with $f'(x_\pm) = -m^2 \mp \epsilon$. Consider two families of functions $u_{-,s}$, $u_{+,s}$ in $C^0_0([0,\pi])$ which are uniformly C^0 bounded and L^1 converge to the constants x_-, x_+ when s tends to 0. From item (c) of Lemma 4.1, for some sufficiently small s_0 we have $\omega_m(u_{-,s_0},\pi) < -m^2$, $\omega_m(u_{+,s_0},\pi) > -m^2$. By continuity, some u in the line segment from u_{-,s_0} to u_{+,s_0} belongs to C_m.

To prove the converse, assume first $f'(0) = -m^2$: from the definition of appropriateness we must then have $f''(0) \neq 0$ and then clearly $-m^2 \in \text{int}(f'(\mathbb{R}))$. Assume instead $f'(0) \neq -m^2$, say $f'(0) > -m^2$, and let $u \in C_m$. For t_- near 0 or π we have $f'(u(t_-)) > -m^2$ and from Lemma 4.1 $\omega'_m(u,t_-) < m$ for such $t_- \neq 0, \pi$. We must then have $\omega'_m(u,t_+) > m$ for some $t_+ \in [0,\pi]$. Thus $f'(u(t_+)) < -m^2$ and the result follows. ∎

We present a rough sketch of the construction of the deformation within C_m of a family of functions u to a final point u_*, as claimed in Proposition 1.1. Actually, from Theorem 2, it suffices that this deformation be continuous in the C^0 norm instead of the H^2 norm. A natural candidate for u_* would be a constant function equal to an arbitrary x_m where $f'(x_m) = -m^2$: in this case, $\omega_m(u,t) = mt$. Unfortunately, this function does not satisfy the Dirichlet boundary conditions: given the family of functions u to be deformed, we construct a fixed $u_* \in C_m$ which is constant equal to x_m in a large interval $[a, \pi - a]$.

Consider now the m-argument of a given $u \in C_m$: $\omega_m(u,\cdot)$ is a continuous function from $[0,\pi]$ to $[0,m\pi]$. These two graphs are shown in Figure 1; in this example $m = 2$. The graphs of the constant functions $u = x_m$ and $\omega_m(u,t) = tm$ are indicated by dotted lines. The idea, to be formalized and implemented in step 4 of the proof, is for the homotopy to squeeze the graph of ω_m between parallel walls advancing towards the line $y = mt$, as shown in Figure 2. A corresponding u is obtained by changing its original value in the region of the domain over which the graph of ω_m has been squeezed—there, the new value of u is x_m. Notice that, in principle, the value of $\omega_m(u,\pi) = m\pi$ for this new u but such u is discontinuous. We must therefore make amends: for a fixed tolerance **tol**, the

Figure 1: Graph of original u and ω_m

region where the graph of ω_m trespasses the wall by more than **tol** is taken to x_m and in the region where the graph of ω_m lies strictly between the walls, u is unchanged. Hence, there is an open region in the domain where u assumes rather arbitrary values in order to preserve its continuity. Steps 2 and 3 guarantee that this open region is uniformly small (for u in the deformed loop): in particular, we have to fudge u so that the graph of $\omega_m(u, \cdot)$ does not include segments parallel to $y = mt$. This *symmetry breaking* is achieved by replacing u by appropriate polynomial approximations. At the end of step 4, we have functions which are equal to x_m in a substantial amount of the domain: the straight line segments joining them to u_* can be deformed onto C_m thus accomplishing the last step of the deformation.

Figure 2: Graph of u and ω_m at some point during step 4

Lemma 4.3 *Given $u \in C^0_D$, $\omega_m(u,\cdot)$ is a C^1 function with $\omega'_m(u,t_0) = m$ and $\omega''_m(u,t_0) = 0$ whenever $\omega_m(u,t_0) = j\pi$, $j \in \mathbb{Z}$. Furthermore, given $u_0 \in \mathbb{R}$ with $f''(u_0) \neq 0$ and a C^1 function $w : (t_0-\epsilon, t_0+\epsilon) \to \mathbb{R}$ with either $w(t_0) \neq j\pi$ or simultaneously $w(t_0) = j\pi$, $w'(t_0) = m$, $w''(t_0) = 0$, there exists, for sufficiently small $\epsilon_1 < \epsilon$, a unique continuous function $u : (t_0-\epsilon_1, t_0+\epsilon_1) \to \mathbb{R}$ with*

$$w'(t) = m - \frac{m^2 + f'(u(t))}{m}\sin^2(w(t)), \qquad u(t_0) = u_0.$$

Proof: The fact that $\omega_m(u,\cdot)$ is a C^1 function follows directly from the differential equation in Lemma 4.1. Given t_0 with $\omega_m(u,t_0) = j\pi$, we have

$$\begin{aligned}
\omega''_m(u,t_0) &= \lim_{t\to t_0} \frac{\omega'_m(u,t) - m}{t - t_0} \\
&= \lim_{t\to t_0} -\frac{m^2 + f'(u(t))}{m}\,\frac{\sin^2(\omega_m(u,t))}{t-t_0} \\
&= C\lim_{t\to t_0} \frac{\sin^2(\omega_m(u,t)) - \sin^2(\omega_m(u,t_0))}{t-t_0} = 0.
\end{aligned}$$

Also, the differential equation in u and w may be written as

$$f'(u(t)) = -m^2 + m\frac{m - w'(t)}{\sin^2(w(t))}.$$

The right hand side is clearly continuous even when $w(t) = j\pi$ which allows for solving in u when f' is invertible. ■

Thus, for example, for concave or convex nonlinearities f, changes in $\omega_m(u,\cdot)$ easily translate back to changes in u. In particular, as mentioned in the introduction, simpler proofs of Theorem 1 are known under these hypothesis.

A technical difficulty in the construction of u_* (and in many points of the deformation process) is making sure that u_* (or a deformed u) belongs to C_m. For this, we make use of the following Lemma.

Consider two smooth functions $u_0 : [0,t_0] \to \mathbb{R}$ and $u_1 : [t_1,\pi] \to \mathbb{R}$ satisfying $u_0(0) = 0$ and $u_1(\pi) = 0$, and consider local m-arguments $\omega_m(u_0,t)$ and $\omega_m(u_1,t)$ starting respectively from 0 at $t = 0$ and $m\pi$ at $t = \pi$. Under the hypothesis of the lemma below, we may *solder* these two chunks in a specific, smoothly defined way, to obtain a smooth function $u : [0,\pi] \to \mathbb{R}$ belonging to C_m (i.e., so that $\omega_m(u,\pi) = m\pi$).

Lemma 4.4 *Let $0 < t_0 < t_1 < \pi$ be real numbers such that $\sin(mt) \neq 0$ for all $t \in [t_0,t_1]$. For sufficiently small $\epsilon > 0$ there exists a smooth function*

$$\Xi_{t_0,t_1} : (-\epsilon,\epsilon) \times (-\epsilon,\epsilon) \times [t_0,t_1] \to \mathbb{R}$$

with the following properties:

(a) $\Xi_{t_0,t_1}(h_0,h_1,t_0) = \Xi_{t_0,t_1}(h_0,h_1,t_1) = x_m$ *for all* $h_0, h_1 \in (-\epsilon, \epsilon)$ *and flat at these points;*

(b) if $\hat{\omega}_m(\Xi_{t_0,t_1}(h_0,h_1,\cdot),t)$ *is defined with initial condition* $\hat{\omega}_m(\Xi_{t_0,t_1}(h_0,h_1,\cdot),t_0) = mt_0 + h_0$ *then* $\hat{\omega}_m(\Xi_{t_0,t_1}(h_0,h_1,\cdot),t_1) = mt_1 + h_1$*;*

(c) $\Xi_{t_0,t_1}(h,h,t) = x_m$ *for all* $h \in (-\epsilon,\epsilon)$*,* $t \in [t_0,t_1]$*.*

Proof: Take $\epsilon > 0$ such that $\sin(mt) \neq 0$ for all $t \in [t_0 - \epsilon/m, t_1 + \epsilon/m]$. Let $\xi : [t_0,t_1] \to [0,1]$ be a smooth bijection with $\xi(t_0) = 0$, $\xi(t_1) = 1$ which is flat at these endpoints. Let

$$w_{h_0,h_1}(t) = mt + h_0 + (h_1 - h_0)\xi(t);$$

we are ready to apply Lemma 4.3 in order to define Ξ_{t_0,t_1} with

$$\hat{\omega}_m(\Xi_{t_0,t_1}(h_0,h_1,\cdot),t) = w_{h_0,h_1}(t).$$

■

The lemma below will be used in step 4 of the proof of Proposition 1.1.

Lemma 4.5 *Let* $g : \mathbb{S}^k \times [0,\pi] \to \mathbb{R}$ *be a smooth function. Suppose there exists* $\epsilon > 0$ *such that for any* $\delta > 0$ *there exists* $y_\delta \in \mathbb{R}$ *and* $\theta_\delta \in \mathbb{S}^k$ *such that* $\mu(A_\delta) > \epsilon$*, where*

$$A_\delta = \{t \in [0,\pi] \mid g(\theta_\delta,t) \in [y_\delta, y_\delta + \delta]\}.$$

Then there exists a point $(\theta_0,t_0) \in \mathbb{S}^k \times [0,\pi]$ *for which the function* $g(\theta_0,\cdot)$ *is flat at* t_0*.*

Proof: Let (y_0,θ_0) be an accumulation point of the sequence (y_{a_n},θ_{a_n}) where $\lim a_n = 0$; without loss, we may suppose that (y_0,θ_0) is the limit of this sequence. Let

$$A_0 = \limsup_n A_{a_n} = \bigcap_n \bigcup_{i \geq n} A_i;$$

from

$$\mu(\limsup_n A_{a_n}) \geq \limsup_n \mu(A_{a_n})$$

(a corollary of Fatou's lemma) we have $\mu(A_0) \geq \epsilon$. On the other hand, from continuity, $g(\theta_0,t) = y_0$ for $t \in A_0$. Thus, any non-isolated point $t_0 \in A_0$ is a flat point for $g(\theta_0,\cdot)$. ■

Proof of Proposition 1.1: Path connectedness of C_m (i.e., the case $k = 0$, for which $\pi_k(C_m)$ has no group structure) will be discussed simultaneously to the verification of the triviality of the homotopy groups of C_m. Keeping with the notation of the previous section, let $\phi : X \to \mathbb{R}$, $\phi(u) = \omega_m(u, \pi) - m\pi$, $M_Y = C_m$ and $M_X = \phi^{-1}(\{0\}) \subseteq X$. As shown in the previous section, it suffices, given a Y-continuous (i.e., H^2-continuous) $\gamma : \mathbb{S}^k \to M_Y$, to construct a X-continuous (i.e., C^0-continuous) $\Gamma : \mathbb{B}^{k+1} \to M_X$ with $\Gamma|_{\mathbb{S}^k = \partial\mathbb{B}^{k+1}} = \gamma$.

We set a more convenient notation: let A^{k+1} be the annulus $\mathbb{S}^k \times [0, \pi]$ and $U(0) : A^{k+1} \to \mathbb{R}$ be the initial map defined by $U(0)(\theta, t) = (\gamma(\theta))(t)$. We want to construct a continuous function $U : [0, 5] \times A^{k+1} \to \mathbb{R}$ with the following properties:

(i) the values at $\{0\} \times A^{k+1}$ are given by the initial map $U(0)$;

(ii) for any $s \in [0, 5]$ and $\theta \in \mathbb{S}^k$, $U(s, \theta, \cdot) \in M_X \subseteq C^0_0([0, \pi])$, i.e., $U(s, \theta, 0) = U(s, \theta, \pi) = 0$ and $\omega_m(U(s, \theta, \cdot), \pi) = m\pi$;

(iii) for $s = 5$ the function U is constant in θ, i.e., for any $\theta_0, \theta_1 \in \mathbb{S}^k$ and $t \in [0, \pi]$, $U(5, \theta_0, t) = U(5, \theta_1, t)$.

We then have

$$\Gamma(r\theta)(t) = \begin{cases} U(5, \theta, t), & \text{if } r \leq 1/2, \\ U(10(1 - r), \theta, t) & \text{if } r \geq 1/2. \end{cases}$$

Step 1 We search for convenient vectors $\varphi(\theta, \cdot)$ in Y along which the derivative

$$-\frac{m}{(v'(U(0, \theta, \cdot), \pi))^2} \int_0^\pi f''(U(0, \theta, \sigma))\varphi(\theta, \sigma)(v(U(0, \theta, \cdot), \sigma))^2 d\sigma,$$

of $\omega_m(U(0, \theta, \cdot), \pi)$ is positive (see Lemma 4.1, (d)). Let $\beta_\delta : [0, \pi] \to [0, 1]$ be a smooth bump equal to zero in $[0, \delta) \cup (\pi - \delta, \pi]$ and equal to one in $(2\delta, \pi - 2\delta)$; this family of bumps may be constructed to be L^1-continuous in δ. Clearly, the choice $\varphi(\theta, t) = -\beta_\delta(t) f''(U(0, \theta, t))$ yields a positive derivative for $\delta = 0$ and therefore, by continuity, also for some positive δ_0 (independent of θ). Furthermore, the smoothness in u of $\omega_m(u, \pi)$ guarantees that there exists $\epsilon > 0$ such that

$$\frac{\partial}{\partial\tau}\omega_m(U(0, \theta, \cdot) + \tau\varphi(\theta, \cdot), \pi) > \epsilon$$

at any point (θ, τ) with $|\tau| < \epsilon$.

We now want to deform each u so that the modified u's are constant equal to x_m on small intervals near $t = 0$ and $t = \pi$. More precisely, we will define small positive real numbers $0 < \delta_2 \ll \delta_1$ and construct U for $0 \leq s \leq 1$ so that

(i) for $\delta_2 \le t \le \delta_1$ and $\pi - \delta_1 \le t \le \pi - \delta_2$ we have $U(1,\theta,t) = x_m$;

(ii) for $t < \delta_2$ and $t > \pi - \delta_2$, $U(1,\theta,t)$ does not depend on θ and satisfies $|U(1,\theta,t)| \le |x_m|$;

(iii) for any $s \in [0,1]$, and any $\theta \in \mathbb{S}^k$ we have $\omega_m(U(s,\theta,\cdot),\pi) = m\pi$, i.e., $U(s,\theta,\cdot) \in C_m$.

In particular, from (i), in the interval $\delta_2 \le t \le \delta_1$ (or $\pi - \delta_1 \le t \le \pi - \delta_2$) the graph of $\omega_m(U(1,\theta,\cdot),t)$ is a straight segment of slope m. We claim that for any sufficiently small δ_1 and δ_2 with $\delta_2 < \delta_1$ this construction can be accomplished. Indeed, for sufficiently small $\delta_1 < \delta_0/2$ we can uniquely define $\xi : [0,1] \times \mathbb{S}^k \times [0,\pi] \to (-\epsilon,\epsilon)$ such that

$$U(s,\theta,t) = (1 - s + s\beta_{\delta_1}(t))U(0,\theta,t) + s(\beta_{\delta_2/2} - \beta_{\delta_1})(t)x_m + \xi(s,\theta,t)\varphi(\theta,t)$$

satisfies $U(s,\theta,\cdot) \in C_m$ for all $s \in [0,1]$ and $\theta \in \mathbb{S}^k$ (here δ_0, ϵ, φ and β are as defined in the previous paragraph). The above formula for U admits the following geometric interpretation. The first two terms of the right hand side parametrize in s a straight line segment from the original u to a modified function satisfying items (i) and (ii). The third term takes care of item (iii) provided the L^1 distance between u and the modified function is so small that their m-arguments ω_m at π differ from less than ϵ.

Step 2 Let $\eta > 0$ be a small number to be specified later. We now define U for $1 \le s \le 2$ so that there exist positive numbers $\delta_2' < \delta_1' < \delta_0' < \pi/m$ such that

(i) for $t \in [\delta_2', \delta_1'] \cup [\pi - \delta_1', \pi - \delta_2']$ we have $U(2,\theta,t) = x_m$ and $\omega_m(U(2,\theta,\cdot),t) = mt$;

(ii) for any θ, $U(2,\theta,\pi - \delta_0') = x_m$ and $\omega_m(U(2,\theta,\cdot),\pi - \delta_0')) = m(\pi - \delta_0') + \eta$;

(iii) for $t \in [0,\delta_1'] \cup [\pi - \delta_0', \pi]$, $U(2,\theta,t)$ does not depend on θ;

(iv) for any $s \in [1,2]$, and any $\theta \in \mathbb{S}^k$, we have $U(s,\theta,\cdot) \in C_m$.

To do this, assume that $\delta_1 < \pi/m$ and $\delta_2 < \delta_1/4$ and let $\delta_2' = \delta_1/2$, $\delta_1' = 3\delta_1/4$, $\delta_0' = 7\delta_1/8$. The L^1-continuity of the m-argument (Lemma 4.1) entails that both $h_- = m\delta_1/4 - \omega_m(U(1,\theta,\cdot),\delta_1/4)$ and $h_+ = m(\pi - \delta_1/4) - \omega_m(U(1,\theta,\cdot),\pi - \delta_1/4)$ can be taken to be arbitrarily small by choosing δ_2 small. Now solder the six chunks: for $s \in [1,2]$ set

$$U(s,\theta,t) = \begin{cases} \Xi_{\delta_1/4,\delta_2'}(h_-, h_-(s), t), & \text{if } \delta_1/4 < t < \delta_2' \\ \Xi_{\delta_1',\delta_1}(h_-(s), h_-, t), & \text{if } \delta_1' < t < \delta_1 \\ \Xi_{\pi-\delta_1,\pi-\delta_0'}(h_+, h_+^0(s), t), & \text{if } \pi - \delta_1 < t < \pi - \delta_0' \\ \Xi_{\pi-\delta_0',\pi-\delta_1'}(h_+^0, h_+^1(s), t), & \text{if } \pi - \delta_0' < t < \pi - \delta_1' \\ \Xi_{\pi-\delta_2',\pi-\delta_1/4}(h_+^1(s), h_+, t), & \text{if } \pi - \delta_2' < t < \pi - \delta_1/4 \\ U(1/4,\theta,t), & \text{otherwise;} \end{cases}$$

where $h_-(s) = (2-s)h_-$, $h_+^0(s) = \eta + (2-s)(h_+ - \eta)$ and $h_+^1(s) = (2-s)h_+$. The number η is chosen to be any positive number sufficiently small to permit soldering. See Figure 3 for a sketch of the graph of $U(2,\theta,\cdot)$ and $\omega_m(U(2,\theta,\cdot),t)$.

Figure 3: Graphs of u and ω_m at the end of step 2

We are now ready to describe u_*:

$$u_*(t) = \begin{cases} U(2,\theta,t), & \text{for } t \in [0,\delta_1'] \cup [\pi - \delta_1', \pi], \\ x_m, & \text{for } t \in [\delta_2', \pi - \delta_2']. \end{cases}$$

Notice that the value of θ in the definition above is immaterial, from property (iii). Moreover, the homotopy from now on will not change the values of the functions $U(s,\theta,\cdot)$ near the endpoints: we shall have $U(s,\theta,t) = u_*(t)$ for all $s \geq 2$, $t \in [0,\delta_2'] \cup [\pi - \delta_2', \pi]$ and $\theta \in \mathbb{S}^k$. The construction of u_* is dependent on the original map $\gamma : \mathbb{S}^k \to M_Y$: still, path connectivity of C_m follows from the same homotopy being described when applied to an original map γ, with $k = 0$.

Step 3 Continuing with the preparations, we change u's in the central interval $[\delta_1', \pi - \delta_0']$ so that there they become polynomials in the variables θ and t. More precisely, for some polynomial function $P : \mathbb{R}^{k+2} \to \mathbb{R}$ (with properties to be specified below) and for $s \in [2,3]$, set

$$U(s,\theta,t) = \begin{cases} (s-2)U(2,\theta,t) + (3-s)P(\theta,t), & \text{for } t \in [\delta_1', \pi - \delta_0'], \\ \Xi_{\pi-\delta_1',\pi-\delta_2'}(h(s,\theta),0,t), & \text{for } t \in [\pi - \delta_1', \pi - \delta_2'], \\ u_*(t), & \text{otherwise}, \end{cases}$$

where $h(s,\theta) = \omega_m(U(s,\theta,\cdot),\pi - \delta_1') - m(\pi - \delta_1')$. In order for this function U to be continuous we must choose P with $P(\theta,\delta_1') = P(\theta,\pi-\delta_0') = x_m$. Lemma 4.4 applies if we require $h(s,\theta) \in (\eta/2, 3\eta/2)$ for all s and θ: this is accomplished from Lemma 4.1 by choosing $P(\theta,t)$ uniformly close to $U(2,\theta,t)$ in the interval $[\delta'_1, \pi-\delta'_0]$ (as we may, from the Stone-Weierstrass theorem).

Step 4 Let

$$A = [3,4]\times\mathbb{S}^k\times[\delta'_1,\pi-\delta'_1]$$

and consider the partition

$$\begin{aligned}
A_I &= \{(s,\theta,t)\in A \mid -(4-s)m\pi \le \omega_m(U(s,\theta,\cdot),t) - mt \le (4-s)m\pi\},\\
A_S &= \{(s,\theta,t)\in A \mid -(4-s)m\pi - \mathbf{tol} \ge \omega_m(U(s,\theta,\cdot),t) - mt\\
&\qquad\qquad \text{or } \omega_m(U(s,\theta,\cdot),t) - mt \ge (4-s)m\pi + \mathbf{tol}\},\\
A_T &= A - (A_I \cup A_S)
\end{aligned}$$

(here the letters I, S and T stand for *invariant*, *squeezed* and *Tietze*). For $(s,\theta,t)\in A$, set

$$U(s,\theta,t) = \begin{cases} U(3,\theta,t), & \text{for } (s,\theta,t)\in A_I,\\ x_m, & \text{for } (s,\theta,t)\in A_S,\end{cases}$$

and now apply the Tietze extension theorem to define U on A_T so that U is continuous in A. Set $U(s,\theta,t) = u_*(t)$ for $t\in[0,\delta'_1]\cup[\pi-\delta'_2,\pi]$ and apply solder in the interval $[\pi-\delta'_1,\pi-\delta'_2]$, i.e., set

$$U(s,\theta,t) = \Xi_{\pi-\delta'_1,\pi-\delta'_2}(\omega_m(U(s,\theta,\cdot),\pi-\delta'_1),0,t).$$

We are left to show that under this construction, for sufficiently small **tol**, solder is applicable, i.e., $\omega_m(U(s,\theta,\cdot),\pi-\delta'_1)$ can be taken as small as desired.

We first prove that given $\epsilon>0$ there exists $\mathbf{tol}>0$ such that for any s_0 and θ_0 the Lebesgue measure of

$$A_{T,s_0,\theta_0} = \{t\in[0,\pi] \mid (s_0,\theta_0,t)\in A_T\}$$

is smaller than ϵ. First consider $A_{T,s_0,\theta_0}\cap[\pi-\delta'_0,\pi-\delta'_1]$: in this interval, $\omega_m(U(3,\theta_0,\cdot),t) - mt$ is strictly decreasing and does not depend on θ_0. Thus, for sufficiently small **tol** we may assume $\mu(A_{T,s_0,\theta_0}\cap[\pi-\delta'_0,\pi-\delta'_1]) < \epsilon/2$. Next consider $A_{T,s_0,\theta_0}\cap[\delta'_1,\pi-\delta'_0]$. Suppose by contradiction that for a fixed $\epsilon>0$ and for all $\mathbf{tol}>0$ we have

$$\mu(A_{T,s_0,\theta_0}\cap[\delta'_1,\pi-\delta'_0]) > \epsilon/2.$$

From Lemma 4.5, $\omega'_m(U(3,\theta_0,\cdot),t)$ is flat in $t=t_0$ for some $\theta_0\in\mathbb{S}^k$. Now, making use of the last item of Lemma 4.1, $U(3,\theta_0,t)$ is flat in $t=t_0$, contradicting its polynomiality.

Step 5

At this point of the construction the loop $U(4,\theta,t)$ is L^1 close to the constant loop u_*. Indeed, consider $U(3,\theta,t)$: this function of θ and t is C^0-bounded by some constant C. Given $\epsilon_1 > 0$ we can choose **tol** in step 4 in such a way that, for all θ,

1. $|U(4,\theta,\cdot) - u_*|_{C^0} < 2C + 1$,
2. $|U(4,\theta,\cdot) - u_*|_{L^1} < \epsilon_1$.

The first follows from compactness of the set of parameters for the function Ξ employed in the soldering in step 4. The second follows from choosing **tol** such that A_T has measure less than $\epsilon_1/(4C+2)$ (as discussed in step 4).

We may now take a family in θ of straight lines joining $U(4,\theta,t)$ to $U(5,\theta,t) = u_*$. More formally, let

$$U(s,\theta,t) = \begin{cases} \Xi_{\pi-\delta_1',\pi-\delta_2'}(h(s,\theta),0,t), & \text{for } t \in [\pi-\delta_1', \pi-\delta_2'], \\ (5-s)U(4,\theta,t) + (s-4)u_*(t), & \text{otherwise}, \end{cases}$$

where, again, $h(s,\theta) = \omega_m(U(s,\theta,\cdot), \pi-\delta_1') - m(\pi-\delta_1')$. The fact that $U(s,\theta,\pi-\delta_1') - m(\pi-\delta_1')$ is appropriately small follows from Lemma 4.1, item (c). ■

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