Refinement of Novikov–Betti numbers and of Novikov homology provided by an angle valued map

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dedicated to the memory of Yuri Petrovich Solovyov

Abstract

To a pair (X,f), X compact ANR and $f : X \to \mathbb{S}^1$ a continuous angle valued map, κ a field and a nonnegative integer r, one assigns a finite configuration of complex numbers z with multiplicities $\delta^f_r(z)$ and a finite configuration of free $\kappa[t^{-1}, t]$–modules $\hat\delta^f_r$ of rank $\delta^f_r(z)$ indexed by the same numbers z. This is in analogy with the configuration of eigenvalues and of generalized eigen-spaces of a linear operator in a finite dimensional complex vector space. The configuration δ^f_r refines the Novikov–Betti number in dimension r and the configuration $\hat\delta^f_r$ refines the Novikov homology in dimension r associated with the cohomology class defined by f.

In the case the field $\kappa = \mathbb{C}$ the configuration $\hat\delta^f_r$ provides by "von-Neumann completion" a configuration $\hat{\hat\delta}^f_r$ of mutually orthogonal closed Hilbert submodules of the L_2–homology of the infinite cyclic cover of X determined by the map f, which is an $L^\infty(\mathbb{S}^1)$-Hilbert module.

Contents

1 Introduction

In [5] and [2] for a pair (X,f), X compact ANR[1], f a real valued map, κ a field and r a nonnegative integer we have assigned a configuration δ^f_r of complex numbers z with multiplicities $\delta^f_r(z)$ and a configuration of vector spaces $\hat\delta^f_r(z)$, indexed by the same set $\{z \mid \delta^f_r(z) \geq 1\} = supp(\delta^f_r)$, with the following properties:

1. $\dim \hat\delta^f_r(z) = \delta^f_r(z)$,

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[1] cf subsection 2.1 for definition

2. $\sum_{z\in supp(\delta^f_r)} \delta^f_r(z) = \beta_r(X;\kappa)$, $\beta_r(X;\kappa) = \dim H_r(X;\kappa)$,

3. $\oplus_{z\in supp(\delta^f_r)} \hat\delta^f_r(z) \simeq H_r(X;\kappa)$. [2]

The integers $\beta_r(X;\kappa)$ are referred to as the Betti numbers of X with coefficients in κ.

Each vector space $\hat\delta^f_r(z)$ appears as a quotient of subspaces of $H_r(X;\kappa)$ well separated in a sense specified later in the paper. This assignment was in analogy with the *spectral package* of a pair (V,T), V a f.d. complex vector space, $T: V \to V$ a linear map, to which one assigns the configurations δ^T and $\hat\delta^T$ defined by the eigenvalues of T with their multiplicity and the collection of generalized eigenspaces corresponding to the eigenvalues of T. It was shown in [5] that :

- **P1**: The assignment $f \rightsquigarrow \delta^f_r$ is continuous,
- **P2**: For M^n a closed topological manifold, one has $\delta^f_r(z) = \delta^f_{n-r}(i\overline{z})$,
- **P3**: If X is homeomorphic to a simplicial complex or a Hilbert cube manifold then for an open and dense set of maps f, one has $\delta^f_r(z) \leq 1$.

When $\kappa = \mathbb{C}$ a Hermitian scalar product on $H_r(X;\mathbb{C})$ (i.e. a Hilbert space structure on $H_r(X;\mathbb{C})$) provides a canonical realization of the vector spaces $\hat\delta^f_r(z)$ as a collection of mutually orthogonal subspaces $\hat{\hat\delta}^f_r(z) \subseteq H_r(M;\mathbb{C})$ such that **P1**, **P2** and **P3** above continue to hold for $\hat{\hat\delta}^f_r$. The configuration δ^f_r can be reformulated as a *characteristic polynomials* $P^f_r(z)$, a monic polynomial whose zeros are the complex numbers z in the support of δ^f_r with multiplicity $\delta^f_r(z)$, of degree equal to the Betti number in dimension r.

The complex numbers z in the support of δ^f_r with multiplicity $\delta^f_r(z)$, equivalently the zeros of $P^f_r(z)$ with their multiplicities, can be calculated in case X is a simplicial complex and f a simplicial map by effective algorithms and are part of the *level persistence invariants* or *bar codes*, exactly those relevant for the global topology of X. They are of interest in data analysis. The configurations $\hat{\hat\delta}^f_r$, when considered for a closed Riemannian manifold, make a real valued map f a provider of an orthogonal decomposition (depending on f) of the space of harmonic forms on the Riemannian manifold. In particular, for a generic f, the components have dimension one providing a *canonical base* in spaces of harmonic forms.

In this paper we present a similar picture for a pair (X,f), X a compact ANR and f an angle valued map $f: X \to \mathbb{S}^1$, with similar virtues. In this case an additional homological data, the cohomology class $\xi^f \in H^1(X;\mathbb{Z})$ determined by the map f, is involved. As expected the results are similar but more subtle and more complex.

In the case of an angle valued map one has to replace the Betti numbers $\beta_r(X;\kappa)$ by the Novikov–Betti numbers $\beta^B(X,\xi;\kappa)$ and the $\kappa-$vector space $H_r(X;\kappa)$ by the Novikov homology $H^N_r(X,\xi;\kappa[t^{-1},t])$. In this paper the Novikov homology is a free $\kappa[t^{-1},t]$-module derived from $H_r(\tilde X;\kappa)$, the homology of the infinite cyclic cover $\tilde X$ associated with ξ and $\beta^N_r(X,\xi;\kappa) = \mathrm{rank}\ \mathrm{H}^N_r(X,\xi;\kappa[t^{-1},t])$. One produces analogous configurations δ^f_r and $\hat\delta^f_r$ which satisfy properties 1., 2., 3. and (analogues of) **P1**, **P2**, **P3** above. In case of $\kappa=\mathbb{C}$, instead of a Hilbert space structure on $H_r(X;\mathbb{C})$ one considers a Hilbert module structure on the von-Neumann completion of $H^N_r(X,\xi;\kappa[t^{-1},t])$, see subsection 2.1, which always exists . This permits to convert the configuration $\hat\delta^f_r$ into a configuration of closed Hilbert submodules $\hat{\hat\delta}^f_r$ with **P1**, **P2** and **P3** continuing to hold as in the case of real valued maps. To formulate the results more precisely one needs first to recall for the reader some algebra and algebraic topology concepts and establish some notations.

[2] $\simeq$ denotes isomorphism

Let κ be a field and $\kappa[t^{-1},t]$ be the ring of Laurent polynomials with coefficients in κ. This ring is a commutative κ–algebra, an integral domain and a principal ideal domain. As a consequence, for any f.g. $\kappa[t^{-1},t]-$module M, the quotient $F(M) := M/T(M)$ with $T(M)$ the submodule of torsion elements, is a f.g. free module. The f.g. module $T(M)$ being torsion is a finite dimensional vector space over κ. The only invariant of $F(M)$ is its rank which, when Q is a field containing $\kappa[t^{-1},t]$, is equal to the dimension of the Q–vector space $M \otimes_{\kappa[t^{-1},t]} Q = F(M) \otimes_{\kappa[t^{-1},t]} Q$.

For a pair $(X, \xi \in H^1(X;\mathbb{Z}))$ one considers the infinite cyclic cover $\pi : \tilde{X} \to X$ associated to ξ, and the deck transformation $\tau : \tilde{X} \to \tilde{X}$, the restriction of the free action $\mu : \mathbb{Z} \times \tilde{X} \to \tilde{X}$ associated to ξ to $1 \times \tilde{X}$. The $\kappa-$vector space $H_r(\tilde{X};\kappa)$ equipped with the isomorphism $t_r : H_r(\tilde{X};\kappa) \to H_r(\tilde{X};\kappa)$ induced by τ becomes a $\kappa[t^{-1},t]$–module. When X is a compact ANR this module is finitely generated. For any Q commutative ring which contains $\kappa[t^{-1},t]$ denote by $H^N_r(X,\xi;Q) := F(H_r(\tilde{X},\kappa)) \otimes_{\kappa[t^{-1},t])} Q$ and refer to this free module as *Novikov homology* with coefficients in Q. When $Q = \kappa[t^{-1},t]]$, the field of Laurent power series, the Novikov homology is a $\kappa[t^{-1},t]]-$vector space which is exactly what Novikov has considered in his approach to Morse theory for an angle valued map. In this paper we consider the case $Q = \kappa[t^{-1},t]$.

The rank of $H^N_r(X,\xi;Q)$ is independent of Q, denoted by $b^N_r(X,\xi;\kappa)$, and referred to as the Novikov–Betti number in dimension r w.r. to κ.

Suppose $\kappa = \mathbb{C}$. The ring $\mathbb{C}[t^{-1},t]$ can be completed to a finite von Neumann algebra $\mathcal{N}$, see subsection 2.1 or [11]. This algebra is exactly $L^\infty(\mathbb{S}^1)$. A f.g free $\mathbb{C}[t^{-1},t]$–module M equipped with a $\mathbb{C}[t^{-1},t]$-inner product can be completed to an $\mathcal{N}$-Hilbert module $\overline{M}$ of finite type, and a collection of split submodules of M to closed Hilbert submodules of $\overline{M}$; therefore a collection N_α of quotients of split submodules of M provides, in a canonical manner, a collection of closed Hilbert submodules $\overline{N}_\alpha$ of $\overline{M}$ as described in subsection 2.1. Different $\mathbb{C}[t^{-1},t]$-inner products on M provide isometric Hilbert modules completions so one can ignore the inner product in the notation $\overline{M}$.

In this paper we start with a pair (X,ξ), X a compact ANR, $\xi \in H^1(X;\mathbb{Z})$ and a field κ. Consider the free $\kappa[t^{-1},t]$–module $H^N_r(X,\xi;\kappa[t^{-1},t])$. When $\kappa = \mathbb{C}$, by using the *von Neumann completion* described in subsection 2.1, a $\mathbb{C}[t^{-1},t]$–inner product on $H^N_r(X,\xi;\mathbb{C}[t^{-1},t])$ permits to pass to the $\mathcal{N}$-Hilbert module $\overline{H^N_r(X,\xi;\mathbb{C}[t^{-1},t])}$ and to convert the configurations $\hat{\delta}^f_r$ into configurations of mutually orthogonal Hilbert submodules of $\overline{H^N_r(X,\xi;\mathbb{C}[t^{-1},t])}$.

Note that the $\mathcal{N}$-Hilbert module $\overline{H^N_r(X,\xi;\mathbb{C}[t^{-1},t])}$ is isomorphic to the $\mathcal{N}$-Hilbert module $H^{L_2}_r(\tilde{X})$ known as L_2-homology defined in case X is a closed Riemannian manifold using the Riemannian metric, or in case X is a finite CW-complex using the cell-structure of X.

The main result of the paper is the following theorem.

Theorem 1.1

1. *To a continuous map $f : X \to \mathbb{S}^1$ one can alssociate a monic polynomial $P^f_r(z)$ with non vanishing roots of degree equal to $b^N_r(X,\xi^f;\kappa)$, equivalently a configuration δ^f_r of non vanishing complex numbers z with multiplicities $\delta^f_r(z) \geq 1$,*

$$\text{Zeros of } P_r(z) \ni z \rightsquigarrow \delta^f_r(z) \in \mathbb{Z}_{\geq 1},$$

   which satisfies **P1**, **P2** *and* **P3**.

2. *One can refine the configuration δ^f_r to the assignment*

$$\text{Zeros of } P_r(z) \ni z \rightsquigarrow (L'(z) \subset L(z)) \in \mathcal{S}(M),\ M = H^N(X,\xi;\kappa[t^{-1},t])$$

   with $\mathcal{S}(M)$ the set of pairs of free split submodudeles of M with $L' \subset L$ such that:

(a) $\bigoplus \hat{\delta}^f_r(z)$ *is isomorphic to* $H^N_r(X,\xi;\kappa[t^{-1},t])$ *where* $\hat{\delta}^f_r(z) = L_r(z)/\mathcal{L}'_r(z)$ *and*

(b) $\mathrm{rank}(\hat{\delta}^f_r(z)) = \delta^f_r(z)$.

3. *In case* $\kappa = \mathbb{C}$ *and a* $\mathbb{C}[t^{-1},t]$*-inner product* [3] *on* $H^N_r(X,\xi;\mathbb{C}[t^{-1},t])$ *is given, by using von-Neumann completion, one can convert the assignment above (the configuration* $\hat{\delta}^f_r$*) into a configuration* $\hat{\hat{\delta}}^f_r(z)$ *of mutually orthogonal closed Hilbert submodules of the* $L^\infty(\mathbb{S}^1)$*-Hilbert module* $\overline{H^N_r(X,\xi;\mathbb{C}[t^{-1},t])}$ *which satisfies* **P1** *and* **P2**. *Up to an isometry of Hilbert module structures the configurations* $\hat{\hat{\delta}}^f_r$ *are independent of the inner product.*

We view the configuration δ^f_r as a refinement of the Novikov-Betti number $b^N_r(X,\xi;\kappa)$ and the configuration $\hat{\delta}^f_r$ as a refinement of the Novikov homology $H^N_r(X,\xi;\mathbb{C}[t^{-1},t])$ or better said as an additional structure on the Novikov homology. In this paper we give only the construction of the configurations δ^f_r, $\hat{\delta}^f_r$ and $\hat{\hat{\delta}}^f_r$; The details of the proof that the configurations $\hat{\hat{\delta}}^f_r$ satisfy **P1**, **P2** and **P3** will be presented in a paper in preparation [3]. The configuration δ^f_r was introduced and studied in [5], however the configurations $\hat{\delta}^f_r$ and $\hat{\hat{\delta}}^f_r$ have not been considered before.

We note that the configuration δ^f_r, or equivalently the polynomial $P^f_r(z)$, i.e. the roots with their multiplicities can be explicitly calculated in case X is a finite simplicial complex and f a simplicial map. Precisely one can produce algorithms with input the simplicial complex and the values of f on vertices and output the zeros of the polynomial $P^f_r(z)$ as a pair of real number (the real and the imaginary part) with their multiplicities. Such algorithm is presented in [4] where the zeros of $P^f_r(z)$ appear as "closed r-bar codes" and "open $(r-1)$-bar codes" [4].This algorithm uses a different definition of the configuration δ^f_r based on bar codes in "level persistence" cf [8] or [4] .

We also note that for $\kappa = \mathbb{C}$ and for X an n–dimensional closed Riemannian manifold the space of L_2-harmonic differential forms of degree $n-r$ on the complete Riemannian manifold $\tilde{X}$ identifies to the L_2-homology in dimension r of $\tilde{X}$ via the Hodge theory on Riemannian manifolds. The configuration $\hat{\hat{\delta}}^f_r$ provides in this case a decomposition of the Hilbert module of harmonic forms which depends continuously on f. For a generic set of continuous functions f, the Hilbert submodules $\hat{\hat{\delta}}^f_r(z)$ have the von Neumann dimension equal to one.

To explain the construction of the configurations δ^f_r, $\hat{\delta}^f_r$ and $\hat{\hat{\delta}}^f_r$ some preliminaries presented in section 2. are necessary. In section 3 one provides the definition of the configurations and a number of intermediate results and properties while in section 3 one indicates the way one verifies the statements in Theorem 1.1.

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2 Preparatory material

2.1 Completion

Let $\mathbb{C}[t^{-1},t]$ be the ring of Laurent polynomials, equivalently the group ring $\mathbb{C}[\mathbb{Z}]$ of the infinite cyclic group. This is an algebra with involution $*$ and trace tr. If $a = \sum_{n\in\mathbb{Z}} a_n t^n$ then:

$$\begin{aligned} *(a) := a^* =& \sum_{n\in\mathbb{Z}} \overline{a}_n t^{-n} \\ tr(a) =& a_0. \end{aligned}$$

[3] see the definition in subsection 2.1

[4] The zero $z = \rho e^{i\theta} \in \mathbb{C}\setminus 0$ represents a closed bar code when $\rho \geq 1$ and an open bar code when $\rho < 1$

with $\overline{a}$ the complex conjugate of the complex number a.

The algebra $\mathbb{C}[\mathbb{Z}]$ can be considered as a sub algebra of the algebra of bounded linear operators on the separable Hilbert space $l_2(\mathbb{Z})$, of square summable sequences $\{a_n, n \in \mathbb{Z} \mid \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty\}$. The linear operator defined by a Laurent polynomial is given by the multiplication of sequences in $l_2(\mathbb{Z})$ with the Laurent polynomial regarded as a sequence with all but finitely many components equal to zero. One denotes by $\mathcal{N}$ the weak closure of $\mathbb{C}[\mathbb{Z}]$ which is a finite von Neumann algebra, with involution and trace extending the ones defined above, cf [11].

This algebra $\mathcal{N}$ is referred to below as the von–Neumann completion of the group ring $\mathbb{C}(\mathbb{Z})$ and is isomorphic to the familiar $L^\infty(\mathbb{S}^1)$ via Fourier series transform (which assigns to a complex valued function defined on $\mathbb{S}^1$ its Fourier series).

Given a free $\mathbb{C}[t^{-1}, t]$–module M a $\mathbb{C}[t^{-1}, t]$-*valued inner product* is a map $\mu : M \times M \to \mathbb{C}[t^{-1}, t]$ which is:

1. $\mathbb{C}[t^{-1}, t]-$linear in the first variable,
2. *symmetric* in the sense that $\mu(x, y) = \mu(y, x)^*$, $x, y \in M$,
3. *positive definite* in the sense that satisfies
   (a) $\mu(x, x) \in \mathbb{C}[t^{-1}, t]_+$ with $\mathbb{C}[t^{-1}, t]_+$ the set of elements of the form aa^* and
   (b) $\mu(x, x) = 0$ iff $x = 0$,

   and satisfies
4. the map $M \to Hom_{\mathbb{C}[t^{-1},t]}(M, \mathbb{C}[t^{-1}, t])$ defined by $\mu(y)(x) = \mu(x, y)$ is one to one.

Clearly $\mathbb{C}[t^{-1}, t]$-valued inner products exist. Indeed, if $e^1, e^2, \cdots e^k$ is a base of M then

$$\mu(\sum a_i e^i, \sum b_j e^j) := \sum a_i(b_i)^*$$

provides such inner product.

By completing the $\mathbb{C}-$vector space M w.r. to the hermitian inner product $< x, y >:= tr(\mu(x, y))$ one obtains a Hilbert space $\overline{M}$ which is an $\mathcal{N}$-Hilbert module cf [11] isomeric to $l_2(\mathbb{Z})^{\oplus k}$, k the rank of M. Two different $\mathbb{C}[t^{-1}, t]$-valued inner products μ_1 and μ_2 lead to the isometric Hilbert modules $\overline{M}_{\mu_1}$ and $\overline{M}_{\mu_2}$. This justifies dropping μ from notation. If one identifies $\mathcal{N}$ to $L^\infty(\mathbb{S}^1)$ and $l_2(\mathbb{Z})^{\oplus k}$ to $L^2(\mathbb{S}^1)^{\oplus k}$ (by interpreting the sequence $\sum_{n\in\mathbb{Z}} a_n t^n$ as the complex valued function $\sum_{n\in\mathbb{Z}} a_n e^{in\theta}$) the $\mathcal{N}-$ module structure on $l_2(\mathbb{Z})^{\oplus k}$ becomes the $L^\infty(\mathbb{S}^1)$-module structure on $(L^2(\mathbb{S}^1))^{\oplus k}$ and is given by the component-wise multiplication of a $k-$tuple of L^2-functions, element in $(L^2(\mathbb{S}^1))^{\oplus k}$, by the L^∞-function in $L^\infty(\mathbb{S}^1)$.

If one has $N \subset M$ a free split submodule of the f.g free $\mathbb{C}[t^{-1}, t]$-module M and μ is an $\mathbb{C}[t^{-1}, t]$-valued inner product on M then $\overline{N}_\mu$ is a closed Hilbert submodule of $\overline{M}_\mu$. Moreover if $N'_i \subseteq N_i \subseteq M$, $i = 1, 2, \cdots$ is a collection of split submodules and N_i/N'_i is a collection of free modules, quotient of submodules of M, then one can canonically convert N_i/N'_i into closed Hilbert submodules of $\overline{M}$ simply by taking the closure of the kernel of the projection $N_i \to N_i/N'_i$ inside N_i. The process of passing from $(\mathbb{C}[t^{-1}, t], M)$ to $(\mathcal{N}, \overline{M})$ is referred to below as *von Neumann completion* and was pioneered in [11] for any group ring $\mathbb{C}[\Gamma]$ and f.g. projective $\mathbb{C}[\Gamma]$-module.

2.2 Configurations and the collision topology on the space of configurations

Let X be a topological space and N a positive integer. Denote by

$$\mathcal{C}_N(X) := \{\delta : X \to \mathbb{Z}_{\geq 0} \mid \sum_{x\in X} \delta(x) = N\}$$

the set of finite configurations of total cardinality N. This set identifies to the space X^N/Σ_N the quotient space of the cartesian $N-$fold product of X by the action of the permutation group Σ_N of $N-$objects.

Let V be a free f.g.module over the commutative unital ring R or a finite type Hilbert module over a finite von Neumann algebra $\mathcal{N}$ and let $\mathcal{P}(V)$ be the set of free split submodules in the first case or of closed Hilbert submodules in the second. One generalizes the set of configurations $\mathcal{C}_N(X)$ to the set $\mathcal{C}_V(X)$. The set $\mathcal{C}_V(X)$ consistis of maps $\hat{\hat{\delta}} : X \to \mathcal{P}(V)$ which satisfy

1. $supp(\hat{\hat{\delta}}) = \{x \in X \mid \hat{\hat{\delta}}(x) \neq 0\}$ is finite
2. if $i(x) : \hat{\hat{\delta}}(x) \to V$ denotes the inclusion of $\hat{\hat{\delta}}(x)$ in V then the map I

$$I := \sum_{x \in supp(\hat{\hat{\delta}})} i(x) : \bigoplus_{x \in supp(\hat{\hat{\delta}})} \hat{\hat{\delta}}(x) \to V$$

is an isomorphism. Denote by

$$e : \mathcal{C}_V(X) \to \mathcal{C}_{\dim V}(X)$$

the map defined by $e(\hat{\hat{\delta}})(x) := \dim \hat{\hat{\delta}}(x)$.

The set $\mathcal{C}_N(X)$ and the set $\mathcal{C}_V(X)$ when $R = \mathbb{C}$ or when V is an $\mathcal{N}-$ Hilbert module carry natural topologies, referred to as the *collision topology*, which make e continuous. One way to describe these topologies is to specify for each δ or $\hat{\hat{\delta}}$ a system of *fundamental neighborhoods*.

If δ has as support the set of points $\{x_1, x_2, \cdots, x_k\}$, a fundamental neighborhood $\mathcal{U}$ of δ is specified by a collection of k disjoint open neighborhoods $U_1, U_2, \cdots U_k$ of $x_1, \cdots x_k$, and consists of $\{\delta' \in \mathcal{C}_N(X) \mid \sum_{x \in U_i} \delta'(x) = \delta(x_i)\}$. Similarly, if $\hat{\hat{\delta}}$ has as support the set of points $\{x_1, x_2, \cdots x_k\}$ with $\hat{\hat{\delta}}(x_i) = V_i \subseteq V\}$, a fundamental neighborhood $\mathcal{U}$ of $\hat{\hat{\delta}}$ is specified by a collection of disjoint open neighborhoods $U_1, U_2, \cdots U_k$ of $x_1, \cdots x_k$, and open neighborhoods $O_1, O_2, \cdots, O_k$ of $V_1, V_2, \cdots, V_k$ in $G_{\dim V_i}(V)$ and consists of

$$\{\hat{\hat{\delta}}' \in \mathcal{C}_V(X) \mid \sum_{x \in U_i} \hat{\hat{\delta}}'(x) \in O_i\}.$$

Here $G_k(V)$ denotes the Grassmanian of $k-$dimensional subspaces of V [5].

Clearly e is continuous and surjective with fiber above δ, the subset of $G_{n_1}(V) \times G_{n_2}(V) \cdots \times G_{n_k}(V)$ consisting of $(V'_1, V'_2, \cdots V'_k), V'_i \in G_{n_i}(V)$ where $n_i = \dim V_i$.

Note that:

1. $\mathcal{C}_N(X) = X^N/\Sigma_N$ is the N–fold symmetric product of X and if X is a metric space with distance D then the collision topology is the topology defined by the induced distance $\underline{D}$ on X^N/Σ_N.
2. If $X = \mathbb{R}^2 = \mathbb{C}$ then $\mathcal{C}_N(X)$ identifies to the set of monic polynomials with complex coefficients. To the configuration δ whose support consists of the points $z_1, z_2, \cdots z_k$ with $\delta(z_i) = n_i$ one associates the monic polynomial $P^f(z) = \prod_i (z - z_i)^{n_i}$. Then $\mathcal{C}_N(X)$ identifies to $\mathbb{C}^N$ as metric spaces.
3. Similarly, if $X = \mathbb{C}^* = \mathbb{C} \setminus 0$ then $\mathcal{C}_N(X)$ identifies to the set of monic polynomials of degree N with non vanishing free coefficient, hence with $\mathbb{C}^{N-1} \times \mathbb{C}^*$ where $\mathbb{C}^* = \mathbb{C} \setminus 0$.

[5] When V is an $\mathcal{N}$–Hilbert module $G_k(V)$ can be identified to the set of $\mathcal{N}$–linear self adjoint projectors whose von Neumann trace is equal to k which inherits the topology induced by the norm of bounded operators in the Hilbert space V.

In this paper we will consider as intermediate step a slightly more general type of configurations involving *quotients of free split submodules* of a free $R-$module or *quotients of closed Hilbert submodules*; actually only the case $R = \kappa[t^{-1}, t]$ will be involved.

Let M be a f.g. free $\kappa[t^{-1}, t]$–module. Denote by $\tilde{\mathcal{S}}(M)$ the set of pairs $(L \supset L')$ each pair with L, L' split submodules of M. Since M is f.g. and free so are L and L' and L/L'.

A finite collection of pairs $(L_r \supset L'_r) \in \tilde{\mathcal{S}}(M),\ r = 1, 2, \cdots k$, is called *well separated* if for any right inverses $i_r : L_r/L'_r \to L_r \subseteq M$ of the projections $L_r \to L_r/L'_r$ the sum of linear map

$$\sum_{1\leq r\leq k} i_r : \bigoplus L_r/L'_r \to M$$

is injective.

For a map $\hat{\delta} : X \to \tilde{\mathcal{S}}(M)$ denote by $supp(\hat{\delta})$ the set

$$supp(\hat{\delta}) := \{x \in X \mid \hat{\delta}(x) = (L(x), L'(x)),\ L(x) \neq 0\}.$$

A finite configuration of quotient of split submodules of M is a map $\hat{\delta} : X \to \tilde{\mathcal{S}}(M)$ which satisfies:

1. $supp(\hat{\delta})$ is finite,
2. The collection of pairs $(L(x) \supset L'(x))$ is well separated.
3. For any right inverses $i(x)'$s the linear map

$$\sum_{x\in supp(\hat{\delta})} i(x) : \bigoplus_{x\in supp(\hat{\delta})} L(x)/L'(x) \to M$$

   is an isomorphism.

When $R = \mathbb{R}$ or $\mathbb{C}$, in the presence of an scalar product (Hilbert space structure) on V, one can canonically pass from a map as above $\hat{\delta}$ to a map $\hat{\hat{\delta}}$ by replacing the pair $L(x) \supset L'(x)$ by the orthogonal complement of $L'(x)$ in $L(x)$. This is also the case when V is a $\mathcal{N}-$Hilbert module.

In view of the subsection 2.1, when $\kappa = \mathbb{C}$, the choice of a $\mathbb{C}[t^{-1}, t]$-valued inner product on M provides a hermitian inner product in M as explained in the previous subsection and the von Newman completion converts any configuration $\hat{\delta}$ into a configuration $\hat{\hat{\delta}}$ of closed Hilbert submodules of $\overline{M}$.

2.3 Preliminary on compact ANR's and tame maps

Tame maps: For a continuous map $f : X \to Y$ between two topological Hausdorff spaces a *regular value* is a point $y \in Y$ for which there exists a neighborhood U of y s.t. for any $y' \in U$ the inclusion $f^{-1}(y') \subset f^{-1}(U)$ is a homotopy equivalence. The values y which are not regular are called *critical* and a map is *tame* if the set of critical values $Cr(f) \subset Y$ is discrete. In case Y is a metric space with distance d, in particular $Y = \mathbb{R}$ or $\mathbb{S}^1$, for a map f one can introduce $\epsilon(f) := \inf_{y_1,y_2\in Cr(f),y_1\neq y_2)} d(y_1, y_2)$. If $Y = \mathbb{R}$ or $\mathbb{S}^1$, X is compact and f is tame then $\epsilon(f) > 0$ and if f is not tame then $\epsilon(f) = 0$.

ANR's and Hilbert cube manifolds: One denotes by $[0, 1]^\infty$ the countable product of the compact interval $[0, 1]$ and call it the *Hilbert cube*. A second countable Hausdorff space is a Hilbert cube manifold if is locally homeomorphic to $[0, 1]^\infty$. In view of fundamental results on the topology of Hilbert cube manifolds due to Edwards, Chapman, West etc, cf [6], *a locally compact space X is an ANR iff the product with $[0, 1]^\infty$ is a Hilbert cube manifold.*

We are concerned in this paper with compact ANRs. A space X is a compact ANR iff stably homeomorphic to a finite simplicial complex, i.e. iff there exists a simplicial complex K such that $X \times [0,1]^\infty$ is homeomorphic to $K \times [0,1]^\infty$. Recall also that two compact Hilbert cube manifolds are homeomorphic iff they are homotopy equivalent cf [6] however not any homotopy equivalence is homotopic to a homeomorphism [6]. The p.l. maps from a simplicial complex K into $\mathbb{R}$ or $\mathbb{S}^1$ are dense in the space of continuous maps with compact open topology. Any p.l map is tame if K is finite. For a compact Hilbert cube manifold the tame maps are also dense in the space of continuous maps (with compact open topology).

We will use these results on compact Hilbert cube manifolds to establish results about general compact ANRs by verifying first their validity for finite simplicial complexes.

3 The configurations δ^f_r, $\hat{\delta}^f_r$ and $\hat{\hat{\delta}}^f_r$.

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Let X be a compact ANR and $f : X \to \mathbb{S}^1$ be a continuous map. An infinite cyclic cover of f is provided by the commutative diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{p} & \mathbb{S}^1 \\ \tilde{f}\uparrow & & \uparrow f \\ \tilde{X} & \xrightarrow{\pi} & X \end{array} \qquad (1)$$

and the free action $\mu : \mathbb{Z} \times \tilde{X} \to \tilde{X}$ with $\pi(\mu(n,x)) = \pi(x)$ inducing an homeomorphism $\pi : \tilde{X}/\mathbb{Z} \to X$ and $\tilde{f}(\mu(n,x)) = \tilde{f}(x) + 2\pi n$. Denote by $\tau : \tilde{X} \to \tilde{X}$ the restriction of μ to $1 \times \tilde{X}$.

Fix κ a field. To ease the writing for a space Y we abbreviate $H_r(Y;\kappa)$ by $H_r(Y)$. The homeomorphism τ induces the isomorphism $t_r : H_r(\tilde{X}) \to H_r(\tilde{X})$ which defines a structure of $\kappa[t^{-1},t]$-module on the $\kappa-$vector space $H_r(\tilde{X})$. The isomorphism t_r represents the multiplication by $t \in \kappa[t^{-1},t]$. Note that $H_r(\tilde{X})$ is a f.g. $\kappa[t^{-1},t]$-module and that $\kappa[t^{-1},t]$ is a principal ideal domain therefore the torsion submodule $T(H_r(\tilde{X}))$ is a finite dimensional $\kappa-$vector space, $H^N_r(X,\xi) := H_r(\tilde{X})/T(H_r(\tilde{X}))$ is a f.g. free $\kappa[t^{-1},t]-$module and $H_r(\tilde{X})$ is isomorphic to $H^N_r(X,\xi) \oplus T(H_r(\tilde{X}))$.

Denote by $\tilde{X}_a = \tilde{f}^{-1}((-\infty,a])$, $\tilde{X}^b = \tilde{f}^{-1}([b,\infty))$. Following [2] one introduces the following notions:

- $\mathbb{I}_a(r) = img(H_r(\tilde{X}_a) \to H_r(\tilde{X}))$, $\mathbb{I}^b(r) = img(H_r(\tilde{X}^b) \to H_r(\tilde{X}))$,
- $\mathbb{I}_{-\infty}(r) := \cap_{a\in\mathbb{R}}\mathbb{I}_a(r), \quad \mathbb{I}^\infty(r) := \cap_{b\in\mathbb{R}}\mathbb{I}^b(r)$,
- $\mathbb{F}_r(a,b) = \mathbb{I}_a(r) \cap \mathbb{I}^b(r)$,
- $\mathbb{F}_r(-\infty,b) = \mathbb{I}_{-\infty}(r) \cap \mathbb{I}^b(r), \quad \mathbb{F}_r(a,\infty) = \mathbb{I}_a(r) \cap \mathbb{I}^\infty(r)$.

With this notation one observes as in [2] that:

Observation 3.1

1. *$t_r : \mathbb{I}_a(r) \to \mathbb{I}_{a+2\pi}(r)$ and $t_r : \mathbb{I}^b(r) \to \mathbb{I}^{b+2\pi}(r)$ are isomorphisms and therefore: $t_r : \mathbb{F}_r(a,b) \to \mathbb{F}_r(a+2\pi, b+2\pi)$ is an isomorphism. Then both $\mathbb{I}_{-\infty}(r)$ and $\mathbb{I}^\infty(r)$ are $\kappa[t^{-1},t]$-submodules.*

[6]but only the simple homotopy equivalences cf [6]

2. *For $a' \leq a$ and $b \leq b'$ one has:*

   $\mathbb{F}_r(a', b') \subseteq \mathbb{F}_r(a, b)$,

   $\mathbb{F}_r(-\infty, b') \subseteq \mathbb{F}_r(a, b)$ *and* $\mathbb{F}_r(a', \infty) \subseteq \mathbb{F}_r(a, b)$.

3. $\cup_{a\in\mathbb{R}}\mathbb{I}_a(r) = \cup_{b\in\mathbb{R}}\mathbb{I}^b(r) = H_r(\tilde{X})$.

Proposition 3.2

1. The dimension of $\mathbb{F}_r(a, b)$ is finite.
2. $\mathbb{I}_{-\infty}(r) = \mathbb{I}^{\infty}(r) = T(H_r(\tilde{X}))$.

Proof:

Item 1. is verified in [2] based on Meyer–Vietoris sequence and on the observations that $\tilde{X}$ is a locally compact ANR and $\tilde{f}$ is a proper map.

Item 2.: If $x \in T(H_r(\tilde{X}))$ then there exists an integer $k \in \mathbb{Z}$ and a polynomial $P(t) = \alpha_n t^n + \alpha_{n-1}t^{n-1} \cdots \alpha_1 t + \alpha_0$, $\alpha_i \in \kappa$, $\alpha_0 \neq 0$ such that $P(t)\, t^k x = 0$.

Let $y = t^k x$. By Observation 3.1 item 3. one has $y \in \mathbb{I}^b$ for some $b \in \mathbb{R}$. Since $P(t)y = 0$ one concludes that $y = -(\alpha_n/\alpha_0)t^{n-1} \cdots - (\alpha_1/\alpha_0)ty$ and then by Observation 3.1 item 1. one has $y \in \mathbb{I}^{b+2\pi}$.

Repeating the same argument one concludes that $y \in I^{b+2\pi k}$ for any k, hence $y \in \mathbb{I}^{\infty}$. Since $x = t^{-k}y$, by Observation 3.1 item 1. one has $x \in \mathbb{I}^{\infty}$. Hence $T(H_r(\tilde{X})) \subseteq I^{\infty}$.

Let $x \in \mathbb{I}^{\infty}$. By Observation 3.1 item 3. one has $x \in \mathbb{I}_a$ for some $a \in \mathbb{R}$ and if $x \in \mathbb{I}^{\infty}$, by Observation 3.1 item 1. , all $x, t^{-1}x, t^{-2}x, \cdots t^{-k}x, \cdots \in \mathbb{I}_a \cap \mathbb{I}^{\infty}$. Since by (1.) the dimension of $\mathbb{I}_a \cap \mathbb{I}^{\infty}$ is finite, there exists $\alpha_{i_1}, \cdots \alpha_{i_k}$ such that $(\alpha_{i_1}t^{-i_1} + \cdots \alpha_{i_k}t^{-i_k})x = 0$. This makes $x \in T(H_r(\tilde{X}))$. Hence $\mathbb{I}^{\infty} \subseteq T(H_r(\tilde{X}))$. Therefore $\mathbb{I}^{\infty} = T(H_r(\tilde{X}))$.

By a similar arguments one concludes that $T(H_r(\tilde{X})) = \mathbb{I}_{-\infty}$.

■

It is convenient to consider a weaker concept of critical values relative to homology with coefficients in the fixed field κ.

Definition 3.3

1. *A real number c is a* **sub level homologically critical values** *if for any $\epsilon > 0$ the inclusion $\mathbb{I}_{c-\epsilon}(r) \subset \mathbb{I}_{c+\epsilon}(r)$ is strict (not equality). Denote by $CR_-(\tilde{f})$ the set of such critical values.*

2. *A real number c is a* **over level homologically critical values** *if for any $\epsilon > 0$ the inclusion $\mathbb{I}^{c+\epsilon}(r) \subset \mathbb{I}^{c-\epsilon}(r)$ is strict (not equality). Denote by $CR^+(\tilde{f})$ the set of such critical values.*

3. *A real number c is a* **homologically critical values** *if it belongs to $CR(\tilde{f}) := CR_-(\tilde{f}) \cup CR^+(\tilde{f})$*

Observation 3.1 can be completed with the following observation.

Observation 3.4 *Suppose $\tilde{f} : \tilde{X} \to \mathbb{R}$ is an infinite cyclic cover of $f : X \to \mathbb{S}^1$, X a compact ANR.*

1. *$CR_-(\tilde{f})$, $CR^+(\tilde{f})$ and then $CR(\tilde{f})$ are discrete sets. Moreover there exists $\tilde{\epsilon}(f) > 0$ such that $\tilde{\epsilon}(f) < |c' - c''|$ for $c', c'' \in CR(\tilde{f}), c' \neq c''$.*

2. *The map f is tame iff $\tilde{f}$ is tame.*

3. *$CR(\tilde{f}) \subseteq Cr(\tilde{f})$ and $\tilde{\epsilon}(f) \geq \epsilon(f)$.*

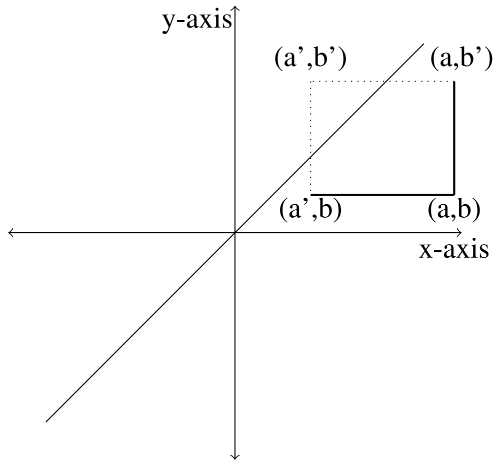

Figure 1: The *box* $B := (a', a] \times [b, b') \subset \mathbb{R}^2$

Item 1. follows from Observation 3.1 and Proposition 3.2, while items 2. and 3. are straightforward consequences of definitions.

Boxes; For $a' < a, b < b'$ one considers the domain $B = (a', a] \times [b, b')$ (see the Figure 1) and call it a finite box. An infinite box is of the form $(-\infty, a] \times [b, b')$ or $(-\infty, a] \times [b, \infty)$ or $(a', a] \times [b, \infty)$.

For a box B denote by:

$$\begin{aligned} \mathbb{F}'_r(B) &:= \mathbb{F}_r(a', b) + \mathbb{F}_r(a, b') \subseteq \mathbb{F}_r(a, b) \\ \mathbb{F}_r(B) &:= \mathbb{F}_r(a, b)/\mathbb{F}'_r(B) \end{aligned} \tag{2}$$

and let $\pi^B_{ab,r} : \mathbb{F}_r(a, b) \to \mathbb{F}_r(B)$ the projection on quotient space.

In view of the definition of $\mathbb{F}_r(a, b)$ and of $\mathbb{F}_r(B)$ it is straightforward to observe (as in [2]) that the following statements are true for either finite or infinite boxes. The linear maps involved are ultimately induced by inclusions $\mathbb{F}_r(a', b') \subseteq \mathbb{F}_r(a, b)$ for $a' \leq a, b \leq b'$ (cf Observation3.1 item 2) and by " passings to quotient spaces".

Observation 3.5

1. *If $a'' < a' < a$, $b < b''$ (possibly $a'' = -\infty, b'' = \infty$) and B_1, B_2 and B are the boxes ,$B_1 = (a'', a'] \times [b, b')$, $B_2 = (a', a] \times [b, b'')$ and $B = (a'', a] \times [b, b')$ (see Figure 2) then the inclusions $B_1 \subset B$ and $B_2 \subset B$ induce the linear maps $i^B_{B_1,r} : \mathbb{F}_r(B_1) \to \mathbb{F}_r(B)$ and $\pi^{B_2}_{B,r} : \mathbb{F}_r(B) \to \mathbb{F}_r(B_2)$ such that the sequence*

$$0 \longrightarrow \mathbb{F}_r(B_1) \xrightarrow{i^B_{B_1,r}} \mathbb{F}_r(B) \xrightarrow{\pi^{B_2}_{B,r}} \mathbb{F}_r(B_2) \longrightarrow 0\ .$$

is exact.

2. *If $a' < a$, $b < b' < b''$ (possibly $a' = -\infty, b'' = +\infty$) and B_1, B_2 and B are the boxes, $B_1 = (a', a] \times [b', b'')$, $B_2 = (a', a] \times [b', b')$ and $B = (a', a] \times [b, b'')$ (see Figure 3) then the inclusions*

$B_1 \subset B$ and $B_2 \subset B$ induce the linear maps $i^{B}_{B_1,r} : \mathbb{F}_r(B_1) \to \mathbb{F}_r(B)$ and $\pi^{B_2}_{B,r} : \mathbb{F}_r(B) \to \mathbb{F}_r(B_2)$ such that the sequence

$$0 \longrightarrow \mathbb{F}_r(B_1) \xrightarrow{i^{B}_{B_1,r}} \mathbb{F}_r(B) \xrightarrow{\pi^{B_2}_{B,r}} \mathbb{F}_r(B_2) \longrightarrow 0$$

is exact.

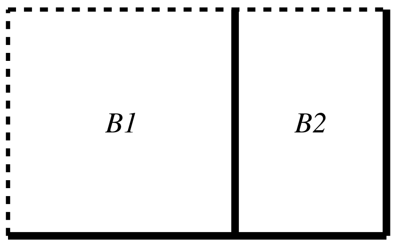

Figure 2

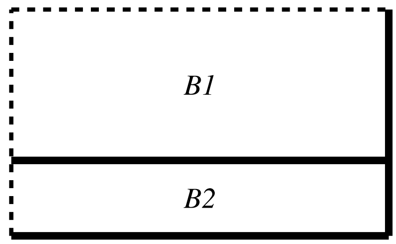

Figure 3

3. *If $B'$and B'' are two boxes with $B' \subseteq B''$ and B' is located in the upper left corner of B'' (see Figure 4) then the inclusion of boxes induces the canonical injective linear maps $i^{B''}_{B',r} : \mathbb{F}_r(B') \to \mathbb{F}_r(B'')$. If $B'$and B'' are two boxes with $B' \subseteq B''$ and B' is located in the lower right corner of B'' (see Figure 5) then the inclusion of boxes induces the canonical surjective linear maps $\pi^{B''}_{B',r} : \mathbb{F}_r(B') \to \mathbb{F}_r(B'')$.*

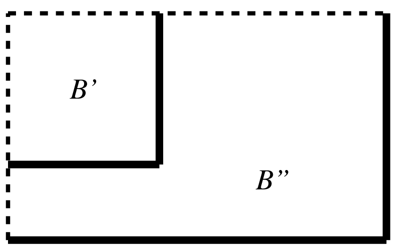

Figure 4

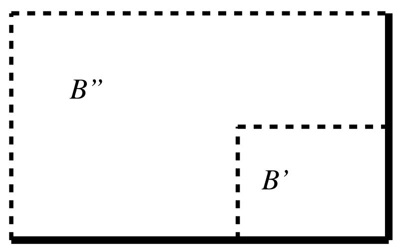

Figure 5

4. *If B is a finite disjoint union of boxes $B = \sqcup B_i$ then $\mathbb{F}_r(B)$ is isomorphic to $\oplus_i \mathbb{F}_r(B_i)$; the isomorphism is not canonical.*

For $\epsilon > 0$ one denotes by $B(a, b; \epsilon) := (a - \epsilon, a] \times [b, b + \epsilon)$ and then for $\epsilon' > \epsilon$ one has the surjective linear maps $\mathbb{F}_r(B(a, b; \epsilon')) \to \mathbb{F}_r(B(a, b; \epsilon))$. In view of Proposition 3.2 item 1. the limit

$$\hat{\delta}^{\tilde{f}}_r(a, b) := \varinjlim_{\epsilon \to 0} \mathbb{F}_r(B(a, b; \epsilon))$$

exists and one defines

$$\delta^{\tilde{f}}_r(a, b) := \dim(\hat{\delta}^{\tilde{f}}_r(a, b)).$$

In view of Observation 3.4 item 1. the limit stabilizes and for ϵ small enough and one has

$$\hat{\delta}^{\tilde{f}}_r(a, b) = \mathbb{F}_r(B(a, b; \epsilon)). \tag{3}$$

It is useful to regard $\hat{\delta}^{\tilde{f}}_r(a, b)$ not only as a vector space but as the quotient of subspaces of $H_r(\tilde{X})$

$$\hat{\delta}^{\tilde{f}}_r(a, b) = \mathbb{F}_r(a, b)/\mathbb{F}'_r(a, b),$$

with

$$\mathbb{F}'_r(a,b) := \mathbb{F}'_r(B(a,b;\epsilon)) = \mathbb{F}_r(a-\epsilon,b) + \mathbb{F}_r(a,b+\epsilon) \subseteq \mathbb{F}_r(a,b) \subseteq H_r(\tilde{X})$$

for ϵ small enough [7]. Note that if at least one of a, b are not homologically critical values then $\mathbb{F}_r(a,b) = \mathbb{F}'_r(a,b)$ and if both a, b are homologically critical values then the stabilization $\mathbb{F}'_r(a,b) = \mathbb{F}'_r(B(a,b;\epsilon))$ holds for any ϵ s.t. $0 < \epsilon < \tilde{\epsilon}(f)$.

Define

$$supp(\delta^{\tilde{f}}_r) = \mathrm{supp}(\hat{\delta}^{\tilde{f}}_r) := \{(a,b) \mid \delta^{\tilde{f}}_r(a,b) \neq 0\}.$$

As a consequence of equality (3) and of Observation 3.5 one has

Proposition 3.6

1. *$\delta^{\tilde{f}}_r(a,b) \neq 0$ implies both $a, b \in CR(\tilde{f})$.*
2. *For any finite or infinite box B the set $\mathrm{supp}(\delta^{\tilde{f}}_r) \cap B$ is finite.*
3. *For any finite or infinite box*

$$\sum_{(a,b)\in supp(\delta^{\tilde{f}}_r)} \delta^{\tilde{f}}_r(a,b) = \dim \mathbb{F}_r(B).$$

Proof:
Item 1. follows from Definition 3.3 and Observation 3.5 item 1. .
Item 2.: If $B \cap supp(\delta^{\tilde{f}}_r)$ is an infinite set then in view of Observation 3.5 item 4. and of the equality (3) $\dim \mathbb{F}_r(B)$ is infinite. This is not possible since this dimension is smaller than $\dim \mathbb{F}_r(a,b)$ with (a,b) the right bottom corner of B, which by Proposition 3.2 is finite.
Item 3. : By Observation 3.5 item 3. and the equality (3), if $B \cap supp(\delta^{\tilde{f}}_r) = \emptyset$ then $\mathbb{F}_r(B) = 0$, and if $B \cap supp(\delta^{\tilde{f}}_r) = (a', b')$ then $\mathbb{F}_r(B) = \hat{\delta}^{\tilde{f}}_r(a', b')$. The statement in full generality follows by observing that any box, finite or infinite, can be divided in a finite collection of disjoint boxes $B = \sqcup B_{i,j}, 1 \leq i \leq m, 1 \leq j \leq n$, s.t $B_{i,j}$ and $B_{i+1,j}$ are in the position indicated in Figure 2, $B_{i,j}$ and $B_{i,j+1}$ in the position indicated in Figure 3 and each $B_{i,j}$ contains at most one element of $B \cap supp(\delta^{\tilde{f}})_r$. The result is established by induction on m and n by applying Observation (3.5) item 1. and item 2. and the particular cases mentioned above. ■.

The above proposition can be strengthen in the following way. For each (a,b) consider the surjective map

$$\pi_r(a,b) : \mathbb{F}_r(a,b) \to \hat{\delta}^{\tilde{f}}_r(a,b)$$

and call *splitting* a right invers of $\pi_r(a,b)$,

$$s_r(a,b) : \hat{\delta}^{\tilde{f}}_r(a,b) \to \mathbb{F}_r(a,b).$$

For each box $B = (\alpha', \alpha] \times [\beta, \beta')$ with $a \in (\alpha', \alpha],\ b \in [\beta, \beta')$ denote by

$$i^B_r(a,b) : \hat{\delta}^{\tilde{f}}_r(a,b) \to \mathbb{F}_r(B) \text{ and } \mathrm{i_r(a,b)} : \hat{\delta}^{\tilde{f}}_{\mathrm{r}}(\mathrm{a,b}) \to \mathbb{H}_{\mathrm{r}}(\tilde{\mathrm{X}})$$

[7] The right side of the equality above is independent of ϵ if ϵ is small enough

the compositions

$$\hat{\delta}^f_r(a,b) \xrightarrow{s_r(a,b)} \mathbb{F}_r(a,b) \xrightarrow{\subseteq} \mathbb{F}_r(\alpha,\beta) \xrightarrow{\pi^B_{\alpha\beta,r}} \mathbb{F}_r(B)$$

and

$$\hat{\delta}^f_r(a,b) \xrightarrow{s_r(a,b)} \mathbb{F}_r(a,b) \xrightarrow{\subseteq} H_r(\tilde{X})\ .$$

The following diagram reviews for the reader the linear maps considered so far

$$\begin{array}{ccccc} H_r(\tilde{X}) & \xleftarrow{\supseteq} & \mathbb{F}_r(a,b) & \underset{\pi_r(a,b)}{\longrightarrow} & \hat{\delta}^{\tilde{f}}_r(a,b) \\ & & \downarrow{\scriptstyle \pi^B_{ab,r}} & \swarrow{\scriptstyle i^B_r(a,b)} & \\ \mathbb{F}_r(B_1) & \xrightarrow{i^B_{B',r}} & \mathbb{F}_r(B) & \underset{\pi^{B_2}_{B,r}}{\longrightarrow} & \mathbb{F}_r(B_2). \end{array} \tag{4}$$

with $i_r(a,b): \hat{\delta}^{\tilde{f}}_r(a,b) \to H_r(\tilde{X})$ and $s_r(a,b): \hat{\delta}^{\tilde{f}}_r(a,b) \to \mathbb{F}_r(a,b)$.

Observe that if $B = B_1 \sqcup B_2$ as in Figure 2 or Figure 3, in view of Observation 3.5, one has

Observation 3.7

1. *If $(a,b) \in B_2$ then $\pi^{B_2}_{B,r} \cdot i^B_r(a,b)$ is injective.*
2. *If $(a,b) \in B_1$ then $\pi^{B_2}_{B,r} \cdot i^B_r(a,b)$ is zero.*

Choose splittings $\{s_r(a,b) \mid (a,b) \in supp(\delta^{\tilde{f}}_r)\}$, and consider the sum of $i_r(a,b)'$s for $(a,b) \in supp(\delta^{\tilde{f}}_r)$.

$$I_r = \sum_{(a,b)\in supp(\delta^{\tilde{f}}_r)} i_r(a,b) : \bigoplus_{(a,b)\in supp(\delta^{\tilde{f}}_r)} \hat{\delta}^f_r(a,b) \to H_r(\tilde{X}).$$

and for a finite or infinite box B the sum

$$I^B_r = \sum_{(a,b)\in supp(\delta^{\tilde{f}}_r)\cap B} i^B_r(a,b) : \bigoplus_{(a,b)\in supp(\delta^{\tilde{f}}_r)\cap B} \hat{\delta}^f_r(a,b) \to \mathbb{F}_r(B).$$

For $\Sigma \subseteq supp(\delta^f_r)$ denote by $I_r(K)$ the restriction of I_r to $\bigoplus_{(a,b)\in\Sigma} \hat{\delta}^f_r(a,b)$ and for $\Sigma \subseteq supp(\delta^f_r) \cap B$ denote by $I^B_r(K)$ the restriction of I^B_r to $\bigoplus_{(a,b)\in\Sigma} \hat{\delta}^f_r(a,b)$. Note that

Observation 3.8

For $B = B_1 \sqcup B_2$ as in Figures 2 or Figure 3 and $\Sigma \subseteq supp\ \delta^{\tilde{f}}_r$ with $\Sigma = \Sigma_1 \sqcup \Sigma_2$, $\Sigma_1 \subseteq B_1, \Sigma_2 \subseteq B_2$ the diagram

$$\begin{array}{ccccc} \mathbb{F}_r(B_1) & \longrightarrow & \mathbb{F}_r(B) & \longrightarrow & \mathbb{F}_r(B_2) \\ \uparrow{\scriptstyle I^{B_1}_r(\Sigma_1)} & & \uparrow{\scriptstyle I^B_r(\Sigma)} & & \uparrow{\scriptstyle I^{B_2}_r(\Sigma_2)} \\ \bigoplus_{(a,b)\in\Sigma_1} \hat{\delta}^{\tilde{f}}_r(a,b) & \longrightarrow & \bigoplus_{(a,b)\in\Sigma} \hat{\delta}^{\tilde{f}}_r(a,b) & \longrightarrow & \bigoplus_{(a,b)\in\Sigma_2} \hat{\delta}^{\tilde{f}}_r(a,b) \end{array}$$

is commutative. In particular if $I^{B_1}_r(\Sigma_1)$ and $I^{B_2}_r(\Sigma_2)$ are injective then so is $I^B_r(\Sigma)$.

Proposition 3.9

1. *For any Σ as above the linear maps $I_r(\Sigma)$, and $I_r^B(\Sigma)$ are injective.*

2. *I_r^B is an isomorphism.*

3. *If π_r is the projection $\pi_r : H_r(\tilde{X}) \to H_r(\tilde{X})/(\mathbb{I}_{-\infty}(r) + \mathbb{I}^{\infty}(r)) = H_r(\tilde{X})/TH_r(\tilde{X})$ then $\pi_r \cdot I_r$ is an isomorphism.*

Proof:

Item 1: If cardinality of Σ is one then the statement is verified by Observation 3.7. If all points of Σ have the same first component the statement follows by induction on the cardinality of Σ using Observation 3.8. Precisely one decomposes B as in Figure 2, $B = B_1 \sqcup B_2$ with B_1 containing $k-1$ elements and B_2 containing one element. Here k is the cardinality of Σ. The statement, being true for $\Sigma \cap B_2$ and by induction hypothesis for $\Sigma \cap B_1$, in view of Observation 3.8, holds true for Σ.

In general one decomposes the set Σ as $\Sigma = \Sigma_1 \sqcup \Sigma_2 \cdots \sqcup \Sigma_k$ with the properties that all elements of Σ_i have the same first component, say a_i, with $a_1 < a_2 \cdots < a_k$. One proceeds by induction on k by decomposing B as in Figure 3, $B = B_1 \sqcup B_2$with B_2 containing the set Σ_1 and B_1 the remaining elements. By the previous step the statement is true for Σ_1 and by induction hypothesis for $\Sigma \cup B_1$. In view of Observation 3.8 the statement holds for Σ.

Item 2.: Injectivity is true by item 1.. The surjectivity follows from equality of the dimension of the source and of the target which follows from Proposition 3.6 .

Item 3.: In view of Observation 3.1 item 3. one has:

$$\varinjlim_{k\to\infty} \mathbb{F}_r(a+k, b-k) = H_r(\tilde{X})$$

and

$$\varinjlim_{k\to\infty} \mathbb{F}_r((-\infty, a+k] \times [b-k, \infty)) := H_r(\tilde{X})/(\mathbb{I}_{-\infty} + \mathbb{I}_{\infty}) = H_r(\tilde{X})/TH_r(\tilde{X}).$$

For a fix (a,b) denote by $B_k := (-\infty, a+k] \times [b-k, \infty)$ and regard $\mathbb{R}^2 := \cup_k B_k$. The statement follows by observing that $\pi_r \cdot I_r = \varinjlim_{\{k\to\infty\}} I_r^{B_k}$. ■

The configurations δ_r^f and $\hat{\delta}_r^f$. Let $\omega : \mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\omega(n,(a,b)) = (a + 2\pi k, b + 2\pi k)$ and consider the quotient space $\mathbb{T}$ which can be identified to $\mathbb{C}^* := \mathbb{C} \setminus 0$ by the map

$$\mathbb{T} \ni \langle a, b\rangle \to z = e^{(b-a)+ia} \in \mathbb{C}^* \setminus 0$$

with $\langle a, b\rangle$ denoting the class of (a,b). In view of the equality $\delta_r^{\tilde{f}}(a,b) = \delta_r^{\tilde{f}}(a+2\pi k, b+2\pi k)$ one defines

$$\delta_r^f(\langle a, b\rangle) = \delta_r^f(z) := \delta_r^{\tilde{f}}(a,b). \tag{5}$$

Consider the commutative diagram with the multiplication by t, (= $t\cdot$), inducing the linear isomorphism $\hat{t}_r(a,b)$

$$\begin{array}{ccccc}
H_r(\tilde{X}) & \xleftarrow{\supseteq} & \mathbb{F}_r(a,b) & \xrightarrow{\pi_r(a,b)} & \hat{\delta}_r^{\tilde{f}}(a,b) \\
\downarrow{\scriptstyle t\cdot = t_r} & & \downarrow{\scriptstyle t_r} & & \downarrow{\scriptstyle \hat{t}_r(a,b)} \\
H_r(\tilde{X}) & \longleftarrow & \mathbb{F}_r(a+2\pi, b+2\pi) & \longrightarrow & \hat{\delta}_r^{\tilde{f}}(a+2\pi, b+2\pi)
\end{array}$$

In view of the equality $t_r(\mathbb{F}_r(a,b)) = \mathbb{F}_r(a+2\pi, b+2\pi)$ the subspaces

$$\begin{aligned}(\mathbb{F}_r)'(\langle a,b\rangle) : &= \sum_k \mathbb{F}'_r(a+2\pi k, b+2\pi k) \subseteq H_r(\tilde{X}) \\ \mathbb{F}_r(\langle a,b\rangle) : &= \sum_k \mathbb{F}_r(a+2\pi k, b+2\pi k) \subseteq H_r(\tilde{X})\end{aligned} \tag{6}$$

are $\kappa[t^{-1},t]$-submodules $H_r(\tilde{X})$. Denote by

$$\hat{\delta}^f_r(\langle a,b\rangle) := \bigoplus_k \hat{\delta}^{\tilde{f}}_r(a+2\pi k, b+2\pi k). \tag{7}$$

This vector space equipped with the isomorphism

$$\hat{t}_r = \hat{t}_r(\langle a,b\rangle) := \bigoplus_k (\hat{t}_r(a+2\pi k, b+2\pi k) : \hat{\delta}^f_r(\langle a,b\rangle) \to \hat{\delta}^f_r(\langle a,b\rangle)$$

becomes a free $\kappa[t^{-1},t]$-module of rank $\delta^f_r(\langle a,b\rangle)$.

The collection of splittings $\{s^{a,b}_r : \hat{\delta}(a,b) \to \mathbb{F}_r(a,b)\}$ which satisfy $t_r \cdot s^{a,b}_r = s^{a+2\pi,b+2\pi}_r \cdot \hat{t}_r(a,b)$ is called *compatible splittings*. Clearly such collections exist. Indeed, it suffices to chose *splittings* only for $\{(a,b) \in supp(\delta^{\tilde{f}}_r), 0 \leq a < 2\pi\}$, observe that any $(a',b') \in supp(\delta^{\tilde{f}}_r)$ is of the form $a' = a+2\pi k, b' = b+2\pi k$ for some integer $k \in \mathbb{Z}$ with $0 \leq a < 2\pi$ and take $s_r(a',b') := (\hat{t}_r)^k \cdot s_r(a,b)(\hat{t}_r)^{-k}$. Choose a collection of compatible splittings.

If one denotes by

$$supp(\delta^f_r) = supp(\hat{\delta}^f_r) := \{\langle a,b\rangle \in \mathbb{T} \mid \delta^f_r(\langle a,b\rangle) \neq 0\},$$

equivalently

$$\{z \in \mathbb{C} \setminus 0 \mid z = e^{ia+(b-a)}, \langle a,b\rangle \in supp(\delta^f_r)\},$$

then the κ-linear map I_r, constructed using *compatible splittings*,

$$I_r : \bigoplus_{\langle a,b\rangle \in supp(\delta^f_r)} \hat{\delta}^f_r(\langle a,b\rangle) \to H_r(\tilde{X})$$

is $\kappa[t^{-1},t]$-linear and Proposition 3.9 implies

Corollary 3.10

1. *The composition of $\pi_r \cdot I_r$ with $\pi_r : H_r(\tilde{X}) \to H^N_r(X,\xi)$ the canonical projection is an isomorphism of free $\kappa[t^{-1},t]$–modules.*

2. *The restriction of I_r to $\oplus_{k\in\mathbb{Z}}\hat{\delta}^{\tilde{f}}_r(a+2\pi k, b+2\pi k) = \hat{\delta}^f_r(\langle a,b\rangle)$ denoted by $I_r(\langle a,b\rangle)$ has the image contained in $\mathbb{F}_r(\langle a,b\rangle)$ and*
$$imgI_r(\langle a,b\rangle)) \cap (\mathbb{F}_r)'(\langle a,b\rangle) = 0.$$
This implies that composed with the restriction of π_r to $\mathbb{F}_r(\langle a,b\rangle)$ is injective since $\pi_r \cdot I_r$ is an isomorphism. Moreover $\pi_r \cdot I_r(\langle a,b\rangle)$ is a split injective $\kappa[t^{-1},t]$-linear map whose image identifies to the free module $\mathbb{F}_r(\langle a,b\rangle)/(\mathbb{F}'_r(\langle a,b\rangle)$.

3. *The collection of integers $\delta^f_r(\langle a,b\rangle)$ and the collection of free $\kappa[t^{-1},t]$-modules $\hat{\delta}^f_r(\langle a,b\rangle)$ provide the configuration δ^f_r of points in $\mathbb{T} = \mathbb{C} \setminus 0$ and the configuration $\hat{\delta}^f_r$ of f.g. free $\mathbb{C}[t^{-1},t]-$modules, actually quotients of split submodules of $H^N_r(X,\xi^f;\kappa[t^{-1},t])$, as stated in introduction.*

4 Proof of Theorem 1.1.

Corollary 3.10, Proposition 3.2 together with the formulae (5), (6) and (7) imply items 1. and 2.

Indeed Proposition 3.2 implies $H^N_r(X, \xi^f; \kappa[t^{-1}, t]) = H_r(\tilde{X})/(\mathbb{I}_{-\infty}(r) + \mathbb{I}^{\infty}(r)) = H_r(\tilde{X})/T(H_r(\tilde{X}))$. The configuration δ^f_r is defined by (5) and the polynomial $P^f_r(z)$ by

$$P^f_r(z) = \prod_{z_i \in supp(\delta^f_r)} (z - z_i)^{\delta^f_r(z_i)}.$$

For $z = e^{ia+(b-a)}$ one takes $L(z) = \mathbb{F}_r(\langle a, b\rangle)$ and $L'(z) = \mathbb{F}'_r(\langle a, b\rangle)$. Corollary 3.10 items 2. and 3. imply that the configuration $\hat{\delta}^f_r$ defined by equality (7) satisfies also $\hat{\delta}^f_r(z) = L(z)/L'(z)$ as $\kappa[t^{-1}, t]$-modules and item 1. and the equality (5) imply that the properties (a) and (b) in Theorem 1.1 item 2. are satisfied.

Suppose $\kappa = \mathbb{C}$. By choosing a $\mathbb{C}[t^{-1}, t]$–valued inner product on $H^N(X, \xi^f; \mathbb{C}[t^{-1}, t])$ and by using the von Neumann completion described in subsection 2.1 one obtains from $\hat{\delta}^f_r$ the configuration $\hat{\hat{\delta}}^f_r(r)$ of closed Hilbert submodules of the $\mathcal{N} = L^{\infty}(\mathbb{S}^1)$-module $\overline{H^N_r(X, \xi : \mathbb{C}[t^{-1}, t])}$ as stated in Introduction. This establishes Item 3.

The verifications of **P1** and **P2** will be discussed in [3]. They are first verified for finite simplicial complexes and simplicial map then for Hilbert cube manifolds and tame maps and then concluded for arbitrary continuous maps defined on arbitrary compact ANRs. This is why we have summarized in subsection (2.3) facts about Hilbert cube manifolds.

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