# TORSIONS FOR MANIFOLDS WITH BOUNDARY AND GLUEING FORMULAS

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Abstract. We extend the definition of analytic and Reidemeister torsion for closed compact Riemannian manifolds, to compact Riemannian manifolds with boundary $(M, \partial M)$, given a parallel flat bundle $\mathcal{F}$ of $\mathcal{A}$-Hilbert modules of finite type and a decomposition of the boundary $\partial M = \partial_- M \cup \partial_+ M$ into disjoint components. When $\mathcal{F}$ is induced from the universal covering of $M$, and the fundamental group of $M$ is infinite (cf. [BFKM]), these torsions are known as the $L_2-$ analytic resp. $L_2-$ Reidemeister torsions. If the system $(M, \partial_- M, \partial_+ M, \mathcal{F})$ is of determinant class we compute the quotient of the analytic and the Reidemeister torsion and prove gluing formulas for both of them. In particular we answer positively Conjecture 7.6 in [LL]. If $\mathcal{F}$ is induced from a $\Gamma-$principal covering where $\Gamma$ is a residually finite group, we derive from the work of Lück (cf.[Lü3]), that the system $(M, \partial_- M, \partial_+ M, \mathcal{F})$ is of determinant class (cf. Theorem A in Appendix).

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## 0. Introduction.

The purpose of this paper is to extend the study of the analytic and the Reidemeister torsion to compact Riemannian manifolds with boundary. The torsions we will consider will be associated to compact manifolds equipped with a parallel flat bundle of $\mathcal{A}$- Hilbert modules of finite type, $\mathcal{F}$, where $\mathcal{A}$ is a finite von Neumann algebra. This torsions involve operators whose spectrum, in general, is not exclusively pure point, but might contain a continuous part as well. In particular, the analytic and Reidemeister $L_2-$torsions are of this type. Among other results we provide a formula for the quotient of the analytic torsion and the Reidemeister torsion. In particular we prove Conjecture 7.6 in [LL]. We also establish glueing formulas for both torsions. These formulas are basic tools for the calculation of these torsions.

In order to formulate our results more precisely, we introduce the following notations. Let $M$ be a compact smooth manifold, not necessarily connected, of dimension $d$, and with boundary $\partial M$. Let $g$ be a Riemannian metric on $M$. Throughout the paper, we will always assume that $g$ has a product structure near the boundary (product-like), i.e. that there exist $\epsilon > 0$ and a diffeomorphism $\Theta_{\partial M} : \partial M \times [0, \epsilon) \to M$ so that

$$\Theta_{\partial M} \restriction_{\partial M \times \{0\}} = id \tag{P1}$$

$$\Theta^*_{\partial M}(g) = g_{\partial M} + dt^2 \tag{P2}$$

where $g_{\partial M}$ denotes the restriction of $g$ to $\partial M$ and $dt^2$ denotes the Euclidean metric on the half open interval $[0, \epsilon)$. Suppose $\partial M$ is a union of two disjoint components, not necessarily connected and possibly empty. Let us specify an ordering for them, $\partial_- M$ and $\partial_+ M$. Then $(M, \partial_- M, \partial_+ M)$ will be referred to as a bordism. A closed manifold $M$ can be regarded as a bordism with $\partial_- M = \partial_+ M = \emptyset$. If it is not ambiguous, we will write $M$ instead of $(M, \partial_- M, \partial_+ M)$ and denote by $-M$ the bordism obtained from $M$ by interchanging $\partial_- M$ and $\partial_+ M$. Next, we introduce the notion of generalized triangulation for a bordism $(M, \partial_- M, \partial_+ M)$.

**Definition.** A pair $\tau = (h, g')$ consisting of a $C^\infty$-function $h : M \to \mathbb{R}$ with range $[a, b] \subseteq \mathbb{R}$, where a and b are elements in $\mathbb{Z}$, and a Riemannian metric $g'$ is said to be a generalized triangulation for $M$ if the following properties hold:

**(T1)**
$$h\big(\Theta_{\partial M}(x,t)\big) = b - t \qquad \big(x \in \partial_+ M, 0 \le t < \epsilon\big)$$
$$h\big(\Theta_{\partial M}(x,t)\big) = a + t \qquad \big(x \in \partial_- M, 0 \le t < \epsilon\big)$$

where $\Theta_{\partial M} : \partial M \times [0, \epsilon) \to M$ denotes the exponential map at $\partial M$, associated to $g'$ (cf. above).

**(T2)**
$$h\big(Cr(h)\big) \subseteq \mathbb{Z},$$

where $Cr(h)$ denotes the set of critical points of $h$. (Notice that we do not require that $h$ is self-indexing. This will be convenient for the glueing construction.)

**(T3)** All critical points of $h$ are nondegenerate and, given any critical point $y$ of index $k$, $y \in Cr_k(h)$, there exists a diffeomorphism $\phi_y : D_{\epsilon_y} \to M$, with $D_{\epsilon_y}$ being the disc in $\mathbb{R}^d$ of radius $\epsilon_y > 0$, centered at 0, so that $\phi_y(0) = y$, $\phi_y^{\#}(g')$ is the Euclidean metric on $D_{\epsilon_y}$ and

$$h\big(\phi_y(x_1, \dots, x_d)\big) = h(y) - \frac{1}{2}\sum_{j=1}^{k} x_j^2 + \frac{1}{2}\sum_{j=k+1}^{d} x_j^2.$$

If for each critical point of h, such a coordinate system exists, the metric $g'$ is called compatible with $h$.

**(T4)** The gradient of $h$ with respect to $g'$, $grad_{g'}h$, satisfies the Morse-Smale condition (cf. [BFKM]).

A smooth simplicial triangulation $t$ of $M$ with the property that $\partial_\pm M$ become subcomplexes induces generalized triangulations on the bordisms $(M, \partial_- M, \partial_+ M)$, $(M, \partial_+ M, \partial_- M)$, $(M, \emptyset, \partial M)$, $(M, \partial M, \emptyset)$ and on the closed manifolds $\partial_\pm M$.

Assume that the following data is given: $(M, \partial_- M, \partial_+ M)$ a bordism; $\tau = (h, g')$ a generalized triangulation; $\mathcal{A}$ a finite von Neumann algebra; $\mathcal{F} = (\mathcal{E}, \nabla)$ a parallel flat bundle where $\mathcal{E} \xrightarrow{p} M$ is a bundle of $\mathcal{A}$-Hilbert modules of finite type and $\nabla$ is a flat connection making the $\mathcal{A}$ -module structure and the inner products $(\cdot,\cdot)$ of $\mathcal{E}_y = p^{-1}(y)$ parallel with respect to $\nabla$.[1]

If $\tau = (h, g')$ is a generalized triangulation of $(M, \partial_- M, \partial_+ M)$ then $\tau_D = (-h, g')$ is a generalized triangulation of $(M, \partial_+ M, \partial_- M)$. It will be refered to as the dual triangulation.

In section 2 we define, similar as in [BFKM], analytic Laplacians $\Delta_q$, acting on $q$-forms with coefficients in $\mathcal{E}$ with relative (or Dirichlet) boundary conditions on $\partial_- M$ and absolute (or Neumann) boundary conditions on $\partial_+ M$, associated to $(M, \partial_- M, \partial_+ M)$, g and $\mathcal{E}$. Further, we define combinatorial Laplacians $\Delta_q^{comb}$, associated to $(M, \partial_- M)$, $\tau$ and $\mathcal{F}$. We say that the triple $\{(M, \partial_- M, \partial_+ M), g, \mathcal{F}\}$ is of $a$- determinant class if all the analytic Laplacians $\Delta_q$ are of determinant class, i.e. $\log det'_N \Delta_q \in \mathbb{R}$. For $\{(M, \partial_- M, \partial_+ M), g, \mathcal{F}\}$ of a-determinant class the analytic torsion $T_{an} = T_{an}(M, \partial_- M, g, \mathcal{F}) > 0$ is defined by

$$\log T_{an} := \frac{1}{2}\sum_{q=0}^{d} (-1)^{q+1} q \log det'_N \Delta_q.$$

Similarly, we say that the triple $\{(M, \partial_- M, \partial_+ M), \tau, \mathcal{F}\}$ is of $c$-determinant class if all the combinatorial Laplacians $\Delta_q^{comb}$ are of determinant class,
i.e. $\log det'_N \Delta_q^{comb} \in \mathbb{R}$. For $\{(M, \partial_- M, \partial_+ M), \tau, \mathcal{F}\}$ of c-determinant class the combinatorial torsion $T_{comb} = T_{comb}(M, \partial_- M, \tau, \mathcal{F}) > 0$ is defined by

$$\log T_{comb} := \frac{1}{2}\sum_{q=0}^{d} (-1)^{q+1} q \log det'_N \Delta_q^{comb}.$$

[1] Given a choice of base points $(x_j)$ in each of the connected components $M_j$ of $M (1 \le j \le k_M)$ $\mathcal{F}$ determines and is determined uniquely by the $(\mathcal{A}, \Gamma_j^{op})-$Hilbert modules $\mathcal{W}_j := \mathcal{E}_{x_j}$ where the action of $\Gamma_j := \pi_1(M_j, x_j)$ on $\mathcal{W}_j$ is given by the parallel transport and where $\Gamma_j^{op}$ denotes the group $\Gamma_j$ with opposite multiplication (cf. [BFKM]).

The Reidemeister torsion $T_{Re} = T_{Re}(M, \partial_- M, g, \tau, \mathcal{F})$ is then defined by

$$\log T_{Re} = \log T_{comb} + \log T_{met}$$

where $T_{met}$ is the metric part of the torsion, defined as in [BFKM] (cf. section 2). $T_{met}$ is equal to 1 if $(M, \partial_- M)$ is $\mathcal{F}$- acyclic. It turns out that $T_{Re}$ does not depend on the triangulation $\tau$. The proof of this fact is not difficult and is the same as in the case of finite dimensional unitary representations. This result is also contained in Theorem 3.1. Nevertheless, we will continue to write $T_{Re}(M, \partial_- M, g, \tau, \mathcal{F})$ in order to emphasize that the Reidemeister torsion is calculated with the triangulation $\tau$.

One can show that the notions of $a$-determinant class and $c$-determinant class are equivalent (cf. [BFKM]) which allows us to introduce the notion of a pair $\{(M, \partial_- M, \partial_+ M), \mathcal{F}\}$ being of determinant class. Given a bordism $(M, \partial_- M, \partial_+ M)$ we say that the system $(M, \partial_- M, \partial_+ M, \mathcal{F})$ is of determinant class if the three pairs $\{(M, \partial_- M, \partial_+ M), \mathcal{F}\}$, $\{(M, \emptyset, \partial M), \mathcal{F}\}$ and $\{\partial M, \mathcal{F} \restriction_{\partial M}\}$ are of determinant class. As we will see in Proposition 2.5 (4) this is equivalent with $\{(M, \emptyset, \partial M), \mathcal{F}\}$ and $\{\partial M, \mathcal{F} \restriction_{\partial M}\}$ being of determinant class. If $\mathcal{F}$ is a parallel flat bundle over a compact manifold $M$ induced from a $\Gamma-$principal covering where $\Gamma$ is a residually finite group, one can derive from the work of Lück (cf.[Lü3]), that for any bordism $(M, \partial_- M, \partial_+ M)$ the system $(M, \partial_- M, \partial_+ M, \mathcal{F})$ is of determinant class (cf. Theorem A in Appendix). In particular, one concludes that if $M$ is connected and has residually finite fundamental group $\Gamma := \pi_1(M)$, the system $(M, \partial_- M, \partial_+ M, \mathcal{F})$ is of determinant class if $\mathcal{F}$ is the Hilbert bundle induced by the $(\mathcal{N}(\Gamma), \Gamma^{op})-$Hilbert module $\ell_2(\Gamma)$ (cf Appendix).

Given a pair $\{(M, \partial_- M, \partial_+ M), \mathcal{F}\}$ of determinant class, denote by $\mathcal{R} = \mathcal{R}(M, \partial_- M, g, \tau, \mathcal{F})$ the quotient of analytic and Reidemeister torsion,

$$\log \mathcal{R} = \log T_{an}(M, \partial_- M, g, \mathcal{F}) - \log T_{Re}(M, \partial_- M, g, \tau, \mathcal{F}).$$

In a forthcoming paper [BFK2], we will show that the quantity $\mathcal{R}$ is well defined without the hypothesis of the pair being of determinant class, a fact which was proved first in [CMM] in the acyclic case[2], and that Theorems 2.10 and 3.1 below hold without this hypothesis.

Extending our earlier results [BFK], [BFKM] and using similar techniques, we prove that $\log \mathcal{R}$ depends only on the data on the boundary:

**Theorem 2.10.** *Assume that for $j = 1, 2, (M_j, \partial_- M_j, \partial_+ M_j)$ is a bordism equipped with a Riemannian metric $g_j$, a generalized triangulation $\tau_j$ and that $\mathcal{F}_j = (\mathcal{E}_j, \nabla_j)$ is a parallel flat bundle of $\mathcal{A}$-Hilbert modules of finite type on $M_j$. Assume also that the systems $(M_j, \partial_- M_j, \partial_+ M_j, \mathcal{F}_j)$ are of determinant class and*

$$(\partial_\pm M_1, g_1 \restriction_{\partial_\pm M_1}, \mathcal{F}_1 \restriction_{\partial_\pm M_1}) = (\partial_\pm M_2, g_2 \restriction_{\partial_\pm M_2}, \mathcal{F}_2 \restriction_{\partial_\pm M_2}).$$

*Then*

$$\mathcal{R}(M_1, \partial_- M_1, g_1, \tau_1, \mathcal{F}_1) = \mathcal{R}(M_2, \partial_- M_2, g_2, \tau_2, \mathcal{F}_2).$$

This theorem allows us to extend the comparison theorem (cf. [BFKM] and [BFK]) of analytic and Reidemeister torsions from closed manifolds to bordisms.

[2]The first author thanks A.Mischenko for providing him with a preliminary version of [CMM].

**Theorem 3.1.** *Assume that the system $(M, \partial_- M, \partial_+ M, \mathcal{F})$ is of determinant class. Then $\log \mathcal{R}$ is given by*

$$\log \mathcal{R}(M, \partial_- M, g, \tau, \mathcal{F}) = \frac{1}{4}\chi(\partial M; \mathcal{F}) \log 2$$

*where $\chi(\partial M; \mathcal{F})$ is the Euler characteristic of $\partial M$ with coefficients in $\mathcal{F}$ and is equal to $\chi(\partial M; \mathcal{F}) = \sum_k \chi(\partial_k M) \cdot dim_N \mathcal{E} \restriction_{\partial_k M}$ where $\partial_k M$ are the connected components of the boundary $\partial M$, $\chi(\partial_k M)$ is the standard Euler characteristic of $\partial_k M$ and $dim_N \mathcal{E} \restriction_{\partial_k M}$ is the von Neumann dimension of the fiber of $\mathcal{E}$ above $\partial_k M$.*

*Remark.* In the case $\mathcal{A}$ is $\mathbb{R}$ or $\mathbb{C}$, Theorem 3.1 is due to W. Lück [Lü1] and, independently, Vishik [Vi] (cf. also [Ch]).

The above result shows that $\mathcal{R}$ does not depend on the partition $\partial M_-, \partial M_+$ of the boundary $\partial M$. However we continue to use the notation $\mathcal{R}(M, \partial_- M, g, \tau, \mathcal{F})$ as the independence of $\partial_- M$ will be proven only towards the end of the paper.

Next we present a gluing formula for the analytic torsion. For $j = 1, 2$, let $M_j = (M_j, \partial_- M_j, \partial_+ M_j)$, $g_j, \tau_j = (h_j, g'_j)$ and $\mathcal{F}_j = (\mathcal{E}_j, \nabla_j)$ be as above and suppose that there exist an isometry

$$\omega : (\partial_+ M_1, g_1 \restriction_{\partial_+ M_1}) \to (\partial_- M_2, g_2 \restriction_{\partial_- M_2})$$

and a connection preserving bundle isometry $\Phi$ above $\omega$ which makes the following diagram commutative

$$\begin{array}{ccc} \mathcal{E}_1 \restriction_{\partial_+ M_1} & \xrightarrow{\Phi} & \mathcal{E}_2 \restriction_{\partial_- M_2} \\ \downarrow & & \downarrow \\ \partial_+ M_1 & \xrightarrow{\omega} & \partial_- M_2. \end{array}$$

Then one can form the bordism $(M = M_1 \cup_\omega M_2, \partial_- M_1, \partial_+ M_2)$ by glueing $\partial_+ M_1$ to $\partial_- M_2$ by $\omega$ and the parallel flat bundle $\mathcal{F}$ by glueing $\mathcal{F}_1$ and $\mathcal{F}_2$ by $(\omega, \Phi)$. The metrics $g_1, g'_1$ and $g_2, g'_2$ determine Riemannian metrics $g$ and $g'$ on $M$, and the functions $h_1$ and $h_2$ determine the $C^\infty$-function $h : M \to \mathbb{R}$ given by

$$h(x) := \begin{cases} h_1(x) & (x \in M_1) \\ b_1 - a_2 + h_2(x) & (x \in M_2) \end{cases}$$

where, for $j = 1, 2$, $h_j(M_j) = [a_j, b_j]$.

**Definition.** Generalized triangulations $\tau_1$ and $\tau_2$ as above are said to be compatible if $\tau = (h, g')$ is a generalized triangulation of $(M, \partial_- M_1, \partial_+ M_2)$.

Notice that by an arbitrary small perturbation, localized in a given neighborhood of $\partial_+ M_1$ one can modify the metric $g'_1$ to $\tilde{g}'_1$ so that the triangulation $\tilde{\tau}_1 := (h_1, \tilde{g}'_1)$ is compatible with $\tau_2$.

Finally the manifolds $M, M_1, M_2$ and the metric g induce a cohomology sequence $\mathcal{H}_{an}(g)$, which is a long weakly exact sequence of $\mathcal{A}-$Hilbert modules, and hence a cochain complex. Different metrics induce isomorphic, but not necessarily isometric complexes. If $\mathcal{H}_{an}(g)$ is of determinant class we denote by $T(\mathcal{H}_{an}(g))$ its torsion.

Similarly, the manifolds $M, M_1, M_2$ and the generalized triangulation $\tau$ induce a cohomology sequence $\mathcal{H}_{comb}(\tau)$, which is a long weakly exact sequence of $\mathcal{A}-$Hilbert modules and thus, again, a cochain complex. Integration theory (cf section 2) provides an isomorphism (which typically is not an isometry) between the two sequences $\mathcal{H}_{an}(g)$ and $\mathcal{H}_{comb}(\tau)$. Both sequences are of the following form:

$$
\begin{aligned}
0 &\to H^0(M_2, \partial_- M_2, \mathcal{F}_2) \to H^0(M, \partial_- M, \mathcal{F}) \to H^0(M_1, \partial_- M_1, \mathcal{F}_1) \to \\
&\to H^1(M_2, \partial_- M_2, \mathcal{F}_2) \to \cdots \to H^k(M_2, \partial_- M_2, \mathcal{F}_2) \to H^k(M, \partial_- M, \mathcal{F}) \to \\
&\to H^k(M_1, \partial_- M_1, \mathcal{F}_1) \to H^{k+1}(M_2, \partial_- M_2, \mathcal{F}_2) \to \dots
\end{aligned}
$$

**Theorem 3.2.** *Assume that for $i = 1, 2$ the system $(M_i, \partial_- M_i, \partial_+ M_i, \mathcal{F}_i)$ is of determinant class. Then the following statements hold:*
*(i) The system $(M, \partial_- M, \partial_+ M, \mathcal{F})$ and the complexes $\mathcal{H}_{comb}, \mathcal{H}_{an}$ are of determinant class;*

$$
(ii) \qquad \log T_{Re}(M, \partial_- M, g, \tau, \mathcal{F}) = \sum_{j=1}^{2} \log T_{Re}(M_j, \partial_- M_j, g_j, \tau_j, \mathcal{F}_j) + \log T(\mathcal{H}_{an}).
$$

$$
(iii) \qquad \log T_{an}(M, \partial_- M, g, \mathcal{F}) = \sum_{j=1}^{2} \log T_{an}(M_j, \partial_- M_j, g_j, \mathcal{F}_j) + \log T(\mathcal{H}_{an}) - \frac{\chi(\partial_+ M_1; \mathcal{F}_1)}{2} \log 2.
$$

*Remark.* In the case where $\mathcal{A} = \mathbb{R}$ this result is due to Vishik [Vi3]. Vishik's proof is, however, very different from ours.

A sightly more general form of this result is contained in Theorem 3.2′ where only some components of $\partial_+ M_1$ and $\partial_- M_2$ are glued together. The last result is Theorem 3.3 which compares the analytic torsions of bordisms with the same underlying compact manifold $M$.

The paper is organized as follows:
In section 1 we prove a number of auxiliary results about cochain complexes of determinant class and a generalization of Milnor's result concerning the torsions of a short exact sequence of finite dimensional complexes to infinite dimension (Proposition 1.13, Theorem 1.14). This section is quite elementary but it is included for the sake of completeness. In section 2 we prove Theorem 2.9 on which all subsequent results depend on. Theorems 3.1, 3.2, 3.2', 3.3 are proven in section 3. In the appendix, we prove that for any bordism $(M, \partial_- M, \partial_+ M)$ with $M$ a compact connected manifold, $\tilde{M} \to M$ a principal covering with residually finite group and $\mathcal{F}$ the parallel flat bundle on $M$, induced by this covering, the system $(M, \partial_- M, \partial_+ M, \mathcal{F})$ is of determinant class. This result is implicit in Lück [Lü3]. Unfortunately, in [Lü3] there are a number of misleading misprints and the definition of the $L^2$-determinant

is incorrect. For the convenience of the reader we present an outline of the proof of this result.

By the same arguments as given in [BFKM, Proposition 5.11] it suffices to prove the above Theorems in the case where the fiber $\mathcal{W}$ of the bundle $\mathcal{E} \to M$ is a free $\mathcal{A}$-module. For the rest of the paper the bundle $\mathcal{E}$ has as a fiber a free $\mathcal{A}$-module.

The results presented in this paper can be extended to the case where $\mathcal{F}$ is a flat bundle with a Hermitian structure which is not necessarily parallel with respect to the connection; in this case the relative torsion $\mathcal{R}$ contains an additional "local quantity" which has been obtained by [BZ] in the case $\partial M = \emptyset$ and $\mathcal{A} = \mathbb{C}$.

Apart from very minor modifications, this paper coincides with an IHES preprint from January 1996.

**Section 1 Linear homological algebra in the von Neumann sense.**

In this section we follow, if not stated otherwise, the notations and use the definitions of Section 1 in [BFKM]. In particular, contrary to the standard notations for cochain complexes, the indice for the degree is denoted as a subscript and not as a superscript.

Throughout this section, let $\mathcal{A}$ be a von Neumann algebra of finite type. By $\mathcal{W}, \mathcal{W}_1, \mathcal{W}_2, \mathcal{U}, \dots$ we denote arbitrary $\mathcal{A}$-Hilbert modules of finite type.

Recall that $f : \mathcal{W}_1 \to \mathcal{W}_2$ is said to be a weak isomorphism if $f$ is 1-1 and its range, $Im(f)$, is dense in $\mathcal{W}_2$. It admits a polar decomposition $f = f_{iso} f_{wiso}$ where $f_{iso} := f(f^*f)^{-1/2} : \mathcal{W}_1 \to \mathcal{W}_2$ is an isometry and $f_{wiso} := (f^*f)^{1/2} : \mathcal{W}_1 \to \mathcal{W}_1$ is a weak isomorphism which is selfadjoint and positive. Therefore $f^2_{wiso} + \epsilon (\epsilon > 0)$ is an isomorphism and one can define $\mathrm{Vol}(f)$ by

$$(1.1) \qquad \log \mathrm{Vol}(f) := \lim_{\epsilon \to 0} \frac{1}{2} \log \det (f^2_{wiso} + \epsilon)$$

where the determinant is the one given by Fuglede-Kadison [FK] and where log denotes the branch of the logarithm with $\log 1 = 0$. Notice that $\frac{1}{2} \log \det (f^2_{wiso}) := \log \mathrm{Vol}(f)$ is in $\mathbb{R} \cup \{-\infty\}$ and

$$(1.2) \qquad \log \mathrm{Vol}(f) = \lim_{\epsilon \to 0} \frac{1}{2} \log \det (f^* f + \epsilon).$$

The weak isomorphism $f$ is said to be of <u>determinant class</u> iff

$$(1.3) \qquad -\infty < \log \mathrm{Vol}(f).$$

Using the spectral distribution function of $f$,

$$F_f(\lambda) := \sup\{ \dim_N \mathcal{L} \,;\, \mathcal{L}\, \text{subspace of}\, \mathcal{W}_1, \|f(x))\| \le \sqrt{\lambda} \|x\| (x \in \mathcal{L})\},$$

one can show that

$$(1.2') \qquad \log \mathrm{Vol}(f) = \int_{0+}^{\infty} \log(\lambda) dF_f(\lambda)$$

and that (1.3) is equivalent to

$$(1.3') \qquad -\infty < \int_{0+}^{1} \log(\lambda) dF_f(\lambda).$$

In case (1.3′) holds, one verifies that

$$\int_0^1 \frac{F_f(\lambda) - F_f(0)}{\lambda} d\lambda = - \int_{0+}^1 \log(\lambda) dF_f(\lambda),$$

using that the boundary term obtained, when integrating by parts, vanishes. It will be convenient to extend the concept of determinant class to an arbitrary morphism $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_2)$. For this purpose we factor $f$ as $f = j \cdot f' \cdot p$ with $p : \mathcal{W}_1 \to \mathcal{W}'_1 = \mathcal{W}_1 / Null f$, the canonical projection and $j : \mathcal{W}'_2 := \overline{Range f} \to \mathcal{W}_2$ the inclusion. Then $f' : \mathcal{W}'_1 \to \mathcal{W}'_2$ is a weak isomorphism. The morphism $f$ is said to be of determinant class if $f'$ is of determinant class.

**Proposition 1.1.**
*(A) Assume that $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_2)$ and $g \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_2, \mathcal{W}_3)$ are both weak isomorphisms. Then $g \cdot f$ is a weak isomorphism. Moreover*

$$\log Vol(g \cdot f) = \log Vol(g) + \log Vol(f). \tag{1.4}$$

*In particular $g \cdot f$ is of determinant class iff both $g$ and $f$ are of determinant class.*
*(B) Assume that $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_1')$ and $g \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_2, \mathcal{W}_2')$ are both weak isomorphisms and $h \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_2, \mathcal{W}_2')$. Then $f' = \begin{pmatrix} f & h \\ 0 & g \end{pmatrix} : \mathcal{W}_1 \oplus \mathcal{W}_2 \to \mathcal{W}_1' \oplus \mathcal{W}_2'$ is a weak isomorphism and*

$$\log Vol(f') = \log Vol(f) + \log Vol(g). \tag{1.5}$$

*(C) If $f_i : \mathcal{W} \to \mathcal{V}_i, i = 1, 2$ are two morphisms and both are of determinant class then $f_1 \oplus f_2 : \mathcal{W} \to \mathcal{V}_1 \oplus \mathcal{V}_2$ is of determinant class.*

*Remark.* : (a) In the case where $\mathcal{W}_1 = \mathcal{W}_2$, statement (A) is verified in [FK].
(b) In the case $f, g$ are isomorphisms and not only weak isomorphisms, statement (B) is proved in [BFKM, Proposition 1.9].

*Proof.* (A) Consider the polar decomposition $f = f_{iso} f_{wiso}$ and $g = g_{iso} g_{wiso}$. Then $gf$ can be written as $gf = \widetilde{g}\widetilde{f}$ where

$$\widetilde{g} := g_{iso} f_{iso} : \mathcal{W}_1 \to \mathcal{W}_3; \ \widetilde{f} := f_{iso}^{-1} g_{wiso} f : \mathcal{W}_1 \to \mathcal{W}_1. \tag{1.6}$$

Note that $\widetilde{g}$ is an isometry. Therefore $\widetilde{g}^* \widetilde{g} = Id_{\mathcal{W}_1}$ and $(\widetilde{g}\widetilde{f})^*(\widetilde{g}\widetilde{f}) = \widetilde{f}^* \widetilde{f}$. From definition (1.1) it then follows that

$$\log \mathrm{Vol}(gf) = \log \mathrm{Vol}(\widetilde{f}). \tag{1.7}$$

The map $\widetilde{f}$ can be decomposed, $\widetilde{f} = (f_{iso}^{-1} g_{wiso} f_{iso}) f_{wiso}$ and we conclude from (1.4), which is applied in the case $\mathcal{W}_1 = \mathcal{W}_2$, that

$$\log \mathrm{Vol}(\widetilde{f}) = \log \mathrm{Vol}(f_{iso}^{-1} g_{wiso} f_{iso}) + \log \mathrm{Vol}(f_{wiso}). \tag{1.8}$$

Taking into account (1.2) one concludes that

$$\log \mathrm{Vol}(f_{iso}^{-1} g_{wiso} f_{iso}) = \log \mathrm{Vol}(g_{wiso}). \tag{1.9}$$

From definition (1.1) we know that $\log \mathrm{Vol}(f) = \log \mathrm{Vol}(f_{wiso})$ and similarly for $g$. Combining (1.7) - (1.9) leads to

$$\log \mathrm{Vol}(gf) = \log \mathrm{Vol}(g) + \log \mathrm{Vol}(f).$$

(B) To verify (B) decompose $f'$ as follows

$$f' = \begin{pmatrix} Id_{\mathcal{W}_i'} & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} Id_{\mathcal{W}_i'} & h \\ 0 & Id_{\mathcal{W}_2} \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & Id_{\mathcal{W}_2} \end{pmatrix} \tag{1.10}$$

and apply (A) together with Proposition 1.9 in [BFKM].

(C) Denote by $F_1(\lambda) := F_{f_1}(\lambda), F_2(\lambda) := F_{f_2}(\lambda)$ and $F(\lambda) := F_{f_1 \oplus f_2}(\lambda)$. It is straightforward to verify that $F(\lambda) - F(0) \leq \sup\{F_1(\lambda) - F_1(0), F_2(\lambda) - F_2(0)\}$. Integration by parts in (1.3′) implies the result. □

Consider a finite $\mathcal{A}$-cochain complex $(\mathcal{C}, d)$ of $\mathcal{A}$-Hilbert modules of finite type

$$0 \to \mathcal{C}_0 \overset{d_0}{\to} \mathcal{C}_1 \overset{d_1}{\to} \cdots \to 0. \tag{1.11}$$

Define the Hodge decomposition $\mathcal{C}_i, \mathcal{C}_i = \mathcal{H}_i \oplus \mathcal{C}_i^+ \oplus \mathcal{C}_i^-$, where

$$\mathcal{C}_i^+ := \overline{d_{i-1}(\mathcal{C}_{i-1})}\,; \quad \mathcal{C}_i^- := \overline{d_i^*(\mathcal{C}_{i+1})}\,; \quad \mathcal{H}_i := Null(d_i) \cap Null(d_{i-1}^*). \tag{1.12}$$

Recall that $\mathcal{H}_i$ is isomorphic to $Null d_i / \overline{Range(d_{i-1})} = H^i(\mathcal{C})$. With respect to this decomposition

$$d_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \underline{d}_i \\ 0 & 0 & 0 \end{pmatrix}; \; \underline{d}_i : \mathcal{C}_i^- \to \mathcal{C}_{i+1}^+. \tag{1.13}$$

Note that $\underline{d}_i$ is a weak isomorphism.

**Definition.** The $\mathcal{A}$-cochain complex $\mathcal{C}$ is said to be of determinant class if, for any $i$, $d_i$ (or, equivalently, $\underline{d}_i$) is of determinant class.

If $\mathcal{C}$ is of determinant class, its torsion $T(\mathcal{C})$ is a positive real number given by

$$\log T(\mathcal{C}) = \sum_j (-1)^j \log \mathrm{Vol}(\underline{d}_j). \tag{1.14}$$

For $k = 1, 2$, let $\mathcal{C}^k$ be $\mathcal{A}_k$-cochain complexes. Denote by $\chi(\mathcal{C}^k)$ the Euler-Poincaré characteristic of $\mathcal{C}^k$. Introduce the $\mathcal{A}_1 \hat{\otimes} \mathcal{A}_2$-cochain complex $\mathcal{C}$ given by $\mathcal{C}_j = \oplus_k (\mathcal{C}_k^1 \hat{\otimes} \mathcal{C}_{j-k}^2)$. Here the tensor product is taken in the category of Hilbert spaces.

In [BFKM, Corollary 1.22] we have verified the following

**Proposition 1.2.** *Suppose $\mathcal{C}^1$ and $\mathcal{C}^2$ are both of determinant class. Then $\mathcal{C}$ is of determinant class and*

$$\log T(\mathcal{C}) = \chi(\mathcal{C}^2)\log T(\mathcal{C}^1) + \chi(\mathcal{C}^1)\log T(\mathcal{C}^2). \tag{1.15}$$

Suppose $f : \mathcal{C}^1 \to \mathcal{C}^2$ is a morphism of chain complexes. By straightforward inspection (cf. [BFKM, section 1]) one sees that, with respect to the Hodge decompositions of $\mathcal{C}_i^1 = \mathcal{H}_i^1 \oplus \mathcal{C}_i^{1,+} \oplus \mathcal{C}_i^{1,-}$ and $\mathcal{C}_i^2 = \mathcal{H}_i^2 \oplus \mathcal{C}_i^{2,+} \oplus \mathcal{C}_i^{2,-}$, the morphisms $f_i : \mathcal{C}_i^1 \to \mathcal{C}_i^2$ are of the form

$$f_i = \begin{pmatrix} f_{i,11} & 0 & f_{i,13} \\ f_{i,21} & f_{i,22} & f_{i,23} \\ 0 & 0 & f_{i,33} \end{pmatrix}. \tag{1.16}$$

Moreover

$$\underline{d}_i^2 f_{i,33} = f_{i+1,22}\underline{d}_i^1. \tag{1.17}$$

Further it is straightforward to conclude from Proposition 1.1 that for any $i \geq 0$ the following statements hold:

(1.18) $f_i$ is a [weak] isomorphism iff $f_{i,11}, f_{i,22}$ and $f_{i,33}$ are all [weak] isomorphisms.

$$\log \mathrm{Vol} f_i = \log \mathrm{Vol} f_{i,11} + \log \mathrm{Vol} f_{i,22} + \log \mathrm{Vol} f_{i,33}. \tag{1.19}$$

(1.20) Assume that $f_i$ is a weak isomorphism. Then $f_i$ is of determinant class iff $f_{i,11}, f_{i,22}, f_{i,33}$ are all of determinant class.

In the sequel, we will occasionally write $H(f_i)$ for $f_{i,11} : \mathcal{H}_i^1 \to \mathcal{H}_i^2$.

**Proposition 1.3.** *Let $f : \mathcal{C}^1 \to \mathcal{C}^2$ be a morphism of cochain complexes.*

*(A) If, for any $i$, $f_i$ is a weak isomorphism and of determinant class, then the following statements hold:*
*(i) For any $i$, $H(f_i)$ is of determinant class;*
*(ii) $\mathcal{C}^1$ is of determinant class iff $\mathcal{C}^2$ is;*

*(iii) if $\mathcal{C}^1$ is of determinant class (and thus, by (ii), $\mathcal{C}^2$ is of determinant class as well), then*

$$\log T(\mathcal{C}^2) = \log T(\mathcal{C}^1) - \sum_i (-1)^i \log \mathit{Vol}(f_i) + \sum_i (-1)^i \log \mathit{Vol}\big(H(f_i)\big). \tag{1.21}$$

*(B) If $f$ is a homotopy equivalence (cf. [BFKM], definition 1.14 ) then $\mathcal{C}^1$ is of determinant class iff $\mathcal{C}^2$ is.*

*Proof.* (A) (i) is obtained from (1.20) and (ii) follows from (1.20), (1.17) and Proposition 1.1. To check (iii), apply Proposition 1.1 to $\underline{d}_i^2 f_{i,33} = f_{i+1,22}\underline{d}_i^1$ to conclude that

$$(1.22)\ (-1)^i \log \mathrm{Vol}\, \underline{d}_2^i = (-1)^i \log \mathrm{Vol} \underline{d}_1^i + (-1)^{i+1} \log \mathrm{Vol} f_{i,33} + (-1)^i \log \mathrm{Vol} f_{i+1,22}$$

which, after summing up and using (1.14), leads to

$$\begin{aligned} \log T(\mathcal{C}^2) - \log T(\mathcal{C}^1) &= -\sum_i (-1)^i \log \mathrm{Vol}(f_{i,22}) - \sum_i (-1)^i \log \mathrm{Vol}(f_{i,33}) \\ &= -\sum_i (-1)^i \log \mathrm{Vol}(f_i) + \sum (-1)^i \log \mathrm{Vol} H(f_i), \end{aligned} \tag{1.23}$$

where for the last equality, we used (1.19).

(B) This result is due to Gromov-Shubin [GS] ( cf. also Proposition 1.18 in [BFKM]).

□

The following, somewhat tedious considerations (up to the end of (1.69)) will be used for the proof of Lemma 1.11, Lemma 1.12, Proposition 1.13 and Theorem 1.14.

Let $\mathcal{C}$ be a cochain complex of the form

$$0 \to \mathcal{C}_0 \overset{d_0}{\to} \mathcal{C}_1 \overset{d_1}{\to} \mathcal{C}_2 \to 0. \tag{1.24}$$

Such a complex is called a three stage complex.

**Definition.** A three stage complex $\mathcal{C}$ is said to be a weakly exact sequence if $d_0$ is injective, $\overline{\text{Range }(d_0)} = Null(d_1)$ and $\overline{\text{Range }(d_1)} = \mathcal{C}^2$.

Note that in this situation $\underline{d}_0 : \mathcal{C}_0 \to \overline{\text{Range }(d_0)}$ and $\underline{d}_1 : Range(d_0)^{\perp} \to \mathcal{C}_2$ are weak isomorphisms where $\overline{\text{Range }(d_0)}^{\perp}$ denotes the orthogonal complement of $Range(d_0)$ in $\mathcal{C}_1$.

In the case where $\mathcal{C}$ is of determinant class, its torsion is given by

$$\log T(\mathcal{C}) = \log \mathrm{Vol}(\underline{d}_0) - \log \mathrm{Vol}(\underline{d}_1). \tag{1.25}$$

**Lemma 1.4.** *Suppose that the three stage cochain complex $\mathcal{C}$ is a weakly exact sequence of the form*

$$0 \to \mathcal{C}_0 \overset{d_0}{\to} \mathcal{C}_1^+ \oplus \mathcal{C}_1^- \overset{d_1}{\to} \mathcal{C}_2 \to 0 \tag{1.26}$$

*where $d_0 = \binom{f}{f_1}$ and $d_1 = (g_1\, g)$. Further assume that $d_1$ is onto. Then the following statements hold:*

*(A) If $f : \mathcal{C}_0 \to \mathcal{C}_1^+$ and $g : \mathcal{C}_1^- \to \mathcal{C}_2$ are weak isomorphisms of determinant class, then the complex is of determinant class and*

$$\log T(\mathcal{C}) = \log\, Vol(f) - \log\, Vol(g). \tag{1.27}$$

*(B) Assume that (1.26) is of determinant class. If $f : \mathcal{C}_0 \to \mathcal{C}_1^+$ [resp. $g : \mathcal{C}_1^- \to \mathcal{C}_2$] is of determinant class, so is $g : \mathcal{C}_1^- \to \mathcal{C}_2$ [resp. $f : \mathcal{C}_0 \to \mathcal{C}_1^+$] and formula (1.27) holds.*

*Proof.* First note that, due to the assumption that $d_1$ is onto, $\underline{d}_1 : \left(Range(d_0)\right)^{\perp} \to \mathcal{C}_2$ is an isomorphism and therefore of determinant class. Moreover, $\log \mathrm{Vol}(\underline{d}_0) \in \mathbb{R} \cup \{-\infty\}$. Therefore, we can define $\log T(\mathcal{C}) = \log \mathrm{Vol}(\underline{d}_0) - \log \mathrm{Vol}(\underline{d}_1) \in \mathbb{R} \cup \{-\infty\}$. Denote the inverse $(\underline{d}_1)^{-1} : \mathcal{C}_2 \to Range(d_0)^{\perp} \subseteq \mathcal{C}_1^+ \oplus \mathcal{C}_1^-$ by $(\underline{d}_1)^{-1} = \binom{h_1}{h}$ and conclude that

$$Id_{\mathcal{C}_2} = (g_1\ g)\begin{pmatrix} h_1 \\ h \end{pmatrix} = g_1 h_1 + g h. \tag{1.28}$$

Consider the complex $\mathcal{C}'$ given by

$$0 \to \mathcal{C}_0 \overset{\binom{Id_{\mathcal{C}_0}}{0}}{\longrightarrow} \mathcal{C}_0 \oplus \mathcal{C}_2 \overset{(0\ Id_{\mathcal{C}_2})}{\longrightarrow} \mathcal{C}_2 \to 0$$

which is of determinant class and satisfies

$$\log T(\mathcal{C}') = 0. \tag{1.29}$$

Further introduce the morphism $f : \mathcal{C} \to \mathcal{C}'$ given by the following diagram

$$
\begin{array}{ccccccccc}
0 & \to & \mathcal{C}_0 & \to & \mathcal{C}_0 \oplus \mathcal{C}_2 & \to & \mathcal{C}_2 & \to & 0 \\
 & & \downarrow Id_{\mathcal{C}_0} & & \downarrow \begin{pmatrix} f & h_1 \\ f_1 & h \end{pmatrix} & & \downarrow Id_{\mathcal{C}_2} & & \\
0 & \to & \mathcal{C}_0 & \overset{\binom{f}{f_1}}{\to} & \mathcal{C}_1^+ \oplus \mathcal{C}_1^- & \overset{(g_1, g)}{\to} & \mathcal{C}_2 & \to & 0.
\end{array}
$$

Using the same arguments as in the proof of Proposition 1.3 one shows that, in view of (1.29),

$$
\log T(\mathcal{C}) = 0 - \log \mathrm{Vol} \begin{pmatrix} f & h_1 \\ f_1 & h \end{pmatrix} + 0. \tag{1.30}
$$

In particular, one concludes that $\mathcal{C}$ is of determinant class iff $\begin{pmatrix} f & h_1 \\ f_1 & h \end{pmatrix}$ is of determinant class.

To compute $\log \mathrm{Vol} \begin{pmatrix} f & h_1 \\ f_1 & h \end{pmatrix}$, note that the weak exactness of $\mathcal{C}$ implies that $g_1 f + g f_1 = 0$. Therefore

$$
\begin{aligned}
\begin{pmatrix} Id_{\mathcal{C}_1^+} & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} f & h_1 \\ f_1 & h \end{pmatrix} &= \begin{pmatrix} f & h_1 \\ g f_1 & gh \end{pmatrix} = \begin{pmatrix} f & h_1 \\ -g_1 f & gh \end{pmatrix} \\
&= \begin{pmatrix} Id_{\mathcal{C}_1^+} & h_1 \\ -g_1 & gh \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & Id_{\mathcal{C}_2} \end{pmatrix}
\end{aligned} \tag{1.31}
$$

and, by Proposition 1.1, one obtains (in $\mathbb{R} \cup \{-\infty\}$)

$$
\log \mathrm{Vol} \begin{pmatrix} f & h_1 \\ f_1 & h \end{pmatrix} + \log \mathrm{Vol}(g) = \log \mathrm{Vol}(f) + \log \mathrm{Vol} \begin{pmatrix} Id_{\mathcal{C}_1^+} & h_1 \\ -g_1 & gh \end{pmatrix}.
$$

According to (1.28), $gh = Id_{\mathcal{C}_2} - g_1 h_1$ and therefore

$$
\begin{pmatrix} Id_{\mathcal{C}_1^+} & h_1 \\ -g_1 & gh \end{pmatrix} = \begin{pmatrix} Id_{\mathcal{C}_1^+} & 0 \\ -g_1 & Id_{\mathcal{C}_2} \end{pmatrix} \begin{pmatrix} Id_{\mathcal{C}_1^+} & h_1 \\ 0 & Id_{\mathcal{C}_2} \end{pmatrix}. \tag{1.32}
$$

Formula (1.32) implies that $\begin{pmatrix} Id_{\mathcal{C}_1^+} & h_1 \\ -g_1 & gh \end{pmatrix}$ is a weak isomorphism of determinant class and

$$
\log \mathrm{Vol} \begin{pmatrix} Id_{\mathcal{C}_1^+} & h_1 \\ -g_1 & gh \end{pmatrix} = 0 + 0. \tag{1.33}
$$

Combining (1.30) - (1.33) statements (A) and (B) follow. $\square$

**Lemma 1.5.** *Let $\mathcal{C}^1, \mathcal{C}^2, \mathcal{C}^3$ be three stage complexes which are short exact sequences, (and thus, in particular, of determinant class)*

$$\mathcal{C}^1 : 0 \to \mathcal{C}^1_0 \overset{d^1_0}{\to} \mathcal{C}^1_1 \overset{d^1_1}{\to} \mathcal{C}^1_2 \to 0$$

$$\mathcal{C}^2 : 0 \to \mathcal{C}^1_0 \oplus \mathcal{C}^3_0 \overset{d^2_0}{\to} \mathcal{C}^1_1 \oplus \mathcal{C}^3_1 \overset{d^2_1}{\to} \mathcal{C}^1_2 \oplus \mathcal{C}^3_2 \to 0$$

$$\mathcal{C}^3 : 0 \to \mathcal{C}^3_0 \overset{d^3_0}{\to} \mathcal{C}^3_1 \overset{d^3_1}{\to} \mathcal{C}^3_2 \to 0$$

*Further assume that the following diagram is commutative,*

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{C}^1_0 & \to & \mathcal{C}^1_1 & \to & \mathcal{C}^1_2 & \to & 0 \\
 & & \downarrow \binom{Id}{0} & & \downarrow \binom{Id}{0} & & \downarrow \binom{Id}{0} & & \\
0 & \to & \mathcal{C}^1_0 \oplus \mathcal{C}^3_0 & \to & \mathcal{C}^1_1 \oplus \mathcal{C}^3_1 & \to & \mathcal{C}^1_2 \oplus \mathcal{C}^3_2 & \to & 0 \\
 & & \downarrow (0\, Id) & & \downarrow (0\, Id) & & \downarrow (0\, Id) & & \\
0 & \to & \mathcal{C}^3_0 & \to & \mathcal{C}^3_1 & \to & \mathcal{C}^3_2 & \to & 0.
\end{array} \tag{1.34}$$

*Then*

$$\log T(\mathcal{C}^2) = \log T(\mathcal{C}^1) + \log T(\mathcal{C}^3). \tag{1.35}$$

*Proof.* We first note that in the Hodge decomposition of $\mathcal{C}^1_1 = \mathcal{C}^{1,+}_1 \oplus \mathcal{C}^{1,-}_1$, and $\mathcal{C}^3_1 = \mathcal{C}^{3,+}_1 \oplus \mathcal{C}^{3,-}_1$ the Hilbert module $\mathcal{C}^{j,+}_1$ is isometric to $\mathcal{C}^j_0$ and $\mathcal{C}^{j,-}_1$ is isometric to $\mathcal{C}^j_2$, $(j \in \{1,3\})$. Therefore one can write $d^j_0 = \binom{\underline{d}^j_0}{0}$; $d^j_1 = (0\, \underline{d}^j_1)$, with $\underline{d}^j_0 : \mathcal{C}^j_0 \to \mathcal{C}^{j,+}_1$ and $\underline{d}^j_1 : \mathcal{C}^{j,-}_1 \to \mathcal{C}^j_2$

Taking the above mentioned isometries into account it follows that $\mathcal{C}^2$ is isometric to the decomposition

$$0 \to \mathcal{C}^1_0 \oplus \mathcal{C}^3_0 \overset{d^2_0}{\to} \mathcal{C}^1_0 \oplus \mathcal{C}^3_0 \oplus \mathcal{C}^1_2 \oplus \mathcal{C}^3_2 \overset{d^2_1}{\to} \mathcal{C}^1_2 \oplus \mathcal{C}^3_2 \to 0$$

and the commutativity of the diagram (1.34) implies that $d^2_0$ must be of the form

$$d^2_0 = \begin{pmatrix} \underline{d}^1_0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \underline{d}^3_0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & h_1 \\ 0 & 0 \\ 0 & h_2 \\ 0 & 0 \end{pmatrix} \tag{1.36}$$

where $h_1 : \mathcal{C}^3_0 \to \mathcal{C}^1_0$ and $h_2 : \mathcal{C}^3_0 \to \mathcal{C}^1_2$ and $d^2_1$ must be of the form

$$d^2_1 = \begin{pmatrix} 0 & 0 & \underline{d}^1_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{d}^3_1 \end{pmatrix} + \begin{pmatrix} 0 & h_3 & 0 & h_4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.37}$$

where $h_3 : \mathcal{C}_0^3 \to \mathcal{C}_2^1$ and $h_4 : \mathcal{C}_2^3 \to \mathcal{C}_2^1$.

The assumption that $\mathcal{C}^2$ is an exact sequence implies that

$$h_3 \underline{d}_0^3 + \underline{d}_1^1 h_2 = 0. \tag{1.38}$$

In order to compute the torsion $T(\mathcal{C}^2)$ we want to apply Proposition 1.3. First note that $\underline{d}_0^1, \underline{d}_0^3, \underline{d}_1^1, \underline{d}_1^3$ are all isomorphism as $\mathcal{C}^1, \mathcal{C}^2, \mathcal{C}^3$ are exact sequences. In view of (1.38),

$$\begin{pmatrix} \underline{d}_0^1 & 0 \\ 0 & \underline{d}_0^3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Id_{\mathcal{C}_0^1} & (\underline{d}_0^1)^{-1} h_1 \\ 0 & Id_{\mathcal{C}_0^3} \end{pmatrix} = \begin{pmatrix} Id & 0 & 0 & 0 \\ 0 & Id & 0 & 0 \\ 0 & (\underline{d}_1^1)^{-1} h_3 & Id & 0 \\ 0 & 0 & 0 & Id \end{pmatrix} \begin{pmatrix} \underline{d}_0^1 & h_1 \\ 0 & \underline{d}_0^3 \\ 0 & h_2 \\ 0 & 0 \end{pmatrix} \tag{1.39}$$

and

$$\begin{aligned} &\begin{pmatrix} Id & -h_4 (\underline{d}_1^3)^{-1} \\ 0 & Id \end{pmatrix} \begin{pmatrix} 0 & h_3 & \underline{d}_1^1 & h_4 \\ 0 & 0 & 0 & \underline{d}_1^3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & \underline{d}_1^1 & 0 \\ 0 & 0 & 0 & \underline{d}_1^3 \end{pmatrix} \begin{pmatrix} Id_{\mathcal{C}_0^1} & 0 & 0 & 0 \\ 0 & Id_{\mathcal{C}_0^3} & 0 & 0 \\ 0 & (\underline{d}_1^1)^{-1} h_3 & Id_{\mathcal{C}_2^1} & 0 \\ 0 & 0 & 0 & Id_{\mathcal{C}_2^3} \end{pmatrix}. \end{aligned} \tag{1.40}$$

Consider the following isomorphic three stage complexes: $\mathcal{C}^2, \mathcal{C}^{21}, \mathcal{C}^{22}$ and the isomorphisms $f' : \mathcal{C}^2 \to \mathcal{C}^{21}$, $f'' : \mathcal{C}^{21} \to \mathcal{C}^{22}$ given by the following commutative diagram (use equality (1.38))

$$\begin{tikzcd}
\mathcal{C}^2 : \; 0 \arrow[r] & \mathcal{C}_0^1 \oplus \mathcal{C}_0^3 \arrow[r, "\begin{pmatrix} \underline{d}_0^1 & h_1 \\ 0 & \underline{d}_0^3 \\ 0 & h_2 \\ 0 & 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} Id & (\underline{d}_0^1)^{-1} h_1 \\ 0 & Id \end{pmatrix}"'] & \mathcal{C}_0^1 \oplus \mathcal{C}_0^3 \oplus \mathcal{C}_2^1 \oplus \mathcal{C}_2^3 \arrow[r, "\begin{pmatrix} 0 & h_3 & \underline{d}_1^1 & h_4 \\ 0 & 0 & 0 & \underline{d}_1^3 \end{pmatrix}"] \arrow[d, "Id"] & \mathcal{C}_2^1 \oplus \mathcal{C}_2^3 \arrow[r] \arrow[d, "\begin{pmatrix} Id & -h_4 (\underline{d}_1^3)^{-1} \\ 0 & Id \end{pmatrix}"] & 0 \\
\mathcal{C}^{21} : \; 0 \arrow[r] & \mathcal{C}_0^1 \oplus \mathcal{C}_0^3 \arrow[r, "\begin{pmatrix} \underline{d}_0^1 & 0 \\ 0 & \underline{d}_0^3 \\ 0 & h_2 \\ 0 & 0 \end{pmatrix}"] \arrow[d, "Id"'] & \mathcal{C}_0^1 \oplus \mathcal{C}_0^3 \oplus \mathcal{C}_2^1 \oplus \mathcal{C}_2^3 \arrow[r, "\begin{pmatrix} 0 & h_3 & \underline{d}_1^1 & 0 \\ 0 & 0 & 0 & \underline{d}_1^3 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} Id & 0 & 0 & 0 \\ 0 & Id & 0 & 0 \\ 0 & (\underline{d}_1^1)^{-1} h_3 & Id & 0 \\ 0 & 0 & 0 & Id \end{pmatrix}"] & \mathcal{C}_2^1 \oplus \mathcal{C}_2^3 \arrow[r] \arrow[d, "Id"] & 0 \\
\mathcal{C}^{22} : \; 0 \arrow[r] & \mathcal{C}_0^1 \oplus \mathcal{C}_0^3 \arrow[r, "\begin{pmatrix} \underline{d}_0^1 & 0 \\ 0 & \underline{d}_0^3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}"] & \mathcal{C}_0^1 \oplus \mathcal{C}_0^3 \oplus \mathcal{C}_2^1 \oplus \mathcal{C}_2^3 \arrow[r, "\begin{pmatrix} 0 & 0 & \underline{d}_1^1 & 0 \\ 0 & 0 & 0 & \underline{d}_1^3 \end{pmatrix}"] & \mathcal{C}_2^1 \oplus \mathcal{C}_2^3 \arrow[r] & 0.
\end{tikzcd}$$

Now apply Proposition 1.3 and use that all three complexes $\mathcal{C}^2, \mathcal{C}^{21}, \mathcal{C}^{22}$ are exact sequences to conclude that

$$\log T(\mathcal{C}^2) = \log T(\mathcal{C}^{21}) = \log T(\mathcal{C}^{22}) \tag{1.41}$$

Moreover,

$$\log T(\mathcal{C}^{22}) = \log T(\mathcal{C}^1) + \log T(\mathcal{C}^3). \tag{1.42}$$

Combining (1.41) and (1.42) leads to (1.35).

□

The proof of Lemma 1.5 actually leads to a slightly stronger result:

**Lemma 1.5′.** *Assume that the three stage complexes $\mathcal{C}^1, \mathcal{C}^2, \mathcal{C}^3$ are weakly exact sequences of determinant class and, in addition, assume that the morphisms $\underline{d}^1_0, \underline{d}^1_1$ and $\underline{d}^3_1$ (or $\underline{d}^1_0, \underline{d}^3_0$ and $\underline{d}^3_1$) are isomorphisms. Then*

$$\log T(\mathcal{C}^2) = \log T(\mathcal{C}^1) + \log T(\mathcal{C}^3).$$

*Remark.* To conclude the result of Lemma 1.5′ in the case where $\underline{d}^1_0$, $\underline{d}^3_0$ and $\underline{d}^3_1$ are isomorphisms one replaces the entry $(\underline{d}^1_1)^{-1}h_3$ in the 4 x 4 matrix which appears in the diagram above by $-h_2(\underline{d}^3_0)^{-1}$.

The following considerations are a preparation for the proof of Theorem 1.14.

Assume that

$$0 \to \mathcal{C}^1 \xrightarrow{f} \mathcal{C}^2 \xrightarrow{g} \mathcal{C}^3 \to 0$$

is an exact sequence of $\mathcal{A}$-cochain complexes. Consider the Hodge decomposition of each of the three complexes, $k = 1, 2, 3$,

$$\mathcal{C}^k_i = \mathcal{H}^k_i \oplus \mathcal{C}^{k,+}_i \oplus \mathcal{C}^{k,-}_i.$$

With respect to these decompositions, $f_i : \mathcal{C}^1_i \to \mathcal{C}^2_i$ and $g_i : \mathcal{C}^2_i \to \mathcal{C}^3_i$ take the form

$$f_i = \begin{pmatrix} f_{i,11} & 0 & f_{i,13} \\ f_{i,21} & f_{i,22} & f_{i,23} \\ 0 & 0 & f_{i,33} \end{pmatrix}, \; g_i = \begin{pmatrix} g_{i,11} & 0 & g_{i,13} \\ g_{i,21} & g_{i,22} & g_{i,23} \\ 0 & 0 & g_{i,33} \end{pmatrix}. \tag{1.43}$$

Moreover

$$\underline{d}^2_i f_{i,33} = f_{i+1,22}\underline{d}^1_i; \; \underline{d}^3_i g_{i,33} = g_{i+1,22}\underline{d}^2_i. \tag{1.44}$$

The exactness of the sequences $0 \to \mathcal{C}^1_k \to \mathcal{C}^2_k \to \mathcal{C}^3_k \to 0$ imply that the following sequences

$$0 \to \mathcal{C}_i^{1,+} \overset{f_{i,22}}{\to} \mathcal{C}_i^{2,+} \overset{g_{i,22}}{\to} \mathcal{C}_i^{3,+} \to 0 \tag{1.45}$$

$$0 \to \mathcal{C}_i^{1,-} \overset{f_{i,33}}{\to} \mathcal{C}_i^{2,-} \overset{g_{i,33}}{\to} \mathcal{C}_i^{3,-} \to 0 \tag{1.46}$$

are three stage complexes with the property that $f_{i,22}$ is 1-1 and $g_{i,33}$ is onto. Using (1.44) one verifies that $f_{i,22}$ has closed image, the range of $g_{i,22}$ is dense in $\mathcal{C}_i^{3,+}$ and $f_{i,33}$ is 1-1. Therefore, the cochain complexes (1.45) and (1.46) have a Hodge decomposition

$$0 \to \mathcal{C}_i^{1,+} \overset{f_{i,22}}{\to} \mathcal{C}_i^{2,+0} \oplus \mathcal{C}_i^{2,++} \oplus \mathcal{C}_i^{2,+-} \overset{g_{i,22}}{\to} \mathcal{C}_i^{3,+} \to 0 \tag{1.47}$$

and

$$0 \to \mathcal{C}_i^{1,-} \overset{f_{i,33}}{\to} \mathcal{C}_i^{2,-0} \oplus \mathcal{C}_i^{2,-+} \oplus \mathcal{C}_i^{2,--} \overset{g_{i,33}}{\to} \mathcal{C}_i^{3,-} \to 0. \tag{1.48}$$

With respect to this decomposition,

$$f_{i,22} = \begin{pmatrix} 0 \\ \alpha_{i,43} \\ 0 \end{pmatrix};\ g_{i,22} = (\,0 \quad 0 \quad \beta_{i,35}\,) \tag{1.50}$$

$$f_{i,33} = \begin{pmatrix} 0 \\ \alpha_{i,74} \\ 0 \end{pmatrix};\ g_{i,33} = (\,0 \quad 0 \quad \beta_{i,48}\,)\,. \tag{1.51}$$

Notice that $\alpha_{i,43}$ and $\beta_{i,48}$ are isomorphisms and $\beta_{i,35}$ and $\alpha_{i,74}$ are weak isomorphisms.

The exact sequence of $\mathcal{A}$-cochain complexes of finite type

$$0 \to \mathcal{C}^1 \overset{f}{\to} \mathcal{C}^2 \overset{g}{\to} \mathcal{C}^3 \to 0$$

induces a long weakly exact sequence $\mathcal{H}$ in cohomology (cf. [CG, p. 10, Thm 2.1])

$$\cdots \to H^i(\mathcal{C}^1) \overset{H(f_i)}{\to} H^i(\mathcal{C}^2) \overset{H(g_i)}{\to} H^i(\mathcal{C}^3) \overset{H(\delta_i)}{\to} H^{i+1}(\mathcal{C}^1) \to \dots \tag{1.52}$$

or, equivalently, in terms of the harmonic spaces $\mathcal{H}_i^k := Null(d_i^k) \cap Null(d_{i-1}^{k*})$

$$\cdots \to \mathcal{H}_i^1 \overset{f_{i,11}}{\to} \mathcal{H}_i^2 \overset{g_{i,11}}{\to} \mathcal{H}_i^3 \overset{\delta_i}{\to} \mathcal{H}_{i+1}^1 \overset{f_{i+1,11}}{\to} \mathcal{H}_{i+1}^2 \to \dots \tag{1.52$'$}$$

In particular the cochain complex $\mathcal{H}$ is acyclic. Consider its Hodge decomposition

$$\begin{aligned}(1.53)\qquad &\dots \to \mathcal{H}_i^{1,+}\oplus\mathcal{H}_i^{1,-} \xrightarrow{\begin{pmatrix} 0 & \alpha_{i,12}\\ 0 & 0\end{pmatrix}} \mathcal{H}_i^{2,+}\oplus\mathcal{H}_i^{2,-} \xrightarrow{\begin{pmatrix} 0 & \beta_{i,12}\\ 0 & 0\end{pmatrix}}\\ &\mathcal{H}_i^{3,+}\oplus\mathcal{H}_i^{3,-} \xrightarrow{\begin{pmatrix} 0 & \gamma_i\\ 0 & 0\end{pmatrix}} \mathcal{H}_{i+1}^{1,+}\oplus\mathcal{H}_{i+1}^{1,-}\to\dots\end{aligned}$$

Notice that $\alpha_{i,12},\beta_{i,12}$ and $\gamma_i$ are weak isomorphisms.

In view of (1.43) - (1.53) $f_i$ and $g_i$ can be written as

$$(1.54)\ f_i=\begin{pmatrix} f_{i,11} & 0 & f_{i,13}\\ f_{i,21} & f_{i,22} & f_{i,23}\\ 0 & 0 & f_{i,33}\end{pmatrix}=\begin{pmatrix} \begin{pmatrix} 0 & \alpha_{i,12}\\ 0 & 0\end{pmatrix} & \begin{pmatrix} 0\\ 0\end{pmatrix} & \begin{pmatrix} \alpha_{i,14}\\ \alpha_{i,24}\end{pmatrix}\\ \begin{pmatrix} \alpha_{i,31} & \alpha_{i,32}\\ \alpha_{i,41} & \alpha_{i,42}\\ \alpha_{i,51} & \alpha_{i,52}\end{pmatrix} & \begin{pmatrix} 0\\ \alpha_{i,43}\\ 0\end{pmatrix} & \begin{pmatrix} \alpha_{i,34}\\ \alpha_{i,44}\\ \alpha_{i,54}\end{pmatrix}\\ \begin{pmatrix} 0 & 0\\ 0 & 0\\ 0 & 0\end{pmatrix} & \begin{pmatrix} 0\\ 0\\ 0\end{pmatrix} & \begin{pmatrix} 0\\ \alpha_{i,74}\\ 0\end{pmatrix}\end{pmatrix}$$

and

$$\begin{aligned}(1.55)\qquad g_i&=\begin{pmatrix} g_{i,11} & 0 & g_{i,13}\\ g_{i,21} & g_{i,22} & g_{i,23}\\ 0 & 0 & g_{i,33}\end{pmatrix}\\ &=\begin{pmatrix} \begin{pmatrix} 0 & \beta_{i,12}\\ 0 & 0\end{pmatrix} & \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} & \begin{pmatrix} \beta_{i,16} & \beta_{i,17} & \beta_{i,18}\\ \beta_{i,26} & \beta_{i,27} & \beta_{i,28}\end{pmatrix}\\ (\beta_{i,31} & \beta_{i,32}) & (0 & 0 & \beta_{i,35}) & (\beta_{i,36} & \beta_{i,37} & \beta_{i,38})\\ (0 & 0) & (0 & 0 & 0) & (0 & 0 & \beta_{i,48})\end{pmatrix}\end{aligned}$$

Recall that $f_i:\mathcal{H}_i^{1,+}\oplus\mathcal{H}_i^{1,-}\oplus\mathcal{C}_i^{1,+}\oplus\mathcal{C}_i^{2,-}\to\mathcal{H}_i^{2,+}\oplus\mathcal{H}_i^{2,-}\oplus\mathcal{C}_i^{2,+0}\oplus\mathcal{C}_i^{2,++}\oplus\mathcal{C}_i^{2,+-}\oplus\mathcal{C}_i^{2,-0}\oplus\mathcal{C}_i^{2,-+}\oplus\mathcal{C}_i^{2,--}$ and $g_i:\mathcal{H}_i^{2,+}\oplus\mathcal{H}_i^{2,-}\oplus\mathcal{C}_i^{2,+0}\oplus\mathcal{C}_i^{2,++}\oplus\mathcal{C}_i^{2,+-}\oplus\mathcal{C}_i^{2,-0}\oplus\mathcal{C}_i^{2,-+}\oplus\mathcal{C}_i^{2,--}\to\mathcal{H}_i^{3,+}\oplus\mathcal{H}_i^{3,-}\oplus\mathcal{C}_i^{3,+}\oplus\mathcal{C}_i^{3,-}$.

We have already noticed that $\alpha_{i,12},\alpha_{i,74},\beta_{i,12},\beta_{i,35}$ are weak isomorphisms, while $\alpha_{i,43}$ and $\beta_{i,48}$ are isomorphisms. Using that $g_i\cdot f_i=0$ and the fact that $\alpha_{i,74}$ and $\beta_{i,35}$ are weak isomorphisms one verifies that

$$(1.56)\qquad \alpha_{i,51}=0;\quad \beta_{i,27}=0$$

$$\begin{aligned}(1.56')\qquad &\beta_{i,12}\alpha_{i,24}+\beta_{i,17}\alpha_{i,74}=0;\quad \beta_{i,31}\alpha_{i,12}+\beta_{i,35}\alpha_{i,52}=0\\ &\beta_{i,31}\alpha_{i,14}+\beta_{i,32}\alpha_{i,24}+\beta_{i,35}\alpha_{i,54}+\beta_{i,37}\alpha_{i,74}=0.\end{aligned}$$

**Lemma 1.6.** *For any $i$,*
*(A) $\alpha_{i,31} : \mathcal{H}_i^{1,+} \to \mathcal{C}_i^{2,+0}$ is an isomorphism;*
*(B) $\beta_{i,26} : \mathcal{C}_i^{2,-0} \to \mathcal{H}_i^{3,-}$ is an isomorphism.*

*Proof.* (A) Inspecting (1.54) and in view of $\alpha_{i,51} = 0$ (cf. 1.56) one concludes that $h_i := f_i \restriction_{\mathcal{H}_i^{1,+} \oplus \mathcal{C}_i^{1,+}}$ has a range which is contained in $\mathcal{C}_i^{2,+0} \oplus \mathcal{C}_i^{2,++}$ and admits a representation of the form

$$h_i := \begin{pmatrix} \alpha_{i,31} & 0 \\ \alpha_{i,41} & \alpha_{i,43} \end{pmatrix} : \mathcal{H}_i^{1,+} \oplus \mathcal{C}_i^{1,+} \to \mathcal{C}_i^{2,+0} \oplus \mathcal{C}_i^{2,++}.$$

From (1.55) one sees that $\mathcal{C}_i^{2,+0} \oplus \mathcal{C}_i^{2,++} \subseteq Null\, g_i$. Due to the exactness $0 \to \mathcal{C}_i^1 \xrightarrow{f_i} \mathcal{C}_i^2 \xrightarrow{g_i} \mathcal{C}_i^3 \to 0$, the map $h_i$ is an isomorphism. Further we already know that $\alpha_{i,43}$ is an isomorphism and therefore conclude that $\alpha_{i,31} : \mathcal{H}_i^{1,+} \to \mathcal{C}_i^{2,+0}$ is an isomorphism as well.
(B) From (1.54) one sees that $\mathcal{C}_i^{2,-0} \oplus \mathcal{C}_i^{2,--} \subseteq (Range\, f_i)^{\perp}$. Therefore, using that $0 \to \mathcal{C}_i^1 \xrightarrow{f_i} \mathcal{C}_i^2 \xrightarrow{g_i} \mathcal{C}_i^3 \to 0$ is exact, $h_i' := g_i \restriction_{\mathcal{C}_i^{2,-0} \oplus \mathcal{C}_i^{2,--}}$ is 1-1 and its image is given by $\mathcal{H}_i^{3,-} \oplus \mathcal{C}_i^{3,-}$, i.e. $h_i' := \begin{pmatrix} \beta_{i,26} & \beta_{i,28} \\ 0 & \beta_{i,48} \end{pmatrix} : \mathcal{C}_i^{2,-0} \oplus \mathcal{C}_i^{2,--} \to \mathcal{H}_i^{3,-} \oplus \mathcal{C}_i^{3,-}$ is bijective. We already know that $\beta_{i,48}$ is an isomorphism and (B) follows. ☐

Further notice that $d_i^2$ takes the following form

$$d_i^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \underline{d}_i^2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \begin{pmatrix} d_{i,36}^2 & 0 & d_{i,38}^2 \\ d_{i,46}^2 & d_{i,47}^2 & d_{i,48}^2 \\ 0 & 0 & d_{i,58}^2 \end{pmatrix} \\ 0 & 0 & 0 \end{pmatrix} \tag{1.57}$$

where we used again (1.44) to verify that $d_{i,56}^2, d_{i,57}^2$ and $d_{i,37}^2$ all vanish and that, for any $i$,

$$\underline{d}_i^3 \beta_{i,48} = \beta_{i+1,35} d_{i,58}^2 \text{ is a weak isomorphism} \tag{1.58}$$

and

$$d_{i,47}^2 \alpha_{i,74} = \alpha_{i+1,43} \underline{d}_i^1 \text{ is a weak isomorphism.} \tag{1.59}$$

Further, considering $\mathcal{C}_i^{2,-} \xrightarrow{g_{i,13}} \mathcal{H}_i^3 \xrightarrow{\delta_i} \mathcal{H}_{i+1}^1 \xrightarrow{f_{i+1,21}} \mathcal{C}_{i+1}^{2,+}$ and taking into account the definition of $\delta_i = \begin{pmatrix} 0 & \gamma_i \\ 0 & 0 \end{pmatrix}$ one obtains

$$d_{i,36}^2 = \alpha_{i+1,31} \gamma_i \beta_{i,26}. \tag{1.60}$$

In a straightforward way one verifies the following

**Lemma 1.7.** *Assume that $F : \mathcal{W}_1 \oplus \mathcal{W}_2 \to \mathcal{V}_1 \oplus \mathcal{V}_2$ is an $\mathcal{A}$-linear map of the form $F = \begin{pmatrix} f & h \\ 0 & g \end{pmatrix}$.*
*(A) If $F$ is 1-1 and has closed range and $f$ is an isomorphism, then $g$ is 1-1 and has closed range.*
*(B) If $F$ is onto and $g$ is an isomorphism, then $f$ is onto.*

**Lemma 1.8.** *For any $i$, $\alpha_{i,12}$ and $\alpha_{i,74}$ are of determinant class iff $\beta_{i,12}$ and $\beta_{i,35}$ are of determinant class.*

*Proof.* Introduce $\varphi_i : \mathcal{H}_i^{1,-} \oplus \mathcal{C}_i^{1,-} \to \mathcal{H}_i^{2,+} \oplus \mathcal{C}_i^{2,-+} \oplus \mathcal{H}_i^{2,-} \oplus \mathcal{C}_i^{2,+-}$ and $\psi_i : \mathcal{H}_i^{2,+} \oplus \mathcal{C}_i^{2,-+} \oplus \mathcal{H}_i^{2,-} \oplus \mathcal{C}_i^{2,+-} \to \mathcal{H}_i^{3,+} \oplus \mathcal{C}_i^{3,+}$ given as follows

$$\psi_i = \begin{pmatrix} 0 & \beta_{i,17} & \beta_{i,12} & 0 \\ \beta_{i,31} & \beta_{i,37} & \beta_{i,32} & \beta_{i,35} \end{pmatrix}; \varphi_i = \begin{pmatrix} \alpha_{i,12} & \alpha_{i,14} \\ 0 & \alpha_{i,74} \\ 0 & \alpha_{i,24} \\ \alpha_{i,52} & \alpha_{i,54} \end{pmatrix}. \tag{1.61}$$

Due to (1.56) - (1.56$'$) we have $\psi_i \cdot \varphi_i = 0$. Moreover $\alpha_{i,12}, \alpha_{i,74}, \beta_{i,12}$ and $\beta_{i,35}$ are all weak isomorphisms. Therefore, for any $i$,

$$(1.62) 0 \to \mathcal{H}_i^{1,-} \oplus \mathcal{C}_i^{1,-} \xrightarrow{\varphi_i} \mathcal{H}_i^{2,+} \oplus \mathcal{C}_i^{2,-+} \oplus \mathcal{H}_i^{2,-} \oplus \mathcal{C}_i^{2,+-} \xrightarrow{\psi_i} \mathcal{H}_i^{3,+} \oplus \mathcal{C}_i^{3,+} \to 0$$

is a weak exact sequence.

Using that $\begin{pmatrix} \alpha_{i,31} & 0 \\ \alpha_{i,41} & \alpha_{i,43} \end{pmatrix}$ is an isomorphism and applying Lemma 1.7 (A) twice to $f_i$ one concludes that $\varphi_i$ has closed image. Similarly, using that $\begin{pmatrix} \beta_{i,26} & \beta_{i,28} \\ 0 & \beta_{i,48} \end{pmatrix}$ is an isomorphism and applying Lemma 1.7 (B) one concludes that $\psi_i$ is onto. Therefore, the sequence (1.62) is, in fact, an exact sequence and thus of determinant class. The statement then follows from Lemma 1.4(B) and Proposition 1.1(B).

□

**Lemma 1.9.** *For any $i, d^2_{i,36}, d^2_{i,47}, d^2_{i,58}$ are weak isomorphisms and satisfy:*
*(A) $d^2_{i,36}$ is of determinant class iff $\gamma_i$ is of determinant class.*
*(B) $d^2_{i,47}$ and $\alpha_{i,74}$ are of determinant class iff $\underline{d_i^1}$ is of determinant class.*
*(C) $d^2_{i,58}$ and $\beta_{i+1,35}$ are of determinant class iff $\underline{d_i^3}$ is of determinant class.*

*Proof.* The fact that $d^2_{i,36}, d^2_{i,47}$ and $d^2_{i,58}$ are weak isomorphisms follows from inspection of (1.57). To prove (A), (B) and (C) one applies Proposition 1.1 and uses formula (1.60) (in case of (A)), formula (1.59) (in case of (B)) and formula (1.58) (in case of (C)) together with the properties that $\alpha_{i+1,31}$ and $\beta_{i,26}$ are isomorphisms (Lemma 1.6) and therefore of determinant class (in case (A)), $\alpha_{i+1,43}$ is an isomorphism (in case (B)) and $\beta_{i,48}$ is an isomorphism (in case (C)).

□

**Lemma 1.10.** *Assume that $\mathcal{C}^1$ and $\mathcal{C}^3$ are of determinant class. Then, for any $i$, the following statements hold:*
*(A) the three stage complexes $\mathcal{D}^i_\pm$ given by $0 \to \mathcal{C}^{1,\pm}_i \to \mathcal{C}^{2,\pm}_i \to \mathcal{C}^{3,\pm}_i \to 0$ are of determinant class and*

$$\log T(\mathcal{D}^i_+) = \log Vol\,\alpha_{i,43} - \log Vol\,\beta_{i,35} \tag{1.63}$$

$$\log T(\mathcal{D}^i_-) = \log Vol\,\alpha_{i,74} - \log Vol\,\beta_{i,48}. \tag{1.64}$$

*(B) Assume, in addition, that the cochain complex $\mathcal{H}$ (cf (1.52)) is of determinant class. Then, for any $i$, the three stage complex $\mathcal{D}^i$, given by $0 \to \mathcal{C}^1_i \xrightarrow{f_i} \mathcal{C}^2_i \xrightarrow{g_i} \mathcal{C}^3_i \to 0$, is of determinant class and*

$$\begin{aligned}(1.65)\ \log T(\mathcal{D}^i) = {} & \log Vol\,\alpha_{i,12} + \log Vol\,\alpha_{i,31} + \log Vol\,\alpha_{i,43} + \log Vol\,\alpha_{i,74} \\ & - \log Vol\beta_{i,12} - \log Vol\beta_{i,26} - \log Vol\beta_{i,35} - \log Vol\beta_{i,48}.\end{aligned}$$

*Proof.* (A) Recall that $\mathcal{D}^i_+$ is given by

$$0 \to \mathcal{C}^{1,+}_i \xrightarrow{f_{i,22}} \mathcal{C}^{2,+}_i \xrightarrow{g_{i,22}} \mathcal{C}^{3,+}_i \to 0$$

where, according to (1.50), $f_{i,22} = \begin{pmatrix} 0 \\ \alpha_{i,43} \\ 0 \end{pmatrix}$ and $g_{i,22} = (0 \quad 0 \quad \beta_{i,35})$. The map $\alpha_{i,43}$ is an isomorphism and therefore of determinant class. Due to Lemma 1.9, the weak isomorphism $\beta_{i,35}$ is of determinant class as well and (1.63) follows.

Similarly one argues for $\mathcal{D}^i_-$ given by

$$0 \to \mathcal{C}^{1,-}_i \xrightarrow{f_{i,33}} \mathcal{C}^{2,-}_i \xrightarrow{g_{i,33}} \mathcal{C}^{3,-}_i \to 0$$

where, according to (1.50),

$$f_{i,33} = \begin{pmatrix} 0 \\ \alpha_{i,74} \\ 0 \end{pmatrix}, \; g_{i,33} = (0 \quad 0 \quad \beta_{i,48}).$$

Notice that $\beta_{i,48}$ is an isomorphism and therefore of determinant class. According to Lemma 1.9, the weak isomorphism $\alpha_{i,74}$ is of determinant class and therefore (1.64) is proved.

(B) The complexes $\mathcal{D}^i$, given by $0 \to \mathcal{C}^1_i \xrightarrow{f_i} \mathcal{C}^2_i \xrightarrow{g_i} \mathcal{C}^3_i \to 0$ are exact and therefore of determinant class. To verify formula (1.65) we want to apply Lemma 1.4.

Decompose $\mathcal{C}^2_i = \mathcal{W}^2_i \oplus \mathcal{V}^2_i$ where

$$\mathcal{W}^2_i := \mathcal{H}^{2,+}_i \oplus \mathcal{C}^{2,+0}_i \oplus \mathcal{C}^{2,++}_i \oplus \mathcal{C}^{2,-+}_i$$

and

$$\mathcal{V}_i^2 := (\mathcal{W}_i^2)^\perp = \mathcal{H}_i^{2,-} \oplus \mathcal{C}_i^{2,+-} \oplus \mathcal{C}_i^{2,-0} \oplus \mathcal{C}_i^{2,--}.$$

Then $f_i = \begin{pmatrix} f_i' \\ f_i'' \end{pmatrix}$ and $g_i = (g_i' \; g_i'')$ have the following representations

$$f_i' = \begin{pmatrix} 0 & \alpha_{i,12} & 0 & \alpha_{i,14} \\ \alpha_{i,31} & \alpha_{i,32} & 0 & \alpha_{i,34} \\ \alpha_{i,41} & \alpha_{i,42} & \alpha_{i,43} & \alpha_{i,44} \\ 0 & 0 & 0 & \alpha_{i,74} \end{pmatrix} \tag{1.66}$$

$$g_i'' = \begin{pmatrix} \beta_{i,12} & 0 & \beta_{i,16} & \beta_{i,18} \\ 0 & 0 & \beta_{i,26} & \beta_{1,28} \\ \beta_{i,32} & \beta_{i,35} & \beta_{i,36} & \beta_{i,38} \\ 0 & 0 & 0 & \beta_{i,48} \end{pmatrix}. \tag{1.67}$$

Due to the assumption that $\mathcal{H}$ is of determinant class and due to Lemma 1.6 and Lemma 1.9, the maps $\alpha_{i,31}, \alpha_{i,12}, \alpha_{i,43}, \alpha_{i,74}, \beta_{i,12}, \beta_{i,35}, \beta_{i,26}$ and $\beta_{i,48}$ are weak isomorphisms of determinant class. Therefore $f_i'$ and $g_i''$ are weak isomorphisms of determinant class. As $g_i$ is onto, we can apply Proposition 1.4 to conclude that

$$\log T(0 \to \mathcal{C}_i^1 \to \mathcal{C}_i^2 \to \mathcal{C}_i^3 \to 0) = \log \mathrm{Vol} f_i' - \log \mathrm{Vol} g_i''. \tag{1.68}$$

Applying Proposition 1.1, we conclude from (1.66) - (1.67)

$$\begin{aligned} \log \mathrm{Vol} f_i' &= \log \mathrm{Vol}\alpha_{i,31} + \log \mathrm{Vol}\alpha_{i,12} + \log \mathrm{Vol}\alpha_{i,43} + \log \mathrm{Vol}\alpha_{i,74} \\ \log \mathrm{Vol} g_i'' &= \log \mathrm{Vol}\beta_{i,12} + \log \mathrm{Vol}\beta_{i,35} + \log \mathrm{Vol}\beta_{i,26} + \log \mathrm{Vol}\beta_{i,48}. \end{aligned} \tag{1.69}$$

Substituting (1.69) into (1.68) leads to the claimed result (1.65). □

Let $f : \mathcal{C}^1 \to \mathcal{C}^2$ be a morphism of cochain complexes. Define the mapping cone of $f$ to be the cochain complex $\mathcal{C}(f) := (\mathcal{C}_i(f), d_i(f))$ where $\mathcal{C}_i(f) = \mathcal{C}_i^2 \oplus \mathcal{C}_{i+1}^1$, $d_i(f) = \begin{pmatrix} d_i^2 & f_{i+1} \\ 0 & -d_{i+1}^1 \end{pmatrix}$. Denote by $S\mathcal{C} = (S\mathcal{C}_i, Sd_i)$ the suspension of a cochain complex $\mathcal{C}$, defined by $S\mathcal{C}_i = \mathcal{C}_{i+1}, Sd_i = -d_{i+1}$.
Consider the suspension $S\mathcal{C}^1$ of $\mathcal{C}^1$ and the morphisms $p(f) : \mathcal{C}(f) \to S\mathcal{C}^1$ and $j(f) : \mathcal{C}^2 \to \mathcal{C}(f)$ given by $p_i(f) = (0 \, Id)$ and $j_i(f) = \begin{pmatrix} Id \\ 0 \end{pmatrix}$. Then

$$0 \to \mathcal{C}^2 \stackrel{j(f)}{\to} \mathcal{C}(f) \stackrel{p(f)}{\to} S\mathcal{C}^1 \to 0$$

is a short exact sequence of cochain complexes. The induced long weakly exact sequence in cohomology (cf. 1.52) becomes in this case

$$\cdots \to H^i(\mathcal{C}^2) \to H^i(\mathcal{C}(f)) \to H^i(S\mathcal{C}^1) \stackrel{H(\delta_i)}{\to} H^{i+1}(\mathcal{C}^2) \to \dots \tag{1.70}$$

Notice that $H^i(S\mathcal{C}^1)$ is equal to $H^{i+1}(\mathcal{C}^1)$ and the connecting homomorphism $H(\delta_i) : H^i(S\mathcal{C}^1) \to H^{i+1}(\mathcal{C}^2)$ is given by the map $H(f_{i+1}) : H^{i+1}(\mathcal{C}^1) \to H^{i+1}(\mathcal{C}^2)$, induced by f in cohomology.

**Lemma 1.11 (A).** *Let $f : \mathcal{C}^1 \to \mathcal{C}^2$ be a morphism of cochain complexes. Assume that the cochain complexes $\mathcal{C}^k, (k = 1, 2)$ are of determinant class as well as the morphisms $H(f_i)\,(i \geq 0)$. Then the cochain complex $\mathcal{C}(f)$ is of determinant class.*

*Proof.* The claimed result follows from applying Proposition 1.1 and Lemma 1.9 (cf the proof of Proposition 1.13(i)). □

Let
$$0 \to \mathcal{C}^1 \xrightarrow{f} \mathcal{C}^2 \xrightarrow{g} \mathcal{C}^3 \to 0$$
be a short exact sequence of cochain complexes. Let $\pi : \mathcal{C}(f) \to \mathcal{C}^3$, $\rho : S\mathcal{C}^1 \to \mathcal{C}(g)$ and $S(f) : S\mathcal{C}^1 \to S\mathcal{C}^2$ be the morphisms defined by $\pi_i := (g_i\, 0)$, $\rho_i := \begin{pmatrix} 0 \\ f_{i+1} \end{pmatrix}$, $S_i(f) := f_{i+1.}$ Clearly $\pi_i \cdot j_i(f) = g_i$ and $p_i(g) \cdot \rho_i = S_i(f)$.

**Lemma 1.11 (B).** *Both $\pi$ and $\rho$ are homotopy equivalences.*

*Proof.* Without loss of generality one can assume that $\mathcal{C}_i^2 = \mathcal{C}_i^1 \oplus \mathcal{C}_i^3$, $f_i = \begin{pmatrix} Id \\ 0 \end{pmatrix}$, $g_i = (0\, Id)$ and $d_i^2 = \begin{pmatrix} d_i^1 & \theta_i \\ 0 & d_i^3 \end{pmatrix}$ with $\theta_i : \mathcal{C}_i^3 \to \mathcal{C}_{i+1}^1$, satisfying

$$d_{i+1}^1 \theta_i + \theta_{i+1} d_i^3 = 0. \tag{1.71}$$

Then $\rho_i = \begin{pmatrix} 0 \\ Id \\ 0 \end{pmatrix}$ and $\pi_i = (0\, Id\, 0)$, as well as $\mathcal{C}_i(f) = \mathcal{C}_i^1 \oplus \mathcal{C}_i^3 \oplus \mathcal{C}_{i+1}^1$ and $\mathcal{C}_i(g) = \mathcal{C}_i^3 \oplus \mathcal{C}_{i+1}^1 \oplus \mathcal{C}_{i+1}^3$. Notice that $d_i(f)$, resp. $d_i(g)$, are given by

$$d_i(f) = \begin{pmatrix} d_i^1 & \theta_i & Id \\ 0 & d_i^3 & 0 \\ 0 & 0 & -d_{i+1}^1 \end{pmatrix}. \tag{1.72}$$

$$d_i(g) = \begin{pmatrix} d_i^3 & 0 & Id \\ 0 & -d_{i+1}^1 & -\theta_{i+1} \\ 0 & 0 & -d_{i+1}^3 \end{pmatrix}. \tag{1.72$'$}$$

Define the morphisms of cochain complexes $\sigma : \mathcal{C}^3 \to \mathcal{C}(f)$ and $\omega : \mathcal{C}(g) \to S\mathcal{C}^1$ by $\sigma_i = \begin{pmatrix} 0 \\ Id \\ -\theta_i \end{pmatrix}$ respectively $\omega_i = (-\theta_i\ -Id\ 0)$ and notice that $\pi_i \cdot \sigma_i = Id$ and $\omega_i \cdot \rho_i = -Id$. It remains to check that $\sigma \cdot \pi$ and $-\rho \cdot \omega$ are both homotopic to $Id$. Since the cochain complexes $Null(\pi)$ and $Null(\omega)$ are, as is verified easily, exact sequences this will follow from Lemma 1.12 below (applied to $0 \to Null(\pi) \to \mathcal{C}(f) \to \mathcal{C}^3 \to 0$, respectively $0 \to Null(\omega) \to \mathcal{C}(g) \to S\mathcal{C}^1 \to 0$). □

**Lemma 1.12.** *Let*

$$0 \to \mathcal{C}^1 \xrightarrow{f} \mathcal{C}^2 \xrightarrow{g} \mathcal{C}^3 \to 0$$

*be a short exact sequence of cochain complexes and let $s : \mathcal{C}^3 \to \mathcal{C}^2$ be a morphism so that $g \cdot s = Id$. If $\mathcal{C}^1$ is an exact sequence, then $s \cdot g$ and $Id$ are homotopic.*

*Proof.* It is to prove (cf. Definition 1.14 in [BFKM]) that there exist morphisms $t_i : \mathcal{C}^2_i \to \mathcal{C}^2_{i-1}$ such that $Id - s_i g_i = d^2_{i-1} t_i + t_{i+1} d^2_i$. Consider the Hodge decomposition of $\mathcal{C}^1, \mathcal{C}^1_i = \mathcal{C}^{1,+}_i \oplus \mathcal{C}^{1,-}_i$ with $d^1_i = \begin{pmatrix} 0 & \underline{d}^1_i \\ 0 & 0 \end{pmatrix}$. In view of the exactness of $\mathcal{C}^1$, $\underline{d}^1_i$ are isomorphisms. Without loss of generality one can assume that $\mathcal{C}^2_i = \mathcal{C}^{1,+}_i \oplus \mathcal{C}^{1,-}_i \oplus \mathcal{C}^3_i$, $f_i = \begin{pmatrix} Id & 0 \\ 0 & Id \\ 0 & 0 \end{pmatrix}$, $g_i = (0\, 0\, Id)$, $s_i = \begin{pmatrix} s^+_i \\ s^-_i \\ Id \end{pmatrix}$ and $d^2_i = \begin{pmatrix} 0 & \underline{d}^1_i & \theta^+_i \\ 0 & 0 & \theta^-_i \\ 0 & 0 & d^3_i \end{pmatrix}$. Note that $d^2_{i+1} \cdot d^2_i = 0$ implies that

$$\underline{d}^1_{i+1} \theta^-_i + \theta^+_{i+1} d^3_i = 0, \ \theta^-_{i+1} d^3_i = 0. \tag{1.73}$$

The condition that s is a morphism of cochain complexes implies that

$$\underline{d}^1_i s^-_i + \theta^+_i = s^+_{i+1} d^3_i, \ \theta^-_i = s^-_{i+1} d^3_i. \tag{1.74}$$

Define $t_i : \mathcal{C}^{1,+}_i \oplus \mathcal{C}^{1,-}_i \oplus \mathcal{C}^3_i \to \mathcal{C}^{1,+}_{i-1} \oplus \mathcal{C}^{1,-}_{i-1} \oplus \mathcal{C}^3_{i-1}$ by

$$t_{i+1} = \begin{pmatrix} 0 & 0 & 0 \\ (\underline{d}^1_i)^{-1} & 0 & -(\underline{d}^1_i)^{-1} \cdot s^+_{i+1} \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.75}$$

One verifies that $Id - s_i g_i = t_{i+1} d^2_i + d^2_{i-1} t_i$. □

Let $0 \to \mathcal{C}^1 \xrightarrow{f} \mathcal{C}^2 \xrightarrow{g} \mathcal{C}^3 \to 0$ be an exact sequence of $\mathcal{A}$-cochain complexes of finite type. Recall that it induces the long weakly exact sequence in cohomology (1.52)

$$\cdots \to H^i(\mathcal{C}^1) \xrightarrow{H(f_i)} H^i(\mathcal{C}^2) \xrightarrow{H(g_i)} H^i(\mathcal{C}^3) \xrightarrow{H(\delta_i)} H^{i+1}(\mathcal{C}^1) \to \dots \tag{1.52}$$

which will be also viewed as a cochain complex and denoted by $\mathcal{H}$.

**Proposition 1.13.** *Let $0 \to \mathcal{C}^1 \xrightarrow{f} \mathcal{C}^2 \xrightarrow{g} \mathcal{C}^3 \to 0$ be an exact sequence of $\mathcal{A}$-cochain complexes of finite type.*
*(i) If $\mathcal{C}^1$, $\mathcal{C}^3$ is of determinant class and $H(\delta_i) : H^i(\mathcal{C}^3) \to H^{i+1}(\mathcal{C}^1)$ is of determinant class for any $i$, then $\mathcal{C}^2$ is of determinant class.*
*(ii) If $\mathcal{C}^1$, $\mathcal{C}^2$ are of determinant class and $H(f_i) : H^i(\mathcal{C}^1) \to H^i(\mathcal{C}^2)$ is of determinant class for any $i$, then $\mathcal{C}^3$ is of determinant class.*

*(iii) If $\mathcal{C}^2$, $\mathcal{C}^3$ are of determinant class and $H(g_i) : H^i(\mathcal{C}^2) \to H^i(\mathcal{C}^3)$ is of determinant class for any $i$, then $\mathcal{C}^1$ is of determinant class.*

*Proof.* (i) Recall that

$$\underline{d}_i^2 = \begin{pmatrix} \begin{pmatrix} d_{i,36}^2 & 0 \\ d_{i,46} & d_{i,47} \end{pmatrix} & \begin{pmatrix} d_{i,38}^2 \\ d_{i,48}^2 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} & d_{i,58}^2 \end{pmatrix}.$$

¿From Proposition 1.1 we conclude that $\underline{d}_i^2$ is of determinant class iff $d_{i,36}^2, d_{i,47}^2$ and $d_{i,58}^2$ are of determinant class. In view of Lemma 1.9 it then follows that $\mathcal{C}^2$ is of determinant class, using that $\underline{d}_i^1, \underline{d}_i^3$ and $\gamma_i$ are of determinant class for any $i$.
(ii) By Lemma 1.11(A) (or statement (i) above), the mapping cone $\mathcal{C}(f)$ is of determinant class. As $\mathcal{C}(f)$ is homotopy equivalent to $\mathcal{C}^3$ (Lemma 1.11(A)) it follows therefore from Proposition 1.3(B) that $\mathcal{C}^3$ is of determinant class.
(iii) Applying again Lemma 1.11(A) and (B) one concludes that the mapping cone $\mathcal{C}(g)$ is of determinant class and homotopy equivalent to $S\mathcal{C}^1$. Thus Proposition 1.3(B) implies that $\mathcal{C}^1$ is of determinant class. □

**Theorem 1.14.** *If three out of the four cochain complexes $\mathcal{C}^1, \mathcal{C}^2, \mathcal{C}^3, \mathcal{H}$ are of determinant class then so is the fourth and one has the following equality*

$$\begin{aligned} \log T(\mathcal{C}^2) &= \log T(\mathcal{C}^1) + \log T(\mathcal{C}^3) + \log T(\mathcal{H}) \\ &\quad - \sum_i (-1)^i \log T(0 \to \mathcal{C}_i^1 \to \mathcal{C}_i^2 \to \mathcal{C}_i^3 \to 0). \end{aligned} \tag{1.75}$$

*Proof.* In view of Proposition 1.13 it remains to prove, apart from formula (1.75), that $\mathcal{H}$ is of determinant class if $\mathcal{C}^1, \mathcal{C}^2$ and $\mathcal{C}^3$ are of determinant class. Notice that the following statements hold:
(i) If $\mathcal{C}^2$ is of determinant class then $d_{i,36}^2$ is of determinant class which implies that $H(\delta_i)$ is of determinant class for any $i$ (cf Lemma 1.9(A)).
(ii) If $\mathcal{C}^3$ is of determinant class then $\mathcal{C}(f)$ is of determinant class (Lemma 1.11(B), Proposition 1.3(B)). Moreover, by Proposition 1.13(i), the morphisms $H^i(S\mathcal{C}^1) \to H^{i+1}(\mathcal{C}^2)$ are of determinant class which is equivalent to $H(f_{i+1})$ being of determinant class (cf (1.70)).
(iii) If $\mathcal{C}^1$ is of determinant class then $\mathcal{C}(g)$ is of determinant class (Lemma 1.11(B), Proposition 1.3(B)) and, by Proposition 1.13(i), $H(g_i) : H^i(\mathcal{C}^2) \to H^i(\mathcal{C}^3)$ is of determinant class for any $i$.
Combining (i)-(iii) one concludes that if $\mathcal{C}^1, \mathcal{C}^2, \mathcal{C}^3$ are of determinant class, then $\mathcal{H}$ is of determinant class.

To prove formula (1.75) consider the commutative diagram

(1.76)

$$0 \to \quad \mathcal{C}_i^{1,-} \quad \overset{\begin{pmatrix} 0 \\ \alpha_{i,74} \\ 0 \end{pmatrix}}{\to} \quad \mathcal{C}_i^{2,-0} \oplus \mathcal{C}_i^{2,-+} \oplus \mathcal{C}_i^{2,--} \quad \overset{(0\,0\,\beta_{i,48})}{\longrightarrow} \quad \mathcal{C}_i^{3,-} \to \quad 0$$

$$\downarrow \underline{d_i^1} \qquad\qquad \downarrow \underline{d_i^2} \qquad\qquad \downarrow \underline{d_i^3}$$

(1.77)

$$0 \to \mathcal{C}_{i+1}^{1,+} \quad \overset{\begin{pmatrix} 0 \\ \alpha_{i+1,43} \\ 0 \end{pmatrix}}{\to} \quad \mathcal{C}_{i+1}^{2,+0} \oplus \mathcal{C}_{i+1}^{2,++} \oplus \mathcal{C}_{i+1}^{2,+-} \quad \overset{(0\,0\,\beta_{i+1,35})}{\longrightarrow} \quad \mathcal{C}_{i+1}^{3,+} \to 0.$$

Notice that $\underline{d_i^1}, \underline{d_i^2}$ and $\underline{d_i^3}$ establish a weak isomorphism of determinant class between the complexes (1.76) and (1.77). By Lemma 1.9 both complexes (1.76) and (1.77) are of determinant class. By Proposition 1.3, for any $i$

$$\begin{aligned} \log T\,(0 \to \ \mathcal{C}_{i+1}^{1,+} \to \ \mathcal{C}_{i+1}^{2,+} \to \ \mathcal{C}_{i+1}^{3,+} \to 0\,) \\ = \log T\,(0 \to \ \mathcal{C}_i^{1,-} \to \ \mathcal{C}_i^{2,-} \to \ \mathcal{C}_i^{3,-} \to 0\,) \\ - \log \mathrm{Vol}(\underline{d_i^1}) + \log \mathrm{Vol}(\underline{d_i^2}) - \log \mathrm{Vol}(\underline{d_i^3}) - \log \mathrm{Vol}(d_{i,36}^2), \end{aligned} \tag{1.78}$$

where we used that $H(\underline{d_i^1}) = 0 = H(\underline{d_i^3})$ and $H(\underline{d_i^2}) = d_{i,36}^2$. Rearrange terms in (1.78), multiply by $(-1)^{i+1}$ and sum over $i$ to obtain

$$\begin{aligned} \sum_i (-1)^i \log \mathrm{Vol}(\underline{d_i^1}) - \sum_i (-1)^i \log \mathrm{Vol}(\underline{d_i^2}) + \sum_i (-1)^i \log \mathrm{Vol}(\underline{d_i^3}) \\ = -\sum_i (-1)^i \log T(0 \to \mathcal{C}_{i+1}^{1,+} \to \mathcal{C}_{i+1}^{2,+} \to \mathcal{C}_{i+1}^{3,+} \to 0) \\ + \sum_i (-1)^i \log T(0 \to \mathcal{C}_i^{1,-} \to \mathcal{C}_i^{2,-} \to \mathcal{C}_i^{3,-} \to 0) \\ - \sum_i (-1)^i \log \mathrm{Vol}(d_{i,36}^2). \end{aligned} \tag{1.79}$$

Recall formula (1.60), $d_{i,36}^2 = \alpha_{i+1,31}\gamma_i\beta_{i,26}$. Using that both $d_{i,36}^2$ and $\gamma_i$ are weak isomorphisms of determinant class and, according to Lemma 1.6 which states that $\alpha_{i+1,31}$ and $\beta_{i,26}$ are isomorphisms we conclude from (1.4) that

$$\log \mathrm{Vol} d_{i,36}^2 = \log \mathrm{Vol} \alpha_{i+1,31} + \log \mathrm{Vol} \gamma_i + \log \mathrm{Vol} \beta_{i,26}. \tag{1.80}$$

Substituting (1.80) into (1.79) and taking into account that

$$\log T(\mathcal{H}) = \sum_i (-1)^i \log \mathrm{Vol}(\alpha_{i,12}) - \sum (-1)^i \log \mathrm{Vol}(\beta_{i,12}) + \sum (-1)^i \log \mathrm{Vol}(\gamma_i),$$

one obtains

$$\begin{aligned}
(1.81)\quad & \log T(\mathcal{C}^1) - \log T(\mathcal{C}^2) + \log T(\mathcal{C}^3) \\
&= -\log T(\mathcal{H}) - \sum_i (-1)^i \log \mathrm{Vol}\alpha_{i+1,31} - \sum_i (-1)^i \log \mathrm{Vol}\beta_{i,26} \\
&\quad - \sum_i (-1)^i \log \mathrm{Vol}\beta_{i,12} + \sum_i (-1)^i \log \mathrm{Vol}\alpha_{i,12} \\
&\quad + \sum_i (-1)^i \log T(0 \to \mathcal{C}_i^{1,-} \to \mathcal{C}_i^{2,-} \to \mathcal{C}_i^{3,-} \to 0) \\
&\quad - \sum_i (-1)^i \log T(0 \to \mathcal{C}_{i+1}^{1,+} \to \mathcal{C}_{i+1}^{2,+} \to \mathcal{C}_{i+1}^{3,+} \to 0).
\end{aligned}$$

By Lemma 1.10,

$$\begin{aligned}
(1.82)\qquad & \log T(0 \to \mathcal{C}_{i+1}^{1,+} \to \mathcal{C}_{i+1}^{2,+} \to \mathcal{C}_{i+1}^{3,+} \to 0) \\
&= \log \mathrm{Vol}(\alpha_{i+1,43}) - \log \mathrm{Vol}(\beta_{i+1,35})
\end{aligned}$$

and

$$(1.83)\quad \log T(0 \to \mathcal{C}_i^{1,-} \to \mathcal{C}_i^{2,-} \to \mathcal{C}_i^{3,-} \to 0) = \log \mathrm{Vol}\alpha_{i,74} - \log \mathrm{Vol}\beta_{i,48}.$$

Changing the index of summation from $i+1$ to $i$ where necessary, one obtains

$$\begin{aligned}
\log T(\mathcal{C}^2) &= \log T(\mathcal{C}^1) + \log T(\mathcal{C}^3) + \log T(\mathcal{H}) \\
&\quad - \sum_i (-1)^i \log \mathrm{Vol}\alpha_{i,31} - \sum (-1)^i \log \mathrm{Vol}\alpha_{i,12} \\
&\quad - \sum (-1)^i \log \mathrm{Vol}\alpha_{i,74} - \sum (-1)^i \log \mathrm{Vol}\alpha_{i,43} \\
&\quad + \sum_i (-1)^i \log \mathrm{Vol}\beta_{i,12} + \sum_i (-1)^i \log \mathrm{Vol}\beta_{i,26} \\
&\quad + \sum_i (-1)^i \log \mathrm{Vol}\beta_{i,35} + \sum_i (-1)^i \log \mathrm{Vol}\beta_{i,48} \\
&= \log T(\mathcal{C}^1) + \log T(\mathcal{C}^3) + \log T(\mathcal{H}) - \sum (-1)^i \log T(0 \to \mathcal{C}^1 \to \mathcal{C}^2 \to \mathcal{C}^3 \to 0)
\end{aligned}$$

where, for the last equality, we used again Lemma 1.10. This proves (1.75). □

## 2. Torsions for compact manifolds with boundary.

### 2.1 Reidemeister and analytic torsion.

Let $(M, g)$ be a compact Riemannian manifold with boundary and $(M, \partial_- M, \partial_+ M)$ a bordism. Further let $\mathcal{A}$ be a unital von Neumann algebra and $\mathcal{E} \xrightarrow{p} M$ be a bundle of $\mathcal{A}$-Hilbert modules of finite type equipped with a flat connection $\nabla$ with the property that the inner products $(\cdot, \cdot)$ of $\mathcal{E}_y = p^{-1}(y)$ are parallel. Such a pair $(\mathcal{E}, \nabla)$ will be denoted by $\mathcal{F}$ and refered to as a parallel flat bundle of $\mathcal{A}$-Hilbert modules.

Given a generalized triangulation $\tau = (h, g')$ of $M$, the collection of the unstable manifolds in $M \backslash \partial M$, $W_y^-$, associated to the critical points $y \in Cr(h)$, and the flow, induced by $grad_{g'} h$, provides a relative CW-complex structure on the pair of spaces $(S, \partial_- M)$, $S = \partial_- M \cup_{y \in Cr(h)} W_y^-$ whose open cells are given by $W_y^-$. The inclusion of pairs $(S, \partial_- M) \subset (M, \partial_- M)$ is a homotopy equivalence. Notice that if $\tau = (h, g')$ is a generalized triangulation for $M = (M, \partial_- M, \partial_+ M)$, then $\tau_D = (-h, g')$ is a triangulation for $-M$. Further if $M$, resp. $N$, are two closed manifolds and $\tau' = (h', g')$, resp. $\tau'' = (h'', g'')$, are generalized triangulations, then $\tau_0 := (h' + h'', g' \oplus g'')$ provides a generalized triangulation on $M \times N$. The CW-complex structure induced by $\tau_0$ is the product of the CW-complex structures induced by $\tau'$ and $\tau''$. Notice that if $(M, \partial_- M, \partial_+ M)$ is a bordism and $\partial M \neq \emptyset$, then $h_0 := h' + h''$ is not constant on the boundary components $\partial_\pm M \times N$, and therefore $\tau_0$ is not a generalized triangulation as defined in the introduction. However it is possible to modify $h_0$ in an arbitrary small neighborhood of $\partial M \times N$, so that the new function h has the same critical points as $h_0$, $\tau := (h, g' \oplus g'')$ is a generalized triangulation and the relative CW-complex structure induced by $\tau$ is the product of the relative CW complex structure induced by $\tau'$ and the CW complex structure induced by $\tau''$. The generalized triangulation $\tau$ will be refered to as a product triangulation $\tau = \tau' \times \tau''$.

Denote by $M = \cup M_j$ the disjoint union of the connected components $M_j$ of a manifold $M$. Let $\widetilde{M}_j$ be the universal cover of $M_j$ and introduce $\widetilde{M} = \cup \widetilde{M}_j$. Let $\widetilde{h}$ be the lift of $h$ to $\widetilde{M}$, and $\widetilde{g}'$ be the pull back of the Riemannian metric $g'$ on $\widetilde{M}$. Then the gradient vector field $grad_{\widetilde{g}'} \widetilde{h}$ satisfies the Morse-Smale condition as well.

For an arbitrary critical point $\widetilde{y} \in Cr(\widetilde{h})$, choose an orientation $\mathcal{O}_{\widetilde{y}}$ of the descending manifold $W_{\widetilde{y}}^-$ and let $\mathcal{O}_h := \{\mathcal{O}_{\widetilde{y}}; \widetilde{y} \in Cr(h)\}$. Given this choice one constructs, as in [BFKM, section 4], the cochain complex $\left(\mathcal{C}^q(M, \partial_- M, \tau, \mathcal{O}_h, \mathcal{F}), \delta_q\right)$ of $\mathcal{A}$-Hilbert modules of finite type.

Denote by $\Delta_q^{comb}$ the combinatorial Laplacians

$$\Delta_q^{comb} := \delta_q^* \delta_q + \delta_{q-1} \delta_{q-1}^*.$$

Here $\delta_q^*$ denotes the adjoint of $\delta_q$. Observe that $\Delta_q^{comb}$ is a bounded, nonnegative selfadjoint $\mathcal{A}$-linear operator on $\mathcal{C}^q$ and for any $\epsilon > 0$ one can define its regularized determinant in the von Neumann sense, by the following Stieltjes integral

$$\log\, det_N(\Delta_q^{comb}+\epsilon) := \int_{0+}^{\infty} \log(\lambda+\epsilon) dN_{\Delta_q^{comb}}(\lambda)$$

where $N_{\Delta_q^{comb}}(\lambda)$ is the spectral distribution function associated to $\Delta_q^{comb}$.

The function $\log\, det_N(\Delta_q^{comb}+\cdot)$ is an element in the vector space $\mathbb{D}$ consisting of equivalence classes $[f]$ of real analytic functions $f:(0,\infty)\to\mathbb{R}$ with $f\sim g$ iff $\lim_{\lambda\to 0}\big(f(\lambda)-g(\lambda)\big)=0$. (The elements of $\mathbb{D}$ represented by the constant functions form a subspace of $\mathbb{D}$ which can be identified with $\mathbb{R}$, the space of real numbers.) We then define $T_{comb}=T_{comb}(M,\partial_- M,\tau,\mathcal{F})$ as the following element in $\mathbb{D}$

$$\log T_{comb} = \frac{1}{2}\sum_q (-1)^{q+1} q \log\, det(\Delta_q^{comb}+\cdot).$$

One can show that $\log T_{comb}$ is independent of the choice of orientations $\mathcal{O}_h$.

To define the analytic torsion we first have to introduce some more notation.

Denote by $\Lambda^q(M;\mathcal{E})$ the $\mathcal{A}$-module of all smooth $q$-forms and let $d_q:\Lambda^q(M;\mathcal{E})\to\Lambda^{q+1}(M;\mathcal{E})$ be the exterior differential induced by the exterior differential on scalar valued $q$-forms and the covariant differentiation given by the connection $\nabla$ on $\mathcal{E}$. Then $\big(\Lambda^*(M;\mathcal{E}),d_*\big)$ is a cochain complex of $\mathcal{A}$-modules. The Riemannian metric $g$ together with the connection $\nabla$ induce the Hodge $*$-operators $J_q:\Lambda^q(M;\mathcal{E})\to\Lambda^{d-q}(M;\mathcal{E})$, denoted by $*$. The operator $J_q$ induces a scalar product $\langle\cdot,\cdot\rangle$ on $\Lambda^q(M;\mathcal{E})$, with $\langle\omega',\omega''\rangle$ given by $\big(\omega',\omega''$ in $\Lambda^q(M;\mathcal{E})\big)$

$$\langle\omega',\omega''\rangle = \int_M \omega'\wedge_{\mathcal{E}} *\omega'' \tag{2.1}$$

where $\wedge\equiv\wedge_{\mathcal{E}}$

is defined by the composition

$$\Lambda^q(M;\mathcal{E})\times\Lambda^k(M;\mathcal{E})\to\Lambda^{q+k}(M;\mathcal{E}\otimes\mathcal{E})\to\Lambda^{q+k}(M).$$

Denote by $d_{q-1}^*:\Lambda^q(M;\mathcal{E})\to\Lambda^{q-1}(M;\mathcal{E})$ the formal adjoint of $d_{q-1}$, i.e

$$d_{q-1}^* = (-1)^{d(q-1)+1} * d_{d-q} *\,.$$

Define the submodules $\Lambda_1^q(M;\mathcal{E})$ and $\Lambda_2^q(M;\mathcal{E})$ of $\Lambda^q(M;\mathcal{E})$

$$\Lambda_1^q(M;\mathcal{E}) := \{\omega\in\Lambda^q(M;\mathcal{E});\ i^{\#}_{\partial_- M}\omega=0;\ i^{\#}_{\partial_+ M}(*\omega)=0\} \tag{2.2}$$

and

$$\begin{aligned}\Lambda_2^q(M;\mathcal{E}) := \{\omega\in\Lambda^q(M;\mathcal{E}); i^{\#}_{\partial_- M}\omega=0;\ i^{\#}_{\partial_+ M}(*\omega)=0;\\ i^{\#}_{\partial_- M}(d^*_{q-1}\omega)=0; i^{\#}_{\partial_+ M}\big(d^*_{d-q-1}(*\omega)\big)=0\}\end{aligned} \tag{2.3}$$

where $i^{\#}_{\partial_\pm M} : \Lambda^*(M;\mathcal{E}) \to \Lambda^*(\partial M_\pm;\mathcal{E})$ are the pullbacks induced by the embeddings $i_{\partial_\pm M} : \partial_\pm M \hookrightarrow M$.

Consider the Laplacians acting on $q$-forms in $\Lambda^q_2(M;\mathcal{E})$,

$$\Delta_q := d_q^* d_q + d_{q-1} d_{q-1}^*.$$

The operators $\Delta_q$ are essentially self-adjoint, nonnegative, elliptic and $\mathcal{A}$-linear. For any $\epsilon > 0$ one can define its regularized determinant in the von Neumann sense,

$$\log\, det_N(\Delta_q + \epsilon) = -\frac{d}{ds}\zeta_{\Delta_q+\epsilon}(s)|_{s=0} - \epsilon \log(tr_N P_q)$$

where $P_q$ denotes the orthogonal projection onto the null space of $\Delta_q$ and $\zeta_{\Delta_q+\epsilon}(s)$ is the meromorphic continuation of the function defined for $\mathcal{R}e(s) > d/2$ by the formula (cf. [BFKM])

$$\zeta_{\Delta_q+\epsilon}(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} e^{-t\epsilon} tr_N e^{-t\Delta_q} dt.$$

Then $\log\, det_N(\Delta_q + \cdot)$ is an element in the vector space $\mathbb{D}$ introduced above and the analytic torsion $T_{an} = T_{an}(M, \partial_- M, g, \mathcal{F})$ is defined as the following element in $\mathbb{D}$

$$\log\, T_{an} = \frac{1}{2}\sum_q (-1)^{q+1} q \log\, det(\Delta_q + \cdot).$$

Denote by $L_2\big(\Lambda^q(M;\mathcal{E})\big)$ the completion of $\Lambda^q(M;\mathcal{E})$ with respect to the scalar product $\langle\cdot,\cdot\rangle$. $L_2\big(\Lambda^q(M;\mathcal{E})\big)$ is an $\mathcal{A}$-Hilbert module. Notice that the completions of $\big(\Lambda^q_j(M;\mathcal{E})\big)$ $(j = 1, 2)$ give the same space $L_2\big(\Lambda^q(M;\mathcal{E})\big)$.

Introduce the space of harmonic forms,

$$\mathcal{H}_q(M;\mathcal{E}) := \{\omega \in \Lambda^q_2(M;\mathcal{E});\ d_q\omega = 0;\ d^*_{q-1}\omega = 0\}.$$

It is an immediate consequence of the definition of $\Lambda^q_j(M;\mathcal{E}), j = 1, 2$, that the Hodge $*$ operators induce isometries (Poincaré duality)

$$\begin{aligned} \Lambda^q_j(M;\mathcal{E}) &\to \Lambda^{d-q}_j(-M;\mathcal{E}) \\ \mathcal{H}_q(M;\mathcal{E}) &\to \mathcal{H}_{d-q}(-M;\mathcal{E}). \end{aligned}$$

(Recall that $-M$ is obtained from the bordism $(M, \partial_- M, \partial_+ M)$ by interchanging the role of $\partial_+ M$ and $\partial_- M$.)

By the theory of elliptic differential $\mathcal{A}$-operators one concludes that, for any $q$, $\mathcal{H}_q(M;\mathcal{E})$ is an $\mathcal{A}$-Hilbert module of finite von Neumann dimension (in fact it is of finite type). Denote by $P_q(\lambda) : \Lambda^q(M;\mathcal{E}) \to \Lambda^q_2(M;\mathcal{E})$ the spectral projections corresponding to the Laplacian $\Delta_q$ and the boundary operator

$$\begin{aligned} B_q : \Lambda^q(M;\mathcal{E}) &\to \Lambda^q(\partial_- M;\mathcal{E}) \times \Lambda^{q-1}(\partial_- M;\mathcal{E}) \times \Lambda^{d-q}(\partial_+ M;\mathcal{E}) \times \Lambda^{d-q-1}(\partial_+ M;\mathcal{E}) \\ \omega &\mapsto \big(i^{\#}_{\partial_- M}(\omega), i^{\#}_{\partial_- M}(d^*_{q-1}\omega), i^{\#}_{\partial_+ M}(*\omega), i^{\#}_{\partial_+ M}(d^*_{d-q-1}(*\omega))\big). \end{aligned} \tag{2.4}$$

**Proposition 2.1.** *(Hodge decomposition) For any $0 \le q \le d$, there exist orthogonal decompositions of $\mathcal{A}$-pre Hilbert modules*

$$(HD) \qquad \Lambda^q(M;\mathcal{E}) = \mathcal{H}_q(M;\mathcal{E}) \oplus \text{ closure } d_{q-1}\big(\Lambda_1^{q-1}(M;\mathcal{E})\big) \oplus \text{ closure } d_q^*\big(\Lambda_1^{q+1}(M;\mathcal{E})\big)$$

*where the word closure refers to the closure with respect to the usual $C^\infty-$topology.*

$$(HD)_1 \qquad \Lambda_1^q(M;\mathcal{E}) = \mathcal{H}_q(M;\mathcal{E}) \oplus \text{ closure } d_{q-1}\big(\Lambda_2^{q-1}(M;\mathcal{E})\big) \oplus \text{ closure } d_q^*\big(\Lambda_2^{q+1}(M;\mathcal{E})\big)$$

*and*

$$\overline{(HD)} \qquad L_2\big(\Lambda_1^q(M;\mathcal{E})\big) = \mathcal{H}_q(M;\mathcal{E}) \oplus L_2\Big(d_{q-1}\big(\Lambda_2^{q-1}(M;\mathcal{E})\big)\Big) \oplus L_2\Big(d_q^*\big(\Lambda_2^{q+1}(M;\mathcal{E})\big)\Big).$$

*Proof.* Firs observe that $\mathcal{H}_q(M;\mathcal{E}), d_{q-1}\big(\Lambda_2^{q-1}(M;\mathcal{E})\big)$ and $d_q^*\big(\Lambda_2^{q+1}(M;\mathcal{E})\big)$ are $\mathcal{A}$-submodules of $\Lambda_1^q(M;\mathcal{E})$ and $\mathcal{H}_q(M;\mathcal{E}), d_{q-1}\big(\Lambda_1^{q-1}(M;\mathcal{E})\big)$ and $d_q^*\big(\Lambda_1^{q+1}(M;\mathcal{E})\big)$ are $\mathcal{A}$-submodules of $\Lambda^q(M;\mathcal{E})$. In each of the two cases the three spaces are, due to Stokes' theorem, pairwise orthogonal with respect to $\langle\cdot,\cdot\rangle$. Moreover $\overline{(HD)}$ follows from $(HD)$ or $(HD)_1$.

Thus it remains to check that an arbitrary element $\omega$ in $\Lambda^q(M;\mathcal{E})$ or $\Lambda_1^q(M;\mathcal{E})$ can be decomposed as stated. With this in mind, we introduce for $\epsilon > 0$ the operator $G_{q;\epsilon} : \Lambda^q(M;\mathcal{E}) \to \Lambda_2^q(M;\mathcal{E})$, which vanishes on the subspace $P_q(\epsilon)\Lambda^q(M;\mathcal{E})$ and is equal to $\Delta_q^{-1}$ (considered with boundary conditions defined by the boundary operator (2.4)) on $\big(Id - P_q(\epsilon)\big)\Lambda^q(M;\mathcal{E})$.

Notice that $\Delta_q G_{q;\epsilon} = Id - P_q(\epsilon)$. Thus, for $\omega \in \Lambda^q(M;\mathcal{E})$, one has

$$\omega = P_q(\epsilon)\omega + d_{q-1}d_{q-1}^*G_{q;\epsilon}\omega + d_q^*d_qG_{q;\epsilon}\omega$$

and therefore

$$(2.5) \qquad \omega - \big(P_q(\epsilon) - P_q(0)\big)\omega = P_q(0)\omega + d_{q-1}d_{q-1}^*G_{q;\epsilon}\omega + d_q^*d_qG_{q;\epsilon}\omega.$$

Since $G_{q;\epsilon}\omega \in \Lambda_2^q(M;\mathcal{E})$

$$(2.5') \qquad d_{q-1}^*G_{q;\epsilon}\omega \in \Lambda_1^{q-1}(M;\mathcal{E}) \text{ and } d_qG_{q;\epsilon}\omega \in \Lambda_1^{q+1}(M;\mathcal{E}).$$

One concludes that (2.5) is a decomposition of type $(HD)$ for $\omega_\epsilon = \omega - (P_q(\epsilon) - P_q(0))\omega$. Let us show that if $\omega \in \Lambda_1^q(M;\mathcal{E})$ then

$$d_{q-1}^*G_{q;\epsilon}\omega \in \Lambda_2^{q-1}(M;\mathcal{E}) \text{ and } d_qG_{q;\epsilon}\omega \in \Lambda_2^{q+1}(M;\mathcal{E}).$$

In view of (2.3),(2.5') and $d \cdot d = 0$ it remains to check that

$$(2.6) \qquad i^{\#}_{\partial_+M}\big( * d_{q-1}d_{q-1}^*G_{q;\epsilon}\omega\big) = 0 \text{ and } i^{\#}_{\partial_-M}\big(d_q^*d_qG_{q;\epsilon}\omega\big) = 0.$$

Observe that

$$*d_{q-1}d^*_{q-1}G_{q;\epsilon}\omega = *\big(\Delta_q - d^*_q d_q\big)G_{q;\epsilon}\omega = *\omega - *P_q(\epsilon)\omega \pm d_{d-q-1} * d_q G_{q;\epsilon}\omega$$

and

$$d^*_q d_q G_{q;\epsilon}\omega = \big(\Delta_q - d_{q-1}d^*_{q-1}\big)G_{q;\epsilon}\omega = \omega - P_q(\epsilon)\omega - d_{q-1}d^*_{q-1}G_{q;\epsilon}\omega.$$

But $i^{\#}_{\partial_+ M} * \omega = 0$ and $i^{\#}_{\partial_- M}\omega = 0$ (as $\omega \in \Lambda^q_1(M;\mathcal{E})$), $i^{\#}_{\partial_+ M} * P_q(\epsilon)\omega = 0$ and $i^{\#}_{\partial_- M}P_q(\epsilon)\omega = 0$ (as $P_q(\epsilon)\omega \in \Lambda^q_2(M;\mathcal{E})$) and $i^{\#}_{\partial_+ M}d_{d-q-1} * d_q G_{q;\epsilon}\omega = d_{d-q-1}i^{\#}_{\partial_+ M} * d_q G_{q;\epsilon}\omega = 0$ and $i^{\#}_{\partial_- M}d_{q-1}d^*_{q-1}G_{q;\epsilon}\omega = d_{q-1}i^{\#}_{\partial_- M}d^*_{q-1}G_{q;\epsilon} * \omega = 0$ (as $G_{q;\epsilon}\omega \in \Lambda^q_2(M;\mathcal{E})$).

Consequently the decompositions $(HD)$ and $(HD)_1$ hold for $\omega_\epsilon = \omega - (P_q(\epsilon) - P_q(0))\omega$. Note that $\lim_{\epsilon\to 0}\omega_\epsilon = \omega$ in $\Lambda^q(M;\mathcal{E})$ with respect to the $C^\infty-$topology and all spaces in $(HD)$ and $(HD)_1$ are closed in this topology. □

Define

$$\Lambda^q(M,\partial_- M;\mathcal{E}) := \{\omega \in \Lambda^q(M;\mathcal{E});\ i^{\#}_{\partial_- M}\omega = 0\}. \tag{2.7}$$

Notice that $(\Lambda^q(M,\partial_- M;\mathcal{E}), d_q)$ is a subcomplex. The integration $Int^{(q)}$ on the $q$-cells of a generalized triangulation $\tau = (h, g')$, which is given by the unstable manifolds of $grad_{g'}h$, defines an $\mathcal{A}$-linear map

$$Int^{(q)} : \Lambda^q(M,\partial_- M;\mathcal{E}) \to \mathcal{C}^q(M,\partial_- M,\tau,\mathcal{O}_h;\mathcal{F})$$

so that $\delta_q Int^{(q)} = Int^{(q+1)}d_q$. Denote by $\pi_q$ the canonical projection $\pi_q : \mathcal{C}^q \to Null\,(\Delta^{comb}_q)$. By a theorem of Dodziuk [Do] of de Rham type, generalized for manifolds with boundary, the map $\pi_q\, Int^{(q)}$, restricted to $\mathcal{H}_q$, is an isomorphism of Hilbert modules. Denote its inverse by $\theta_q$. Define the metric part $T_{met} = T_{met}(M,\partial_- M, g, \tau, \mathcal{F})$ of the Reidemeister torsion by

$$\log T_{met} := \frac{1}{2}\sum_q (-1)^q \log\, det_N(\theta^*_q\theta_q).$$

$\log T_{met}$ is independent of the choice of the orientations $\mathcal{O}_h$. The Reidemeister torsion $T_{Re} = T_{Re}(M,\partial_- M, g, \tau, \mathcal{F})$ is defined as an element in $\mathbb{D}$ by

$$\log T_{Re} = \log\, T_{comb} + \log\, T_{met}.$$

In the remaining of this subsection we will discuss the behavior of the torsions with respect to the Poincaré duality and the product formula for torsions.

Suppose $M$ is oriented. Then the orientations $\mathcal{O}_h$ of the unstable manifolds corresponding to the gradient vector field $grad_{g'}h$ induce orientations on the stable manifolds of $grad_{g'}h$. These manifolds can be identified with the stable manifolds of $grad_{g'}(-h)$ with the orientations $\mathcal{O}_{-h}$. Since the critical points of index $k$ of $h$

are the critical points of index $d-k$ of $-h$ one obtains isometries of Hilbert modules $J_q^{comb}: \mathcal{C}^q(M,\partial_- M,\tau;\mathcal{F}) \to \mathcal{C}^{d-q}(M,\partial_+ M,\tau_D;\mathcal{F})$ ( which induce Poincaré duality) and verifies that

$$J_{q+1}^{comb} \cdot \delta_{q;\tau} = (-1)^{q(d-q)} \delta^*_{d-q-1;\tau_D} \cdot J_q^{comb} \tag{2.8}$$

where $\delta_{q;\tau}$ resp. $\delta_{q;\tau_D}$ are the differentials in the cochain complexes $\mathcal{C}(M,\partial_- M,\tau;\mathcal{F})$ resp. $\mathcal{C}(M,\partial_+ M,\tau_D;\mathcal{F})$ and $\delta^*_{q;\tau}$ denotes the adjoint of $\delta_{q;\tau}$. As in section 4 of [BFKM] denote by $\overline{Int}_\tau^{(q)}$ the restriction of $\pi_{q,\tau} Int_\tau^{(q)}$ to $\mathcal{H}_q(M;\mathcal{E})$, where $\pi_{q,\tau}$ is the orthogonal projection on $Null(\Delta_{q,\tau}^{comb})$, and by $(\overline{Int}_\tau^{(q)})^*$ the adjoint of $\overline{Int}_\tau^{(q)}$. Formula (2.8) implies that $J_q^{comb}(Null(\Delta_{q,\tau}^{comb})) = Null(\Delta^{comb}_{d-q,\tau_D})$ The same arguments as in [BFKM] Proposition 5.15 lead to

$$J_q = (\overline{Int}_{\tau_D}^{(d-q)})^* J_q^{comb} \overline{Int}_\tau^{(q)} \tag{2.9}$$

By using 2.8 and 2.9 one obtains

**Proposition 2.2.** *The following identities hold in* $\mathbb{D}$ :
*(i)* $\log T_{comb}(M,\partial_- M,\tau;\mathcal{F}) = (-1)^{d+1} \log T_{comb}(M,\partial_+ M,\tau_D;\mathcal{F})$;
*(ii)* $\log T_{an}(M,\partial_- M,g;\mathcal{F}) = (-1)^{d+1} \log T_{an}(M,\partial_+ M,g;\mathcal{F})$;
*(iii)* $\log T_{met}(M,\partial_- M,g,\tau;\mathcal{F}) = (-1)^{d+1} \log T_{met}(M,\partial_+ M,g,\tau_D;\mathcal{F})$.

**Proposition 2.3.** *Let* $(M,\partial_- M,\partial_+ M)$ *be a bordism equipped with a Riemannian metric* $g$*, a generalized triangulation* $\tau$ *and a parallel flat bundle* $\mathcal{F}$ *of* $\mathcal{A}$*-Hilbert modules of finite type. Let* $N$ *be a closed manifold equipped with a Riemannian metric* $g_N$*, a generalized triangulation* $\tau_N$ *and a parallel flat bundle* $\mathcal{F}_N$ *of* $\mathcal{A}_N$*−Hilbert modules of finite type. Denote by* $\underline{\tau}$ *a product triangulation and by* $\underline{\mathcal{A}}$ *the von Neumann algebra* $\underline{\mathcal{A}} = \mathcal{A}\hat{\otimes}\mathcal{A}_N$*. Then* $\mathcal{F}\hat{\otimes}\mathcal{F}_N$ *is a parallel flat bundle of* $\underline{\mathcal{A}}$*−Hilbert modules of finite type.*
*If the pairs* $\{(M,\partial_- M,\partial_+ M),\mathcal{F}\}$ *and* $\{N,\mathcal{F}_N\}$ *are of determinant class, so is the pair* $\{(M\times N,\partial_- M\times N,\partial_+ M\times N),\mathcal{F}\hat{\otimes}\mathcal{F}_N\}$ *and the following equalities hold:*

$$\begin{aligned} &\log T_{an}(M\times N,(\partial_- M)\times N, g\oplus g_N,\mathcal{F}\hat{\otimes}\mathcal{F}_N) = \chi(M,\partial_- M;\mathcal{F}) \\ &\quad \log T_{an}(N,g;\mathcal{F}_N) + \chi(N;\mathcal{F}_N)\log T_{an}(M,\partial_- M,g;\mathcal{F}) \end{aligned} \tag{2.10}$$

$$\begin{aligned} &\log T_{Re}(M\times N,(\partial_- M)\times N, g\oplus g_N,\underline{\tau},\mathcal{F}\hat{\otimes}\mathcal{F}_N) = \chi(M,\partial_- M;\mathcal{F}) \\ &\quad \log T_{Re}(N,g,\tau_N;\mathcal{F}_N) + \chi(N;\mathcal{F}_N)\log T_{Re}(M,\partial_- M,g,\tau;\mathcal{F}). \end{aligned} \tag{2.10'}$$

The proof of Proposition 2.3 is identical to the one of Proposition 4.1 in [BFKM].

### 2.2 Determinant class.

Following [BFKM, section 4.1] we introduce the following

**Definition.** (1) The triple $\{(M, \partial_- M, \partial_+ M), \tau, \mathcal{F}\}$ is said to be of $c$-determinant class iff for $0 \leq k \leq d$

$$-\infty < \int_{0+}^{1} \log \lambda dN_{\Delta_k^{comb}}(\lambda).$$

(2) The triple $\{(M, \partial_- M, \partial_+ M), g, \mathcal{F}\}$ is said to be of $a$-determinant class iff for $0 \leq k \leq d$

$$-\infty < \int_{0+}^{1} \log \lambda dN_{\Delta_k}(\lambda).$$

As in [BFKM] Proposition 5.6, one can conclude from work of Gromov-Shubin [GS] and using Hellfer Sjöstrand theory (cf. also [Ef1],[Ef2]) that the following result holds:

**Proposition 2.4.** *([GS]) The triple $\{(M, \partial_- M, \partial_+ M), g, \mathcal{F}\}$ is of a-determinant class iff $\{(M, \partial_- M, \partial_+ M), \tau, \mathcal{F}\}$ is of c-determinant class.*

Proposition 2.4 allows us to introduce the following

**Definition.** (i) The pair $\{(M, \partial_- M, \partial_+ M), \mathcal{F}\}$ is said to be of determinant class if there exists a generalized triangulation $\tau$ such that $\{(M, \partial_- M, \partial_+ M), \tau, \mathcal{F}\}$ is of $c$-determinant class.
(ii) The system $(M, \partial_- M, \partial_+ M, \mathcal{F})$ is said to be of determinant class if the three pairs $\{(M, \partial_- M, \partial_+ M), \mathcal{F}\}, \{(M, \emptyset, \partial M), \mathcal{F}\}, \{\partial M, \mathcal{F} \restriction_{\partial M}\}$ are of determinant class.

Notice that a system $(M, \partial_- M, \partial_+ M, \mathcal{F})$ being of determinant class implies that the two pairs $\{\partial_\pm M, \mathcal{F} \restriction_{\partial_\pm M}\}$ are of determinant class as well as, by Poincaré duality, that the two pairs $\{(M, \partial_+ M, \partial_- M), \mathcal{F}\}$ and $\{(M, \partial M, \emptyset), \mathcal{F}\}$ are of determinant class.

**Proposition 2.5.** *(1) If the system $(M, \partial_- M, \partial_+ M, \mathcal{F})$ is of determinant class then the long weakly exact sequences in cohomology with coefficients in $\mathcal{F}$ associated to $(M, \partial M)$ and $(M, \partial_\pm M)$ are of determinant class.*
*(2) Suppose that the systems $(M_i, \partial_- M_i, \partial_+ M_i, \mathcal{F}_i)$ $(i = 1, 2)$ have the property that $(\partial_+ M_1, \mathcal{F}_1 \restriction_{\partial_+ M_1}) = (\partial_- M_2, \mathcal{F}_2 \restriction_{\partial_- M_2})$. Let $(M, \partial_- M, \partial_+ M, \mathcal{F})$ be the system where $M$ is obtained by glueing $M_1$ to $M_2$ along $\partial_+ M_1 = \partial_- M_2$ and $\mathcal{F}$ defined by $\mathcal{F}_i$ on $M_i$. If the systems $(M_i, \partial_- M_i, \partial_+ M_i, \mathcal{F}_i)$ are of determinant class then so is the system $(M, \partial_- M, \partial_+ M, \mathcal{F})$.*
*(3) Suppose that each connected component of a closed Riemannian manifold $N$ is simply connected and let $\tilde{\mathcal{F}}$ denote the pullback of $\mathcal{F}$ by the canonical projection of $M \times N$ on $M$. Then the pair $\{(M, \partial_- M, \partial_+ M), \mathcal{F}\}$ is of determinant class iff the pair $\{(M \times N, \partial_- M \times N, \partial_+ M \times N), \tilde{\mathcal{F}}\}$ is.*
*(4) The system $(M, \partial_- M, \partial_+ M, \mathcal{F})$ is of determinant class iff the system $(M, \emptyset, \partial M, \mathcal{F})$ is of determinant class.*

*Proof.* (1) We verify the statement for $(M, \partial_- M)$. Similar arguments can be used for $(M, \partial M)$ and $(M, \partial_+ M)$. Choose generalized triangulations $\tau_- = (h_-, g'_-)$ of

$\partial_- M$ and $\tau = (h, g')$ of $(M, \partial_- M, \partial_+ M)$. The descending manifolds of $grad_{g'_-} h_-$ in $\partial_- M$ and of $grad_{g'} h$ in $M \backslash \partial_- M$ provide a cell structure for the space $X = S \cup \partial_- M$, where $S$ is the union of all open cells of $\tau$ (i.e. of all descending manifolds of $grad_{g'} h$ in $M \backslash \partial_- M$). By an arbitrary small change of the metric $g'_-$ which keeps the CW-complex structure of $\partial_- M$ isomorphic to the unperturbed one, one can arrange that the cell structure is in fact a CW-complex structure, denoted by $\tilde{\tau}$. Choose orientations $\mathcal{O}_h$ resp. $\mathcal{O}_{h_-}$ for the cells of $\tau$ resp. $\tau_-$. These orientations define orientations $\mathcal{O}$ for the cells of $\tilde{\tau}$. Consider the cochain complex $(\mathcal{C}^q(X, \tilde{\tau}, \mathcal{O}; \mathcal{F}), \delta_q)$ of $\mathcal{A}$ -Hilbert modules of finite type associated to the CW-complex $(X, \tilde{\tau})$ and $\mathcal{F}$. The short exact sequence of cochain complexes

$$0 \to \mathcal{C}^*(M, \partial_- M, \tau, \mathcal{O}_h; \mathcal{F}) \to \mathcal{C}^*(X, \tilde{\tau}, \mathcal{O}; \mathcal{F}) \to \mathcal{C}^*(\partial_- M, \tau_-, \mathcal{O}_{h_-}; \mathcal{F} \upharpoonright_{\partial_- M}) \to 0$$

induces a long cohomology sequence which is weakly exact, hence a cochain complex. This complex will be called the cohomology sequence of the pair $(M, \partial_- M)$. Different generalized triangulations give rise to isomorphic (but not necessarily isometric) cochain complexes. Using subdivisions, one can show that any generalized triangulation $\tau' = (h', g'')$ of the bordism $(M, \emptyset, \partial M)$ produces a cochain complex $\mathcal{C}^*(M, \tau', \mathcal{O}_{h'}; \mathcal{F})$ of $\mathcal{A}$-Hilbert modules which is homotopy equivalent to $\mathcal{C}^*(X, \tilde{\tau}, \mathcal{O}; \mathcal{F})$. By hypotheses, $\mathcal{C}^*(M, \tau', \mathcal{O}_{h'}; \mathcal{F})$ is of determinant class and therefore, by Proposition 1.3(B), $\mathcal{C}^*(X, \tilde{\tau}, \mathcal{O}; \mathcal{F})$ is of determinant class as well. Thus, by Theorem 1.14, the cohomology sequence of the pair $(M, \partial_- M)$ is of determinant class.

(2) We verify that the pair $\{(M, \emptyset, \partial M), \mathcal{F}\}$ is of determinant class. Similar arguments can be applied to the pair $\{(M, \partial_- M, \partial_+ M), \mathcal{F}\}$. (Notice that, by assumption, $\{\partial M, \mathcal{F} \upharpoonright_{\partial M}\}$ is of determinant class.) Choose generalized triangulations $\tau_1 = (h_1, g'_1)$ for $(M_1, \partial_+ M_1, \partial_- M_1)$, $\tau_2 = (h_2, g'_2)$ for $(M_2, \partial_- M_2, \partial_+ M_2)$ and $\tau_0 = (h_0, g'_0)$ for $\partial_+ M_1 = \partial_- M_2$. The open cells of $\tau_0, \tau_1, \tau_2$ provide a CW-complex structure $\tilde{\tau}$ (if necessary, modify $g'_0$ slightly) for the space $Y = \partial_+ M_1 \cup S_1 \cup S_2$ where $S_1$ resp. $S_2$ are the union of the open cells of $\tau_1$ resp. $\tau_2$. Choose orientations $\mathcal{O}_{h_0}, \mathcal{O}_{h_1}, \mathcal{O}_{h_2}$ for the cells of $\tau_0, \tau_1, \tau_2$ which define orientations $\mathcal{O}$ for the cells of $\tilde{\tau}$, and consider the associated cochain complex $(\mathcal{C}^q(Y, \tilde{\tau}, \mathcal{O}; \mathcal{F}), \delta_q)$ of $\mathcal{A}-$Hilbert modules of finite type. Consider the short exact sequence of cochain complexes

$$\begin{aligned} 0 \to \mathcal{C}^q(M_1, \partial_+ M_1, \tau_1, \mathcal{O}_{h_1}; \mathcal{F}_1) \oplus \mathcal{C}^q(M_2, \partial_- M_2, \tau_2, \mathcal{O}_{h_2}; \mathcal{F}_2) \to \mathcal{C}^q(Y, \tilde{\tau}, \mathcal{O}; \mathcal{F}) \to \\ \to \mathcal{C}^q(\partial_+ M_1, \tau_0, \mathcal{O}_{h_0}; \mathcal{F} \upharpoonright_{\partial_+ M_1}) \to 0. \end{aligned}$$

Notice that the connecting homomorphisms in the induced cohomology sequence, $H^q(\partial_+ M_1, \tau_0; \mathcal{F} \upharpoonright_{\partial_+ M_1}) \to H^{q+1}(M_1, \partial_+ M_1, \tau_1; \mathcal{F}_1) \oplus H^{q+1}(M_2, \partial_- M_2, \tau_2; \mathcal{F}_2)$, are the sum of the connecting homomorphisms in the cohomology sequences of the pairs $(M_1, \partial_+ M_1)$ and $(M_2, \partial_- M_2)$ and hence of determinant class by (1) and Proposition 1.1 (C). Therefore, Proposition 1.13 (i) together with the made assumptions implies that the complex $\mathcal{C}^*(Y, \tilde{\tau}, \mathcal{O}; \mathcal{F})$ is of determinant class. By Propostion 1.3(B), the homotopy equivalence of $\mathcal{C}^*(Y, \tilde{\tau}, \mathcal{O}; \mathcal{F})$ with the cochain complex associated to an arbitrary generalized triangulation of the bordism $(M, \emptyset, \partial M)$ implies that the pair $\{(M, \partial_- M, \partial_+ M), \mathcal{F}\}$ is of determinant class.
(3) follows from Proposition 1.2.

(4) In view of the definition of a system being of determinant class it suffices to prove that the pair $\{(M, \partial_- M, \partial_+ M); \mathcal{F}\}$ is of determinant class if the system $(M, \emptyset, \partial M, \mathcal{F})$ is of determinant class. First notice the following two facts:
(i) The disjoint union of two systems of determinant class is a system of determinant class and
(ii) if $\{N, \mathcal{F}\}$ is a pair of determinant class ($N$ a closed manifold) and $\mathcal{F}_{[a,b]}$ denotes the pullback of $\mathcal{F}$ by the projection $p : N \times [a, b] \to N$ ($[a, b]$ a compact interval on $\mathbb{R}$) then $(N \times [a, b], N \times a, N \times b; \mathcal{F}_{[a,b]})$ is a system of determinant class.

For $i = 1, 2$ define bordisms $(N_i, \partial_- N_i, \partial_+ N_i)$

$$(N_1, \partial_- N_1, \partial_+ N_1) := (M, \emptyset, \partial M) \sqcup (\partial_- M \times [-1, 0], \partial_- M \times \{-1\}, \partial_- M \times \{0\})$$

$$(N_2, \partial_- N_2, \partial_+ N_2) := (\partial_- M \times [0, 1], \partial_- M \times \{0, 1\}, \emptyset)$$
$$\sqcup (\partial_+ M \times [0, 1], \partial_+ M \times \{0\}, \partial_+ M \times \{1\})$$

and whose parallel flat bundle $\mathcal{G}_i$ of $\mathcal{A}-$Hilbert modules are equal to $\mathcal{F}$ on $M$ and to $(\mathcal{F} \restriction_{\partial_\pm M})_{[a,b]}$ on the other components. (Recall that $A \sqcup B$ denotes the disjoint union of two sets A and B.)
The statements (i) and (ii) above imply that the systems $(N_i, \partial_- N_i, \partial_+ N_i; \mathcal{G}_i)$ are of determinant class. Since $(\partial_+ N_1, \mathcal{G}_1 \restriction_{\partial_+ N_1}) = (\partial_- N_2, \mathcal{G}_2 \restriction_{\partial_- N_2})$ one can glue these two systems to obtain a system $(N, \partial_- N, \partial_+ N, \mathcal{G})$ which by (2) is of determinant class. The result follows once we observe that $(N, \partial_- N, \partial_+ N, \mathcal{G})$ is isomorphic to $(M, \partial_- M, \partial_+ M, \mathcal{F})$. $\square$

**2.3 Witten deformation of the de Rham complex for manifolds with boundary.**

Given a triangulation $\tau = (h, g')$ of $M$, consider the Witten deformation $d_q(t)$ of $d_q$,

$$d_q(t) = e^{-th} d_q e^{th} = d_q + t dh \wedge . \tag{2.11}$$

Noticing that $d^*_{q-1}(t) = (-1)^{d(q-1)+1} * d_{d-q}(-t) *$ we define the spaces $\Lambda^*_{j;t}(M; \mathcal{E}) \equiv \Lambda^*_j(M; \mathcal{E})(t)$ ($j = 1, 2$) and $\mathcal{H}_{*;t}(M; \mathcal{E})$ by replacing in the definition (2.2) - (2.3) the exterior derivative $d_q$ and its adjoint $d^*_q$ with $d_q(t)$ respectively $d^*_q(t)$. We point out that $\Lambda^q_1(M; \mathcal{E}) = \Lambda^q_{1;t}(M; \mathcal{E})$ is independent of $t$.

Using the same proof, the above Proposition 2.1 can be generalized as follows:

**Proposition 2.1'.** *For any $t \in \mathbb{R}$ and $0 \le q \le d$, there exists an orthogonal decomposition of $\mathcal{A}$-pre Hilbert modules*

$$(HD)^t \Lambda^q(M; \mathcal{E}) = \mathcal{H}_{q;t}(M; \mathcal{E}) \oplus \mathit{closure}\, d_{q-1}(t)\big(\Lambda^{q-1}_{1;t}(M; \mathcal{E})\big) \oplus \mathit{closure}\, d^*_q(t)\big(\Lambda^{q+1}_{1;t}(M; \mathcal{E})\big),$$

$$(HD)^t_1 \Lambda^q_1(M; \mathcal{E}) = \mathcal{H}_{q;t}(M; \mathcal{E}) \oplus \mathit{closure}\, d_{q-1}(t)\big(\Lambda^{q-1}_{2;t}(M; \mathcal{E})\big) \oplus \mathit{closure}\, d^*_q(t)\big(\Lambda^{q+1}_{2;t}(M; \mathcal{E})\big)$$

*(where the word closure refers to closure with respect to the $C^\infty-$topology) and*

$$(\overline{HD})^t \quad L_2(\Lambda^q(M;\mathcal{E})) = \mathcal{H}_{q;t}(M;\mathcal{E}) \oplus L_2\Big(d_{q-1}(t)(\Lambda^{q-1}_{2;t}(M;\mathcal{E}))\Big) \oplus L_2\Big(d^*_q(t)(\Lambda^{q+1}_{2;t}(M;\mathcal{E}))\Big).$$

Consider the differential boundary operator

$$(2.4) \quad \begin{aligned} B_q(t) : \Lambda^q(M;\mathcal{E}) &\to \Lambda^q(\partial_- M;\mathcal{E}) \times \Lambda^{q-1}(\partial_- M;\mathcal{E}) \times \Lambda^{d-q}(\partial_+ M;\mathcal{E}) \times \Lambda^{d-q-1}(\partial_+ M;\mathcal{E}) \\ \omega &\mapsto \big(i^\#_{\partial_- M}(\omega), i^\#_{\partial_- M}(d^*_{q-1}(t)\omega), i^\#_{\partial_+ M}(*\omega), i^\#_{\partial_+ M}(d^*_{d-q-1}(t)(*\omega))\big). \end{aligned}$$

One verifies that, for any $t \in \mathbb{R}$, $\big(\Delta_q(t), B_q(t)\big)$ is an elliptic boundary value problem which is essentially selfadjoint and nonnegative. Hence the spectrum $\sigma_q(t) := Spec\,\big(\Delta_q(t), B_q(t)\big)$ of $\big(\Delta_q(t), B_q(t)\big)$ is contained in $[0,\infty)$. Following the proof of Proposition 5.2 in [BFKM] one concludes that there exist constants $C > 0, C' > 0$ and $t_0 > 0$ so that for $t \geq t_0$ and $0 \leq q \leq d$,

$$\sigma_q(t) \cap (e^{-tC}, C't) = \emptyset.$$

Choose $t_0$ so large that

$$e^{-t_0 C} < 1 < C't_0$$

and introduce the orthogonal projections $Q_q(t) : \Lambda^q(M;\mathcal{E}) \to \Lambda^q_{2;t}(M;\mathcal{E})$,

$$Q_q(t) := \frac{1}{2\pi i}\int_{S^1} \big(\lambda - \Delta_q(t)\big)^{-1} d\lambda$$

for $t \geq t_0$, where $\big(\lambda - \Delta_q(t)\big)^{-1}$ is viewed as a map

$$\big(\lambda - \Delta_q(t)\big)^{-1} : \Lambda^q(M;\mathcal{E}) \to \Lambda^q_{2;t}(M;\mathcal{E})$$

and $S^1$ is the circle in the complex $\lambda$-plane, centered at 0, with radius 1.

Introduce the following $\mathcal{A}$-submodules of $\Lambda^q_1(M;\mathcal{E})$,

$$\Lambda^q_t(M;\mathcal{E})_{sm} := Q_q(t)\big(\Lambda^q(M;\mathcal{E})\big)$$

and

$$\Lambda^q_t(M;\mathcal{E})_{la} := \big(Id - Q_q(t)\big)\big(\Lambda^q_1(M;\mathcal{E})\big).$$

Notice that $\Lambda^q_t(M;\mathcal{E})_{sm} \subseteq \Lambda^q_{2;t}(M;\mathcal{E})$ is a $\mathcal{A}$-Hilbert submodule of finite von Neumann dimension. Further $\big(\Lambda^q_t(M;\mathcal{E})_{sm}, d_q(t)\big)$ and $\big(\Lambda^q_t(M;\mathcal{E})_{la}, d_q(t)\big)$ is a subcomplex of $\big(\Lambda^q(M,\partial_- M;\mathcal{E}), d_q(t)\big)$.

Let us now describe $\big(\Lambda^q_t(M;\mathcal{E})_{sm}, d_q(t)\big)$ in more detail.

Recall that the cochain complex $\big(\mathcal{C}^q = \mathcal{C}^q(M,\partial_- M,\tau,\mathcal{O}_h;\mathcal{F}), \delta_q\big)$ is given by

$$\mathcal{C}^q = \underset{x \in Cr_q(h)}{\oplus} \mathcal{E}_x$$

where $Cr_q(h) = \{x_{q;j}; 1 \leq j \leq m_q\}$ denotes the set of critical points of $h$ which are of index $q$. Given $q$ and $1 \leq j \leq m_q := \#Cr_q(h)$, denote by $e_{q;j,i}(1 \leq i \leq \ell)$ an orthonormal basis of $\mathcal{E}_{x_{q;j}}\ (x_{q;j} \in Cr_q(h))$. (Recall that $\mathcal{E}_x$ is assumed to be a free $\mathcal{A}-$Hilbert module (cf introduction).) Define $E_{q;j,i} \in \mathcal{C}^q$ by

$$E_{q;j,i}(x_{q;j'}) = \delta_{j'j} e_{q;j,i}$$

where $\delta_{j'j}$ denotes the Kronecker delta. With respect to this basis, $\delta_q$ is given by

$$\delta_q(E_{q;i,j}) = \sum_{\substack{1 \leq i' \leq m_{q+1} \\ 1 \leq j' \leq \ell}} \gamma_{q;ij,i'j'} E_{q+1;i'j'}.$$

Generalizing results of Helffer-Sjöstrand, one constructs for $t \geq t_0$ an orthonormal base $\varphi_{q;i,j}(t)$ of $\Lambda^q_t(M;\mathcal{E})_{sm}$ to conclude that $\Lambda^q_t(M;\mathcal{E})_{sm}$ is a free $\mathcal{A}$-Hilbert module of finite type.

Let us recall from [BFKM] the notion of a $H$-neighborhood $U_{qj} \equiv U_{x_{q;j}}$ of a critical point $x_{q;j} \in Cr_q(h)$:

**Definition.** $U_{qj} \subseteq M \backslash \partial M$ is said to be an $H$-neighborhood of $x_{q;j}$ if there exist a disc $B_{2\alpha} := \{x \in \mathbb{R}^d; |x| < 2\alpha\}$ and diffeomorphisms $\varphi : B_{2\alpha} \to U_{qj}$ and $\Phi : B_{2\alpha} \times \mathcal{W} \to \mathcal{E} \restriction_{U_{qj}}$ with the following properties:

(i) $\varphi(0) = x_{q;j}$;

(ii) when expressed in the coordinates induced by $\varphi$, $h$ is of the form

$$h(x) = h(x_{q;j}) - \frac{1}{2}\sum_{k=1}^{q} x_k^2 + \frac{1}{2}\sum_{k=q+1}^{d} x_k^2;$$

(iii) the pullback $\varphi^*(g')$ of the Riemannian metric $g'$ is the Euclidean metric;

(iv) $\Phi$ is a trivialization of $\mathcal{E} \restriction_{U_{qj}}$.

For later use, we call the coordinates provided by $\varphi$ $H$-coordinates and define $U'_{qj} := \varphi(B_\alpha)$.

A collection $(U_x)_{x \in Cr(h)}$ of $H$-neighborhoods is called a system of $H$-neighborhoods if, in addition, they are pairwise disjoint.

Taking into account that $h$ is not necessarily a self indexing Morse function one obtains the following version of Theorem 5.7 in [BFKM]. (We recall that we assume throughout this section that the $\mathcal{W}_j$'s are free $\mathcal{A}$-Hilbert modules.)

**Theorem 2.6.** *Assume $(U_x)_{x \in Cr(h)}$ is a system of $H$-neighborhoods such that, for any $q$ and $j \neq j'$*

$$U_{x_{q;j}} \cap W^-_{q;j'} = \emptyset$$

*where $W^-_{q;j'}$ denotes the descending manifold associated to the critical point $x_{q;j'}$ and the gradient flow $grad_{g'}h$.*
*Then there exists $t_1 \geq t_0$ such that for $t > t_1$, the elements $\varphi_{q;j,i}(t)$, constructed in [BFKM] with $1 \leq j \leq m_q, 1 \leq i \leq \ell$ form an orthonormal basis of the $\mathcal{A}$-Hilbert module $\Lambda^q_t(M;\mathcal{E})_{sm}$ with the following properties:*

*(i) There exist $C > 0, \eta > 0$ so that for $t \geq t_1, 1 \leq r \leq \ell$,*

$$\sup_{x\in M\setminus U_{x_{q,j}}} \|\varphi_{q;j,r}(t)(x)\| \leq Ce^{-t\eta}.$$

*(ii) When expressed in $H$-coordinates on $U_{x_{q;j}} \cap W^-_{q,j}$, the $q$-forms $\varphi_{q;j,r}(t)$ satisfy the following estimate*

$$\varphi_{q;j,r}(t)(x) = \left(\frac{t}{\pi}\right)^{d/4} e^{-t|x|^2/2}\left(dx_1 \wedge \cdots \wedge dx_q \otimes e_{q;j,r} + 0(\frac{1}{t})\right).$$

*(iii) With respect to this basis,*

$$Int^{(q)}\big(e^{th}\varphi_{q;j,r}(t)\big) = (\frac{t}{\pi})^{\frac{d-2q}{4}} e^{th(x_{q;j})}\big(E_{q;j,r} + 0(\frac{1}{t})\big).$$

*(iv) With respect to this basis*

$$d_q(t)\varphi_{q;j,r}(t) = \sum_{\substack{1\leq j'\leq m_{q+1}\\ 1\leq r'\leq \ell}} \eta_{q;jr,j'r'}(t)\varphi_{q+1;j',r'}(t)$$

*and the coefficients $\eta_{q;jr,j'r'}$ satisfy*

$$\eta_{q;jr,j'r'}(t) = \left((\frac{t}{\pi})^{1/2}\gamma_{q;jr,j'r'} + 0(1)\right) e^{-t\left(h(x_{q+1,i'})-h(x_{q;i})\right)}.$$

Consider the isomorphism of $\mathcal{A}$-Hilbert modules

$$f_k(t) : \Lambda^k_t(M;\mathcal{E})_{sm} \to \mathcal{C}^k(M,\partial_-M,\tau,\mathcal{O}_h;\mathcal{F})$$

given by

$$f_k(t)\big(\varphi_{k;jr}(t)\big) := \left((\frac{\pi}{t})^{\frac{d-2k}{4}} e^{-th(x_{q;j})}\right) Int^{(k)}(e^{th}\varphi_{k;jr}(t)). \tag{2.12}$$

Then the maps $f_k(t)$ provide an isomorphism between $\big(\Lambda^k_t(M;\mathcal{E})_{sm}, \widetilde{d}_k(t)\big)$ and $\big(\mathcal{C}^k(M,\partial_-M,\tau,\mathcal{O}_h;\mathcal{F}),\delta_k\big)$. As a consequence of Theorem 2.6 (iv), $f_k(t)$, when expressed with respect to the basis $(\varphi_{k;ir})_{i,r}$ of $\Lambda^k_t(M;\mathcal{E})_{sm}$ and the basis $(E_{k;ir})_{i,r}$ of $\mathcal{C}^k(M,\partial_-M,\tau,\mathcal{O}_h;\mathcal{F})$, is of the form

$$f_k(t) = Id + 0(\frac{1}{t}) \quad (t\to\infty).$$

**2.4 Asymptotic expansions and a comparison theorem.**

Suppose that $(M, \partial_- M, \partial_+ M)$ is a bordism, $g$ a Riemannian metric, $\tau = (h, g')$ a generalized triangulation and $\mathcal{F}$ a parallel flat bundle.
Denote by $\log T(M, \partial_- M, g, h, \mathcal{F})(t)$ the analytic torsion defined by the Laplace operators $\Delta_q(t)$ associated to $d_q(t)$ and the metric $g$. For $t$ large enough let $\log T_{sm}(M, \partial_- M, g, h, \mathcal{F})(t)$ and $\log T_{la}(M, \partial_- M, g, h, \mathcal{F})(t)$ be the torsion of the subcomplex $\big(\Lambda^q_t(M; \mathcal{E})_{sm}, d_q(t)\big)$ respectively $\big(\Lambda^q_t(M; \mathcal{E})_{la}, d_q(t)\big)$.
Both,
$\log T(M, \partial_- M, g, h, \mathcal{F})(t)$ and $\log T_{sm}(M, \partial_- M, g, h, \mathcal{F})(t)$ are elements in the vector space $\mathbb{D}$ and $\log T_{la}(M, \partial_- M, g, h, \mathcal{F})(t)$ is a real number. In the case the pair $\{(M, \partial_- M, \partial_+ M), \mathcal{F}\}$ is of determinant class the Laplace operators $\Delta_q(t)$ associated to $d_q(t)$ and the metric $g$ are of determinant class. As in [BFKM] one proves the following two theorems:

**Theorem A.** *Let $M = (M, \partial_- M, \partial_+ M)$ be a bordism, $g$ a Riemannian metric, $\tau = (h, g)$ a generalized triangulation and $\mathcal{F}$ a parallel flat bundle of $\mathcal{A}-$Hilbert modules so that the pair $\{(M, \partial_- M, \partial_+ M), \mathcal{F}\}$ is of determinant class. Then the following statements hold:*

*(i) The functions* $\log T(M, \partial_- M, g, h, \mathcal{F})(t)$, $\log T_{sm}(M, \partial_- M, g, h, \mathcal{F})(t)$ *and* $\log T_{la}(M, \partial_- M, g, h, \mathcal{F})(t)$ *admit asymptotic expansions for $t \to \infty$ of the form*

$$a_0 + \sum_{j=1}^{d+1} a_j t^j + b \log t + o(1).$$

*(ii) The asymptotic expansion of* $\log T(M, \partial_- M, g, h, \mathcal{F})(t)$ *is of the form*

$$\begin{aligned} &\log T(M, \partial_- M, g, h, \mathcal{F})(0) - \log T_{met}(M, \partial_- M, \tau, g, \mathcal{F}) \\ &\quad + (\log \pi)\Big(\sum_{q=0}^{d} (-1)^q \frac{d-2q}{4} \beta_q\Big) + \Big(\sum_{q=0}^{d} (-1)^{q+1} \frac{d-2q}{4} \beta_q\Big) \log t \\ &\quad + \Big(\sum_{q=0}^{d} (-1)^{q+1} q \beta_q\Big) t + \sum_{j=1}^{d+1} \Big(\sum_{q=0}^{d} (-1)^q p_{q,j}\Big) t^j + o(1) \end{aligned}$$

*where $p_{q,j}$ can be written as a sum, $p_{q,j} = p^I_{q,j} + p^{II}_{q,j}$ with $p^I_{q,j}$ being a local term on $M$ and $p^{II}_{q,j}$ being a local term on $\partial M$.*

*(iii) The asymptotic expansion of* $\log T_{sm}(M, \partial_- M, g, h, \mathcal{F})(t)$ *is of the form*

$$\log T_{comb}(M, \partial_- M, \tau, \mathcal{F}) + \frac{1}{2} \sum_{q=0}^{d} (-1)^q q (m_q \ell - \beta_q)(2t - \log t + \log \pi) + o(1).$$

**Theorem B.** *Assume that the pairs $\{(M_j, \partial_- M_j, \partial_+ M_j), \mathcal{F}_j\}$ $(j = 1, 2)$ are of determinant class and $\tau_j = (h_j, g_j)$ are generalized triangulations so that there exist neighborhoods $\mathcal{U}_j$ of $\partial M_j \cup Cr(h_j)$ in $M_j$ and a diffeomorphism $\Psi : \mathcal{U}_1 \to \mathcal{U}_2$ with*

*$\Psi(\partial_\pm M_1) = \partial_\pm M_2$, $\Psi\big(Cr_q(h_1)\big) = Cr_q(h_2)$ for $0 \le q \le d$, as well as $\Psi^* g_2 = g_1$ and $h_2 \circ \Psi = h_1$ on $\mathcal{U}_1$.*

*Then the free term*

$$FT\big(\log T_{la}(M_1, \partial_- M_1, g_1, h_1, \mathcal{F}_1)(t) - \log T_{la}(M_2, \partial_- M_2, g_2, h_2, \mathcal{F}_2)(t)\big)$$

*of the asymptotic expansion of*

$$\log T_{la}(M_1, \partial_- M_1, g_1, h_1, \mathcal{F}_1)(t) - \log T_{la}(M_2, \partial_- M_2, g_2, h_2, \mathcal{F}_2)(t)$$

*for $t \to \infty$ is given by*

$$\int_{M_1 \setminus Cr(h_1)} a_0(h_1, \epsilon = 0, x_1) - \int_{M_2 \setminus Cr(h_2)} a_0(h_2, \epsilon = 0, x_2)$$

*where $a_0(h_1, \epsilon, x_1)$ and $a_0(h_2, \epsilon, x_2)$ are densities (forms of degree $d$) and are given by explicit local formulas; the difference of the two integrals has to be taken in the same way as in [BFK, section 0, Remarks after Theorem 3].*

Introduce

$$\begin{aligned} &A(M_1, M_2, \tau_1, \tau_2, \mathcal{F}_1, \mathcal{F}_2) \\ &\quad := \int_{M_1 \setminus Cr(h_1)} a_0(h_1; \epsilon = 0, x_1) - \int_{M_2 \setminus Cr(h_2)} a_0(h_2; \epsilon = 0, x_2). \end{aligned}$$

**Corollary C.** *Assume in addition to the hypotheses in Theorem B that there exists a system $(M_3, \partial_- M_3, \partial_+ M_3, \mathcal{F}_3)$ with the following properties $(j = 1, 2)$:*
*(i) $\partial_\pm M_3 = \partial_\pm M_j$ and $\mathcal{F}_3 \restriction_{\partial_\pm M_3} = \mathcal{F}_j \restriction_{\partial_\pm M_j}$;*
*(ii) the pair $\{N_j, \tilde{\mathcal{F}}_j\}$ is of determinant class where $N_j$ denotes the closed manifold obtained by gluing $M_j$ to $M_3$ and $\tilde{\mathcal{F}}_j$ is given by $\mathcal{F}_j$ on $M_j$ and by $\mathcal{F}_3$ on $M_3$.*
*Then*

$$A(M_1, M_2, \tau_1, \tau_2, \mathcal{F}_1, \mathcal{F}_2) = 0$$

*Proof.* Let a be the minimum of $h_1$ (which is also the minimum of $h_2$) and let b be its maximum. Choose a smooth function $h_3 : M_3 \to \mathbb{R}$ with $h_3(M_3) \subset [a-1, b+1]$ as well as $h_3 \restriction_{\partial_- M_3} = a$, $h_3 \restriction_{\partial_+ M_3} = b$ (cf. Figure 1), and a metric $g_3$ on $M_3$ so that, for $i = 1, 2$, $h_i$ together with $h_3$ defines a Morse function $\tilde{h}_i$ on $N_i$ and $g_i$ together with $g_3$ defines a metric $\tilde{g}_i$ on $N_i$. Further choose $g_3$ (if necessary, modify $g_3$ in an arbitrary small neighborhood of $\partial M_3$) so that $(\tilde{h}_i, \tilde{g}_i)$ is a generalized triangulation for $N_i$. Since $\{N_i, \mathcal{F}_i\}, (i = 1, 2)$ are, by assumption, of determinant class, one concludes from Theorem A

$$\begin{aligned} &FT\big(\log T(N_1, \tilde{g}_1, \tilde{h}_1, \tilde{\mathcal{F}}_1)(t) - \log T(N_2, \tilde{g}_2, \tilde{h}_2, \tilde{\mathcal{F}}_2)(t)\big) \\ &\quad = \log T_{an}(N_1, \tilde{g}_1, \tilde{h}_1, \tilde{\mathcal{F}}_1) - \log T_{an}(N_2, \tilde{g}_2, \tilde{h}_2, \tilde{\mathcal{F}}_2) \\ &\qquad - \log T_{met}(N_1, \tilde{\tau}_1, \tilde{g}_1, \tilde{\mathcal{F}}_1) + \log T_{met}(N_2, \tilde{\tau}_2, \tilde{g}_2, \tilde{\mathcal{F}}_2) \end{aligned}$$

and

$$\begin{aligned}&FT\big(\log T_{sm}(N_1,\tilde g_1,\tilde h_1,\tilde{\mathcal F}_1)(t)-\log T_{sm}(N_2,\tilde g_2,\tilde h_2,\tilde{\mathcal F}_2)(t)\big)\\&\quad=\log T_{comb}(N_1,\tilde g_1,\tilde{\mathcal F}_1)-\log T_{comb}(N_2,\tilde\tau_2,\tilde{\mathcal F}_2).\end{aligned}$$

As $N_1$ and $N_2$ are both closed manifolds of determinant class we conclude from [BFKM] that

$$\begin{aligned}\log T_{an}(N_j,\tilde g_j,\tilde{\mathcal F}_j)&=\log T_{comb}(N_j,\tilde\tau_j,\tilde{\mathcal F}_j)\\&\qquad+\log T_{met}(N_j,\tilde\tau_j,\tilde g_j,\tilde{\mathcal F}_j).\end{aligned}$$

Combining the above equalities with the identity

$$\begin{aligned}FT\big(\log T(N_j,\tilde g_j,\tilde h_j,\tilde{\mathcal F}_j)(t)\big)&=FT\big(\log T_{sm}(N_j,\tilde g_j,\tilde h_j,\tilde{\mathcal F}_j)(t)\big)\\&\quad+FT\big(\log T_{la}(N_j,\tilde g_j,\tilde h_j,\tilde{\mathcal F}_j)(t)\big)\end{aligned}$$

one obtains, with $A$ defined as above,

$$A(N_1,N_2,\tilde\tau_1,\tilde\tau_2,\tilde{\mathcal F}_1,\tilde{\mathcal F}_2)=0.$$

In view of the fact that A is local, one concludes that

$$A(M_1,M_2,\tau_1,\tau_2,\mathcal F_1,\mathcal F_2)=0.$$

□

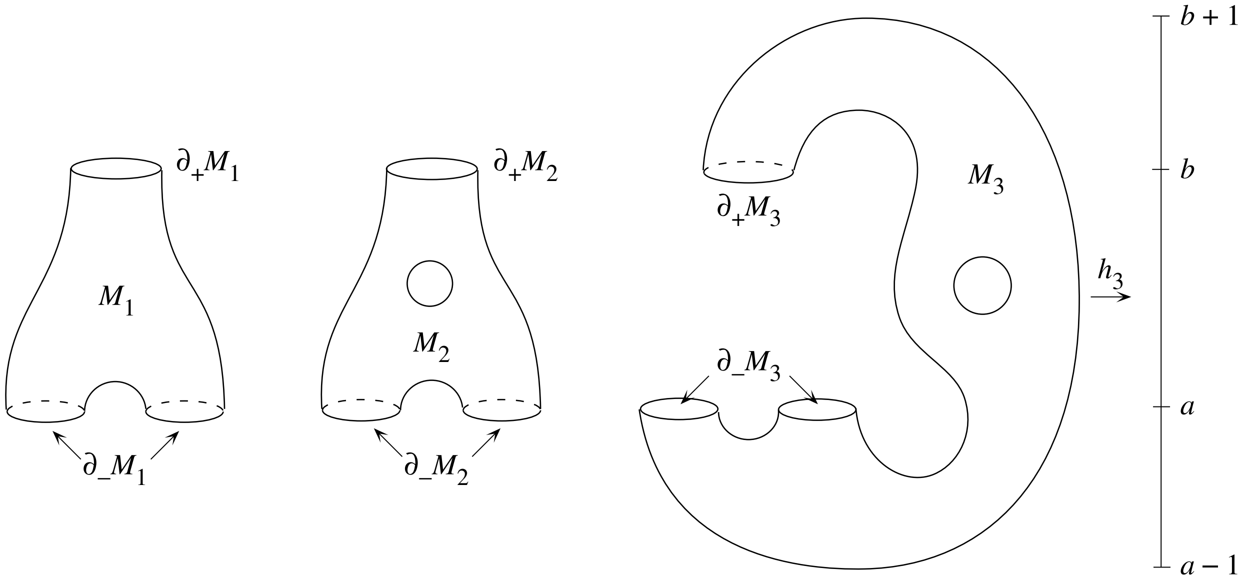


Figure 1

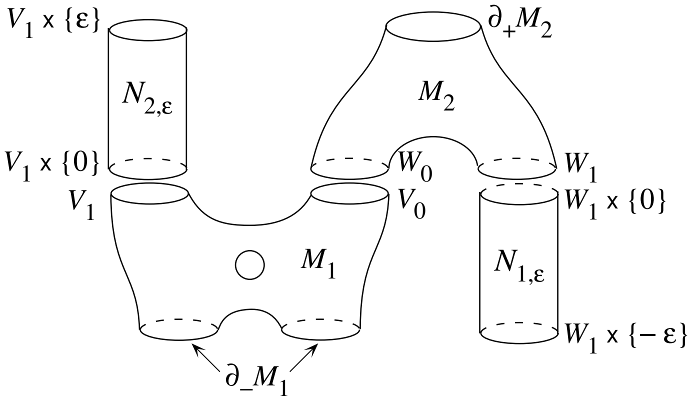


Figure 2

Consider the pair $\{(M, \partial_- M, \partial_+ M), \mathcal{F}\}$ and assume that it is of determinant class. Define $\mathcal{R}(M, \partial_- M, g, \tau, \mathcal{F})$ by

$$\log \mathcal{R}(M, \partial_- M, g, \tau, \mathcal{F}) := \log T_{an}(M, \partial_- M, g, \mathcal{F}) - \log T_{Re}(M, \partial_- M, \tau, g, \mathcal{F}).$$

Our aim is to show that the ratio $\mathcal{R}$ depends only on the data on the boundary at least in the case when the system $(M, \partial_- M, \partial_+ M, \mathcal{F})$ is of determinant class.

First observe that the result concerning the metric anomaly of the analytic torsion [BFKM, Lemma 6.11] extends to manifolds with boundary:

**Proposition 2.8.** *Assume that the pair $\{(M, \partial_- M, \partial_+ M), \mathcal{F}\}$ is of determinant class and that $g(s), a \leq s \leq b$, is a smooth 1-parameter family of Riemannian metrics on $M$ so that $g(s)$ is independent of $s$ in a collar neighborhood of $\partial M$. Then*

$$\frac{d}{ds} \log T_{an}\big(M, \partial_- M, g(s), \mathcal{F}\big) = \frac{d}{ds} \log T_{met}\big(M, \partial_- M, g(s), \tau, \mathcal{F}\big).$$

*Proof.* Arguing as in [RS] one has

$$\begin{aligned} &\frac{d}{ds} \log T_{an}\big(M, \partial_- M, g(s), \mathcal{F}\big) \\ &\quad = \frac{d}{ds} \log T_{met}\big(M, \partial_- M, g(s), \tau, \mathcal{F}\big) + \sum_{q=0}^{d} (-1)^q q \hat{\rho}_q \end{aligned}$$

where $\hat{\rho_q} = \int_M \rho_q(x, s)$ and $\rho_q(x, s)$ is a density (i.e. a d-form) on $M$, which can be calculated in terms of the symbol of $\Delta_q(s)$ acting on $\Lambda^q(M; \mathcal{E})$. Due to our assumption that $g(s)$ does not depend on s near the boundary, $\rho_q(x, s)$ vanishes for $x$ near the boundary. If $d = dim\, M$ is odd one can then use a parity argument to conclude that $\rho_q(x, s) \equiv 0$ for $0 \leq q \leq d$. If $d = dim\, M$ is even, $\sum_{q=0}^{d} (-1)^q q \hat{\rho}_q = 0$. Indeed take the double of $M$ and use that $\log T_{an}$ and $\log T_{met}$ are 0 for closed manifolds of even dimension.

**Corollary 2.9.** *Suppose that the pair $\{(M, \partial_- M, \partial_+ M), \mathcal{F}\})$ is of determinant class and that $\tau = (h, g')$ is a generalized triangulation. If $g_1$ and $g_2$ are two Riemannian metrics on $M$ which agree on $\partial M$, then*

$$\mathcal{R}(M, \partial_- M, g_1, \tau, \mathcal{F}) = \mathcal{R}(M, \partial_- M, g_2, \tau, \mathcal{F}).$$

**Theorem 2.10.** *Assume that, for $j = 1, 2$, $M_j = (M_j, \partial_- M_j, \partial_+ M_j)$ is a bordism equipped with a Riemannian metric $g_j$, a generalized triangulation $\tau_j$ and a parallel flat bundle $\mathcal{F}_j = (\mathcal{E}_j, \nabla_j)$ of $\mathcal{A}$-Hilbert modules of finite type on $M_j$.Further assume that the systems $(M_j, \partial_- M_j, \partial_+ M_j, \mathcal{F}_j)$ are of determinant class and*

$$(\partial_\pm M_1, g_1 \restriction_{\partial_\pm M_1}, \mathcal{F}_1 \restriction_{\partial_\pm M_1}) = (\partial_\pm M_2, g_2 \restriction_{\partial_\pm M_2}, \mathcal{F}_2 \restriction_{\partial_\pm M_2}).$$

*Then*

$$\mathcal{R}(M_1, \partial_- M_1, g_1, \tau_1, \mathcal{F}_1) = \mathcal{R}(M_2, \partial_- M_2, g_2, \tau_2, \mathcal{F}_2).$$

*Proof.* The proof proceeds in several steps.

(I) In the case where the hypotheses of Corollary C are satisfied the claim follows from Theorem A.

(II) Next assume that only the hypotheses of Theorem B are satisfied. Consider $S^2$ equipped with a generalized triangulation $\tau_{S^2} = (h_0, g_0)$ and denote by $\underline{\tau}_j = (\underline{h}_j, \underline{g}'_j)$ a product triangulation on $M_j \times S^2$. We claim that, for $j = 1, 2$, $(M_j \times S^2, g_j \times g_0, \underline{\tau}_j, \underline{\mathcal{F}_j})$, where $\underline{\mathcal{F}_j}$ is the pullback of $\mathcal{F}_j$ by the projection $M_j \times S^2 \to M_j$, satisfies the assumptions of Corollary C. Indeed denote by $D^3$ the unit disc in $\mathbb{R}^3$ and choose for $M_3$ the disjoint union of $\partial_- M_1 \times D^3$ with $\partial_+ M_1 \times D^3$. Then $\partial_\pm M_3 = \partial_\pm M_1 \times S^2$. Let $\mathcal{F}_3$ be the pull back of $\mathcal{F}_1 \restriction_{\partial M_1}$ by the projection $\partial M_1 \times D^3 \to \partial M_1$. Since the pairs $\{\partial_\pm M_i, \mathcal{F}_i \restriction_{\partial_\pm M_i}\}$ are of determinant class, the system $(M_3, \partial_- M_3, \partial_+ M_3, \mathcal{F}_3)$ is of determinant class and therefore, by Proposition 2.3, the pairs $\{N_i, \tilde{\mathcal{F}}_i\}$ (as defined in Corollary C) are of determinant class. One concludes then by (I) that

$$\mathcal{R}(M_1 \times S^2, \partial_- M_1 \times S^2, g_1 \times g_0, \underline{\tau}_1, \underline{\mathcal{F}_1}) = \mathcal{R}(M_2 \times S^2, \partial_- M_2 \times S^2, g_2 \times g_0, \underline{\tau}_2, \underline{\mathcal{F}_2}).$$

By applying Proposition 2.5 (3) and Proposition 2.3 (formulas (2.10)- (2.10')), one obtains

$$\mathcal{R}(M_1, \partial_- M_1, g_1, \tau_1, \mathcal{F}_1) = \mathcal{R}(M_2, \partial_- M_2, g_2, \tau_2, \mathcal{F}_2).$$

(III) Applying Corollary 2.9 it suffices to prove the statement in the case where $g_j$ and $\tau_j = (h_j, g'_j)$ $(j = 1, 2)$ have the additional property that $g_j = g'_j$.

(IV) (A) It suffices to prove the result under the additional hypothesis that $\chi(M_1, \partial_- M_1) = \chi(M_2, \partial_- M_2)$. Indeed, if $dim M_1$ is odd, $\chi(M_1, \partial_- M_1) = \chi(M_2, \partial_- M_2)$. If $dim M_1$ is even, then

$$\chi(M_1, \partial_- M_1) = \chi(M_2, \partial_- M_2) + 2k.$$

If, in addition, $k > 0$ we replace, without loss of generality, $M_1$ by the disjoint union of $M_2$ with k copies of $(S^d, \mathcal{F})$, where $\mathcal{F}$ is the trivial parallel flat bundle with the same fiber as $\mathcal{F}_1$. (Observe that $\mathcal{R} = 0$ for $(S^d, \mathcal{F})$. ) In the case when $k < 0$ interchange the role of $M_1$ and $M_2$.

(B) Under the additional hypotheses $\chi(M_1, \partial_- M_1) = \chi(M_2, \partial_- M_2)$ the result can be proved as follows: By the invariance of the Reidemeister torsion under a subdivision of a generalized triangulation (cf. Lemma 6.12 in [BFKM] for the case of a closed manifold) it suffices to prove that there exist subdivisions $\hat{\tau}_j$ of $\tau_j$ so that the assumptions of Theorem B are satisfied. But such subdivisions exist if $\chi(M_1, \partial_- M_1) = \chi(M_2, \partial_- M_2)$ (cf. [BFKM]). □

## 3. Applications to torsions.

### 3.1 Comparison of analytic and Reidemeister torsion.

In this subsection we provide a formula for the ratio of analytic and Reidemeister torsion of a compact Riemannian manifold with boundary as introduced at the end of section 2.

We begin by some auxiliary considerations. Assume that $(M,g)$ is a closed Riemannian manifold, $\tau = (h,g')$ a generalized triangulation, and $\mathcal{F} = (\mathcal{E},\nabla)$ a parallel flat bundle on $M$. Consider the cylinder $M_I = M \times I$ where $I$ is a compact interval $[a,b]$ in $\mathbb{R}$ with $a,b \in 4\mathbb{Z}$. Denote by $g_0$ the standard metric on $[a,b]$ and by $\tau_i = (h_i, g_0)$, $(i=1,2)$, the generalized triangulations of the bordisms $([a,b],\{a\},\{b\})$ resp. $([a,b],\emptyset,\{a,b\})$ with $h_1(x) = x$ resp. $h_2 = \frac{1}{2}(x-(b+a)/2)^2$. ( To satisfy condition (T1) in the definition of a generalized triangulation, one can perturb $h_2$ slightly so that it is linear near the boundary of the interval.) Notice that the generalized triangulation $\tau_1$ has no cells. Denote by $\mathcal{T}$ the trivial 1-dimensional complex vector bundle with trivial connection. $\mathcal{T}$ is a parallel flat bundle over $[a,b]$. An easy calculation gives

$$\begin{gathered}\log T_{comb}(I,\{a\},\tau_1,\mathcal{T}) = 0,\ \log T_{an}(I,\{a\},g_0,\mathcal{T}) = \frac{1}{2}\log 2,\\ \log T_{met}(I,\{a\},\tau_1,g_0,\mathcal{T}) = 0,\end{gathered} \tag{3.1}$$

$$\begin{gathered}\log T_{comb}(I,\emptyset,\tau_2,\mathcal{T}) = 0,\ \log T_{an}(I,\emptyset,g_0,\mathcal{T}) = \frac{1}{2}(\log 2 + \log(b-a)),\\ \log T_{met}(I,\emptyset,\tau_2,g_0,\mathcal{T}) = \frac{1}{2}\log(b-a).\end{gathered} \tag{3.1$'$}$$

Assume that the pair $\{M,\mathcal{F}\}$ is of determinant class. Then, by Proposition 2.3, the systems $(M_I,\emptyset,M\times\partial I,\mathcal{F}_I)$ and $(M_I, M\times\{a\}, M\times\{b\},\mathcal{F}_I)$ are of determinant class (where $\mathcal{F}_I$ is the pull back of $\mathcal{F}$ by the projection $M\times I \to M$) and

$$\begin{gathered}\log T_{comb}(M_I, M\times\{a\},\tau\oplus\tau_1,\mathcal{F}_I) = 0,\\ \log T_{an}(M_I, M\times\{a\}, g\oplus g_0,\mathcal{F}_I) = \chi(M;\mathcal{F})\frac{\log 2}{2}\\ \log T_{met}(M_I, M\times\{a\}, g\oplus g_0,\tau\oplus\tau_2,\mathcal{F}_I) = 0,\end{gathered} \tag{3.2}$$

$$\begin{gathered}\log T_{comb}(M_I,\emptyset,\tau\oplus\tau_2,\mathcal{F}_I) = \log T_{comb}(M,\tau,\mathcal{F}),\\ \log T_{an}(M_I,\emptyset,g\oplus g_0,\mathcal{F}_I) = \frac{\chi(M;\mathcal{F})}{2}(\log 2 + \log(b-a)) + \log T_{an}(M,g,\mathcal{F}),\\ \log T_{met}(M_I,\emptyset,g\oplus g_0,\tau\oplus\tau_2,\mathcal{F}_I) = \frac{\chi(M;\mathcal{F})}{2}\log(b-a) + \log T_{met}(M,g,\tau,\mathcal{F})\end{gathered} \tag{3.2$'$}$$

(where we have used that $\chi(I,\emptyset)=1$ and $\chi(I,\{a\})=0$).

Since M is closed and the pair $\{M,\mathcal{F}\}$, is of determinant class one obtains, by [BFKM],

$$\log T_{an}(M,g,\mathcal{F})=\log T_{Re}(M,g,\tau,\mathcal{F}).$$

Combining with the formulas (3.2)-(3.2'), one obtains

$$\log \mathcal{R}(M_I, M\times\{a\}, g\otimes g_0, \tau\oplus\tau_1, \mathcal{F}_I)=\frac{\chi(M;\mathcal{F})\log 2}{2}. \tag{3.3}$$

$$\log \mathcal{R}(M_I, \emptyset, g\otimes g_0, \tau\oplus\tau_2, \mathcal{F}_I)=\frac{\chi(M;\mathcal{F})\log 2}{2}. \tag{3.3$'$}$$

**Theorem 3.1.** *Assume that the system $(M,\partial_- M,\partial_+ M,\mathcal{F})$ is of determinant class. Then*

$$\log \mathcal{R}(M,\partial_- M, g,\tau,\mathcal{F})=\frac{1}{4}\chi(\partial M;\mathcal{F})\log 2. \tag{3.4}$$

*Remark.* In the case where $\mathcal{A}=\mathbb{R}$, the result, as it stands, is due independently to Lück ([Lü1]) and Vishik [Vi] (cf. also [Ch]). However, even in the case $\mathcal{A}=\mathbb{R}$, the proof presented here is new, elementary and short.

*Proof.* The proof proceeds in two steps.

(A) We first consider the case where $M=(M,\emptyset,\partial M)$. Take a disjoint union $M\sqcup M$ of two copies of $M$ and consider the bordism $\big(M\sqcup M,\emptyset,(\partial M)\sqcup(\partial M)\big)$. Further introduce $(\partial M)_I:=\partial M\times I$ and consider the bordism $\big((\partial M)_I,\emptyset,\partial((\partial M)_I)\big)$. Notice that $\partial((\partial M)_I)=(\partial M\times\{0\})\sqcup(\partial M\times\{1\})=(\partial M)\sqcup(\partial M)$. Since by hypothesis $(\partial M,\mathcal{F}\restriction_{\partial M})$ is of determinant class so is $(\partial M\times I,\emptyset,(\mathcal{F}\restriction_{\partial M})_I)$ (cf Proposition 2.5(3)). Therefore, by Theorem 2.10,

$$\log\mathcal{R}(\partial M\times I,\emptyset,g_I,\tau_I,(\mathcal{F}\restriction_{\partial M})_I)=\log\mathcal{R}(M\sqcup M,\emptyset,g\sqcup g,\tau\sqcup\tau,\mathcal{F}\sqcup\mathcal{F})$$

where $(\mathcal{F}\restriction_{\partial M})_I$ is defined as above, $g_I:=g\restriction_{\partial M}\oplus g_0$ and $\tau_I$ is a product triangulation. Notice that

$$\log\mathcal{R}(M\sqcup M,\emptyset,g\sqcup g,\tau\sqcup\tau,\mathcal{F}\sqcup\mathcal{F})=2\log\mathcal{R}(M,\emptyset,g,\tau,\mathcal{F}),$$

and, by (3.3'),

$$\log\mathcal{R}(\partial M\times I,\emptyset,g_I,\tau_I,\mathcal{F}_I\restriction_{\partial M})=\frac{\chi(\partial M;\mathcal{F})}{2}\log 2.$$

Combining the three equalities above, we obtain

$$\log \mathcal{R}(M, \emptyset, g, \tau, \mathcal{F}) = \frac{\chi(\partial M; \mathcal{F})}{4} \log 2$$

and statement (3.4) is proved in the case where $\partial_- M = \emptyset$.

(B) In view of Proposition 2.3 it suffices to check (3.4) for the system $(M \times S^2, \partial_- M \times S^2, \partial_+ M \times S^2, \tilde{\mathcal{F}})$ which is, by Proposition 2.5 (3), of determinant class because $(M, \partial_- M, \partial_+ M, \mathcal{F})$ is. Here $\tilde{\mathcal{F}}$ denotes the pullback of $\mathcal{F}$ by the projection $M \times S^2 \to M$. In view of Theorem 2.10 it suffices to prove the statement (3.4) for the two systems $(\partial_+ M \times D^3, \emptyset, \partial_+ M \times S^2, (\mathcal{F} \restriction_{\partial_+ M})\tilde{})$ and $(\partial_- M \times D^3, \partial_- M \times S^2, \emptyset, (\mathcal{F} \restriction_{\partial_- M})\tilde{})$ where $(\mathcal{F} \restriction_{\partial_\pm M})\tilde{}$ denotes the pullback of $\mathcal{F} \restriction_{\partial_\pm M}$ by the projections $\partial_\pm M \times D^3 \to \partial_\pm M$. For the first system it suffices to apply (A) and for the second system, (3.4) follows from Proposition 2.2 (Poincaré duality) and (A). □

### 3.2. Glueing formulas (Part I).

In this subsection we present a gluing formula for the analytic and Reidemeister torsions. Let $(M_j, \partial_- M_j, \partial_+ M_j)$, be two bordisms equipped with Riemannian metrics $g_j$, generalized triangulations $\tau_j = (h_j, g'_j)$ and parallel flat bundles $\mathcal{F}_j = (\mathcal{E}_j, \nabla_j)$. Suppose that there exists an isometry

$$\omega : (\partial_+ M_1, g_1 \restriction_{\partial_- M_1}) \to (\partial_- M_2, g_2 \restriction_{\partial_+ M_2}) \tag{3.5}$$

and a connection preserving bundle isomorphism $\Phi$ above $\omega$ which makes the following diagram commutative

$$\begin{array}{ccc} \mathcal{E}_1 \restriction_{\partial_- M_1} & \xrightarrow{\Phi} & \mathcal{E}_2 \restriction_{\partial_+ M_2} \\ \downarrow & & \downarrow \\ \partial_- M_1 & \xrightarrow{\omega} & \partial_+ M_2. \end{array} \tag{3.6}$$

Then one can form a bordism $(M := M_1 \cup_\omega M_2, \partial_- M_1, \partial_+ M_2)$ by gluing $\partial_+ M_1$ to $\partial_- M_2$ by $\omega$ and a parallel flat bundle $\mathcal{F}$ by gluing $\mathcal{F}_1$ and $\mathcal{F}_2$ by $(\omega, \Phi)$. The metrics $g_1, g'_1$ and $g_2, g'_2$ determine Riemannian metrics $g$, $g'$ on $M$, and the functions $h_1$ and $h_2$ determine the $C^\infty$-function $h : M \to \mathbb{R}$ given by

$$h(x) := \begin{cases} h_1(x) & (x \in M_1) \\ b_1 - a_2 + h_2(x) & (x \in M_2) \end{cases} \tag{3.7}$$

where, for j=1,2, $h_j(M_j) = [a_j, b_j]$.

Notice that if $\tau = (h, g')$ is not a generalized triangulation (i.e. violates (T4)) one can modify, by an arbitrary small perturbation and localized in a given neighborhood of $\partial_+ M_1$, the metric $g'_1$ to $\widetilde{g}'_1$ so that the triangulation $\widetilde{\tau}_1 := (h_1, \widetilde{g}'_1)$ is compatible with $\tau_2$ and $\widetilde{\tau}_1$ and $\tau_1$ provide the same relative CW-complex structure. Therefore, without loss of generality, we may assume that $\tau = (h, g')$ is a generalized triangulation of $(M, \partial_- M_1, \partial_+ M_2)$.

Choose orientations $\mathcal{O}_h$ for $M$. They induce orientations $\mathcal{O}_{h_i}$ for $M_i$. Now consider the short exact sequence of cochain complexes

$$\begin{aligned}0 \to \mathcal{C}^*(M_2, \partial_- M_2, \tau_2, \mathcal{O}_{h_2}; \mathcal{F}_2) &\xrightarrow{i_*} \mathcal{C}^*(M, \partial_- M_1, \tau, \mathcal{O}_h; \mathcal{F}) \\ &\xrightarrow{r_*} \mathcal{C}^*(M_1, \partial_- M_1, \tau_1, \mathcal{O}_{h_1}; \mathcal{F}_1) \to 0,\end{aligned} \tag{3.8}$$

where $i_*$ is the map which extends the cochains, defined on the cells of $M_2$, to cochains defined on all cells of $M$ by assigning the value zero on the cells of $M_1$ and where $r_*$ is the map which restricts the cochains, defined on the cells of $M$, to the cells of $M_1$. The sequence (3.8) induces a long weakly exact sequence $\mathcal{H}_{comb}(\tau)$ in cohomology of $\mathcal{A}$- Hilbert modules (cf (1.52))

$$\begin{aligned}\cdots \to H^q(M_2, \partial_- M_2, \tau_2; \mathcal{F}_2) \to H^q(M, \partial_- M_1, \tau; \mathcal{F}) \to H^q(M_1, \partial_- M_1, \tau_1; \mathcal{F}_1) \to \\ H^{q+1}(M_2, \partial_- M_2, \tau_2; \mathcal{F}_2) \to \cdots\end{aligned} \tag{3.9}$$

A similar sequence (depending on the Riemannian metric $g$), denoted by $\mathcal{H}_{an}(g)$, can be obtained using de Rham cohomology $H^q(M, \partial_- M, g; \mathcal{F})$ instead of the combinatorial one,

$$\begin{aligned}\cdots \to H^q(M_2, \partial_- M_2, g_2; \mathcal{F}_2) \to H^q(M, \partial_- M, g; \mathcal{F}) \to H^q(M_1, \partial_- M_1, g_1; \mathcal{F}_1) \to \\ H^{q+1}(M_2, \partial_- M_2, g_2; \mathcal{F}_2) \to \cdots\end{aligned} \tag{3.10}$$

As we have seen in section 2, the integration theory provides isomorphisms $(\theta_j)^{-1}$ of $\mathcal{A}$-Hilbert modules from de Rham cohomology to the combinatorial cohomology. Moreover the $(\theta_j)$ define an isomorphism of cochain complexes from $\mathcal{H}_{comb}(\tau)$ to $\mathcal{H}_{an}(g)$.

**Theorem 3.2.** *Assume that, for i=1,2, the system $(M_i, \partial_- M_i, \partial_+ M_i, \mathcal{F}_i)$ is of determinant class. Then the following statements hold:*

(i) *The system $(M, \partial_- M, \partial_+ M, \mathcal{F})$ and the complexes $\mathcal{H}_{comb}, \mathcal{H}_{an}$ are of determinant class;*

$$\log T_{Re}(M, \partial_- M, g, \tau, \mathcal{F}) = \sum_{j=1}^{2} \log T_{Re}(M_j, \partial_- M_j, g_j, \tau_j, \mathcal{F}_j) + \log T(\mathcal{H}_{an}). \tag{ii}$$

$$\log T_{an}(M, \partial_- M, g, \mathcal{F}) = \sum_{j=1}^{2} \log T_{an}(M_j, \partial_- M_j, g_j, \mathcal{F}_j) + \log T(\mathcal{H}_{an}) - \frac{\chi(\partial_+ M_1; \mathcal{F}_1 \restriction_{\partial_+ M_1})}{2} \log 2. \tag{iii}$$

*Remark.* In the case where $\mathcal{A} = \mathbb{R}$, the result (ii) is due to Milnor [Mi]. Work related to (i) can be found in [LL]. In the case where $\mathcal{A} = \mathbb{R}$, (iii) is due to Vishik [Vi] . Vishik's proof, however, is completely different than ours.

*Proof.* (i) Proposition 2.5 implies that the system $(M, \partial_- M, \partial_+ M, \mathcal{F})$ is of determinant class. Theorem 1.14 applied to the short exact sequence of cochain complexes (3.8) implies that $\mathcal{H}_{comb}$ is of determinant class. In view of the fact that $\theta$ defines an isomorphism of cochain complexes, Proposition 1.3 implies that $\mathcal{H}_{an}$ is of determinant class as well.

(ii) Concerning the torsion of the sequence (3.8), notice that

$$\begin{aligned}\log T(0 \to \mathcal{C}^q(M_2, \partial_- M_2, \tau_2, \mathcal{O}_{h_2}; \mathcal{F}) &\xrightarrow{i_q} \mathcal{C}^q(M, \partial_- M, \tau, \mathcal{O}_h; \mathcal{F}) \\ &\xrightarrow{r_q} \mathcal{C}^q(M_1, \partial_- M_1, \tau_1, \mathcal{O}_{h_1}; \mathcal{F}) \to 0) = 0\end{aligned}$$

since $\log Vol(i_q) = 0$ and $\log Vol(r_q \restriction (Null(r_q)^\perp) = 0$. Theorem 1.14 implies that

$$\log T_{comb}(M, \partial_- M, \tau, \mathcal{F}) = \sum_{j=1}^{2} \log T_{comb}(M_j, \partial_- M_j, \tau_j, \mathcal{F}_j) + \log T(\mathcal{H}_{comb}). \tag{3.11}$$

From Proposition 1.3 (C) and in view of the fact that $\theta$ is an isomorphism of cochain complexes of $\mathcal{A}-$Hilbert modules one obtains

$$\begin{aligned}\log T_{met}(M, \partial_- M, g, \tau, \mathcal{F}) = &\sum_{j=1}^{2} \log T_{met}(M_j, \partial_- M_j, g_j, \tau_j, \mathcal{F}_j) \\ &+ \log T(\mathcal{H}_{an}) - \log T(\mathcal{H}_{comb}).\end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) one obtains (ii).

(iii) follows from (ii) and Theorem 3.1. □

### 3.3. Glueing formulas (Part II).

In this subsection we extend Theorem 3.2 for partial glueing, i.e. to a situation where not necessarily all of the components of $\partial_+ M_1$ and $\partial_- M_2$ are glued together.

Suppose that $(M_i, \partial_- M_i, \partial_+ M_i; \mathcal{F}_i), i = 1, 2$ are two systems with $\partial_+ M_1$ resp. $\partial_- M_2$ consisting of two disjoint components $\partial_+ M_1 = V_0 \sqcup V_1$, resp. $\partial_- M_2 = W_0 \sqcup W_1$. Assume that there exist a diffeomorphism $\omega_0 : V_0 \to W_0$ and a connection preserving isomorphism above $\omega_0$, $\Phi_0 : \mathcal{F}_1 \restriction_{V_0} \to \mathcal{F}_2 \restriction_{W_0}$ . Then $M_1$ and $M_2$ can be glued together to give rise to the bordism $(M, \partial_- M, \partial_+ M)$ with $M := M_1 \cup_\omega M_2$ and boundary components $\partial_- M = \partial_- M_1 \sqcup W_1$ and $\partial_+ M = V_1 \sqcup \partial_+ M_2$. Using $\Phi_0$, $\mathcal{F}_1$ can be glued with $\mathcal{F}_2$ to obtain a parallel flat bundle $\mathcal{F}$ on $M$. Suppose, in addition, that Riemannian metrics $g_1$ on $M_1$ and $g_2$ on $M_2$ are given so that $\omega^\#(g_2 \restriction_{W_0}) = g_1 \restriction_{V_0}$ . The metrics $g_1$ and $g_2$ define a Riemannian metric $g$ on $M$. One then obtains the following (weakly exact) sequence $\mathcal{H}$ in de Rham cohomology

$$\begin{aligned}\cdots \to &H^q(M_2, \partial_- M_2, g_2; \mathcal{F}_2) \to H^q(M, \partial_- M, g; \mathcal{F}) \to \\ &\to H^q(M_1, \partial_- M_1, g_1; \mathcal{F}_1) \to H^{q+1}(M_2, \partial_- M_2, g_2; \mathcal{F}_2) \to \cdots\end{aligned} \tag{3.19}$$

This sequence is the cohomology sequence induced by the exact sequence of cochain complexes (cf (2.7))

$$(3.20)\; 0 \to \Lambda^*(M_2, \partial_- M_2; \mathcal{F}_2) \to \Lambda^*(M, \partial_- M; \mathcal{F}) \to \Lambda^*(M_1, \partial_- M_1; \mathcal{F}_1) \to 0.$$

(To be completely correct, $\Lambda^q(M_2, \partial_- M_2; \mathcal{F}_2)$ should be the smaller space of smooth $q-$forms which are the restriction to $M_2$ of $q-$forms which are defined on all of M, but vanish on $M_1$.)

**Theorem 3.2$'$.** *If the systems $(M_i, \partial_- M_i, \partial_+ M_i, \mathcal{F}_i)$ (i=1,2) are of determinant class then so are the system $(M, \partial_- M, \partial_+ M, \mathcal{F})$ and the cohomology sequence $\mathcal{H}$, given by (3.19). Moreover*

$$\log T_{an}(M, \partial_- M, g, \mathcal{F}) = \sum_{j=1}^{2} \log T_{an}(M_j, \partial_- M_j, g_j, \mathcal{F}_j) + \log T(\mathcal{H}) - \chi(V_0; \mathcal{F}_1 \restriction_{V_0}) \frac{\log 2}{2}.$$

*Proof.* We will derive the stated results from Theorem 3.2. For that purpose introduce, for $\epsilon > 0$, the systems $(N_{i,\epsilon}, \partial_- N_{i,\epsilon}, \partial_+ N_{i,\epsilon}, \mathcal{G}_{i,\epsilon}), i = 1, 2$, defined by

$$N_{1,\epsilon} := W_1 \times [-\epsilon, 0],\; \partial_- N_{1,\epsilon} := W_1 \times \{-\epsilon\},\; \partial_+ N_{1,\epsilon} := W_1 \times \{0\};$$

$$N_{2,\epsilon} := V_1 \times [0, \epsilon],\; \partial_- N_{2,\epsilon} := V_1 \times \{0\},\; \partial_+ N_{2,\epsilon} := V_1 \times \{\epsilon\};$$

$$\mathcal{G}_{1,\epsilon} := (\mathcal{F}_2 \restriction_{W_1})_{[-\epsilon, 0]},\; \mathcal{G}_{2,\epsilon} := (\mathcal{F}_1 \restriction_{V_1})_{[0,\epsilon]},$$

and let $g_{i,\epsilon}$ be the metrics on $N_{1,\epsilon}$ defined by $g_{1,\epsilon} := g_2 \restriction_{W_1} \oplus g_0$ and $g_{2,\epsilon} := g_1 \restriction_{V_1} \oplus g_0$ (where $g_0$ is the standard Euclidean metric) (see Figure 2).
Denote by $(M_{i,\epsilon}, \partial_- M_{i,\epsilon}, \partial_+ M_{i,\epsilon}, \mathcal{F}_{i,\epsilon})$ the disjoint union of the systems $(M_i, \partial_- M_i, \partial_+ M_i, \mathcal{F}_i)$ and $(N_{i,\epsilon}, \partial_- N_{i,\epsilon}, \partial_+ N_{i,\epsilon}, \mathcal{G}_{i,\epsilon})$ and let $\omega_\epsilon : \partial_+ M_{1,\epsilon} := V_0 \sqcup V_1 \sqcup W_1 \to \partial_- M_{2,\epsilon} := W_0 \sqcup V_1 \sqcup W_1$ be the diffeomorphism which is equal to $\omega_0$ on $V_0$ and to the identity on $V_1 \sqcup W_1$. Moreover $\omega_0$ and $\Phi_0$ induce an isomorphism of parallel flat bundles

$$\Phi_\epsilon : \mathcal{F}_{1,\epsilon} \restriction_{\partial_+ M_{1,\epsilon}} \to \mathcal{F}_{2,\epsilon} \restriction_{\partial_- M_{2,\epsilon}} .$$

Following the glueing instructions described in subsection 3.2, one forms the system $(M_\epsilon, \partial_- M_\epsilon, \partial_+ M_\epsilon, \mathcal{F}_\epsilon)$ and defines the Riemannian metric $g_\epsilon$ given by $g_i$ on $M_i$ and $g_{i,\epsilon}$ on $N_{i,\epsilon}$.
There exists a commutative diagram connecting the cohomology sequence $\mathcal{H}$, defined in (3.19), with the sequence (3.10) for the bordism $(M_\epsilon, \partial_- M_\epsilon, \partial_+ M_\epsilon)$, which we denote by $\mathcal{H}_\epsilon$,

$$\begin{array}{ccccc}
H^q(M_2, \partial_- M_2, g_2; \mathcal{F}_2) & \to & H^q(M, \partial_- M, g; \mathcal{F}) & \to & H^q(M_1, \partial_- M_1, g_1; \mathcal{F}_1) \\
\downarrow (i_{2,\epsilon})_q & & \downarrow (i_\epsilon)_q & & \downarrow (i_{1,\epsilon})_q \\
H^q(M_{2,\epsilon}, \partial_- M_{2,\epsilon}, g_{2,\epsilon}; \mathcal{F}_{2,\epsilon}) & \to & H^q(M_\epsilon, \partial_- M_\epsilon, g_\epsilon; \mathcal{F}_\epsilon) & \to & H^q(M_{1,\epsilon}, \partial_- M_{1,\epsilon}, g_{1,\epsilon}; \mathcal{F}_{1,\epsilon}).
\end{array}$$

In this diagram, the horizontal lines represent the cohomology sequence (3.19) and (3.10) respectively and the vertical arrows are isomorphisms of Hilbert modules of finite type. The isomorphisms $(i_{1,\epsilon})_q$, $(i_\epsilon)_q$ and $(i_{2,\epsilon})_q$ can be described as the composition

$$(i_{1,\epsilon})_q := (i'_{1,\epsilon})_q \cdot ((i''_{1,\epsilon})_q)^{-1},\ (i_{2,\epsilon})_q := (i'_{2,\epsilon})_q \cdot ((i''_{2,\epsilon})_q)^{-1},$$

$$(i_\epsilon)_q = (i'_\epsilon)_q \cdot ((i''_\epsilon)_q)^{-1}$$

where $(i'_{k,\epsilon})_q$ and $(i''_{k,\epsilon})_q$ are induced by the inclusions (k=1,2)

$$\begin{aligned} i'_{k,\epsilon} &: (M_k, \partial_- M_k) \to (M_{k,\epsilon}, \partial_- M_{k,\epsilon} \cup N_{k,\epsilon}) \text{ and} \\ i''_{k,\epsilon} &: (M_{k,\epsilon}, \partial_- M_{k,\epsilon}) \to (M_{k,\epsilon}, \partial_- M_{k,\epsilon} \cup N_{k,\epsilon}) \end{aligned}$$

and where $(i'_\epsilon)_q$ and $(i''_\epsilon)_q$ are induced by the inclusions

$$\begin{aligned} i'_\epsilon &: (M, \partial_- M) \to (M_\epsilon, \partial_- M_\epsilon \cup N_{1,\epsilon}) \text{ and} \\ i''_\epsilon &: (M_\epsilon, \partial_- M_\epsilon) \to (M_\epsilon, \partial_- M_\epsilon \cup N_{1,\epsilon}). \end{aligned}$$

Since the harmonic forms, which represent the cohomology classes in $H^*(M_k, \partial_- M_k, g_k; \mathcal{F}_k)$ and $H^*(M_{k,\epsilon}, \partial_- M_{k,\epsilon}, g_{k,\epsilon}; \mathcal{F}_{k,\epsilon})$ are in fact the same, the homomorphisms $(i_{k,\epsilon})_q$ are isometries for any $\epsilon$ (k=1,2). Notice that one can provide a smooth family of diffeomorphisms $\varphi_\epsilon : M \to M_\epsilon$, with $\varphi_0 = id$ so that $\varphi_\epsilon^{\#}(g_\epsilon) = \tilde{g}_\epsilon$ is a smooth family of Riemannian metrics on $M$ with $\tilde{g}_0 = g$. This implies that

$$\lim_{\epsilon \to 0} \log T_{an}(M_\epsilon, \partial_- M_\epsilon, g_\epsilon, \mathcal{F}_\epsilon) = \log T_{an}(M, \partial_- M, g, \mathcal{F}).$$

Using $\varphi_\epsilon$, one can also show that

$$\lim_{\epsilon \to 0} \log Vol((i_\epsilon)_q) = 0.$$

In view of Proposition 1.3 A(iii) $\mathcal{H}$ is of determinant class iff $\mathcal{H}_\epsilon$ is of determinant class (for one and then for any $\epsilon > 0$) and if so

$$\log T(\mathcal{H}) = \log T(\mathcal{H}_\epsilon) + \sum_q (-1)^q \log Vol(i_\epsilon)_q.$$

Note that if the systems $(M_i, \partial_- M_i, \partial_+ M_i, \mathcal{F}_i), i = 1, 2$, are of determinant class, then the systems $(M_{i,\epsilon}, \partial_- M_{i,\epsilon}, \partial_+ M_{i,\epsilon}, \mathcal{F}_{i,\epsilon}), i = 1, 2$, are of determinant class as well. From (3.2) one obtains

$$\log T_{an}(M_{i,\epsilon}, \partial_- M_{i,\epsilon}, g_{i,\epsilon}, \mathcal{F}_{i,\epsilon}) = \log T_{an}(M_i, \partial_- M_i, g_i, \mathcal{F}_i) + \chi_i \cdot \frac{\log 2}{2}$$

with $\chi_1 = \chi(W_1; \mathcal{F}_2 \restriction_{W_1})$ and $\chi_2 = \chi(V_1; \mathcal{F}_1 \restriction_{V_1})$. Then by Theorem 3.2

$$\begin{aligned} \log T_{an}(M_\epsilon, \partial_- M_\epsilon, g_\epsilon, \mathcal{F}_\epsilon) = &\sum_{i=1,2} \log T_{an}(M_i, \partial_- M_i, g_i, \mathcal{F}_i) + \\ &+ \log T(\mathcal{H}_\epsilon) - \chi(V_0, \mathcal{F}_1 \restriction_{V_0}) \cdot \frac{\log 2}{2} \end{aligned}$$

The result follows by passing to the limit $\epsilon \to 0$. $\square$

### 3.4. Comparison of the analytic torsion for different boundary conditions.

Suppose $(M, g)$ is a Riemannian manifold whose boundary $\partial M$ is a disjoint union of three components $\partial_1 M$, $\partial_2 M$, and $\partial_3 M$ and $\mathcal{F}$ is a parallel flat bundle of $\mathcal{A}-$Hilbert modules on $M$. In this subsection we will compare the analytic torsions of $(M, \partial_1 M, g, \mathcal{F})$ and $(M, \partial_1 M \cup \partial_3 M, g, \mathcal{F})$.

Introduce $g_3 := g \restriction_{\partial_3 M}$ and $\mathcal{F}_3 := \mathcal{F} \restriction_{\partial_3 M}$ and denote by $\mathcal{H}'$ the cohomology sequence

$$\begin{aligned} &\cdots \to H^q(M, \partial_1 M \cup \partial_3 M, g; \mathcal{F}) \to H^q(M, \partial_1 M, g; \mathcal{F}) \to \\ &H^q(\partial_3 M, g_3; \mathcal{F}_3) \to H^{q+1}(M, \partial_1 M \cup \partial_3 M, g; \mathcal{F}) \to \cdots \end{aligned} \tag{3.21}$$

which is induced by the short exact sequence

$$0 \to \Lambda^*(M, \partial_1 M \cup \partial_3 M; \mathcal{F}) \to \Lambda^*(M, \partial_1 M; \mathcal{F}) \to \Lambda^*(\partial_3 M; \mathcal{F}_3) \to 0.$$

**Theorem 3.3.** *The system $(M, \partial_1 M, \partial_2 M \cup \partial_3 M, \mathcal{F})$ is of determinant class iff the system $(M, \partial_1 M \cup \partial_3 M, \partial_2 M, \mathcal{F})$ is of determinant class and if so*

$$\begin{aligned} \log T_{an}(M, \partial_1 M, g, \mathcal{F}) &= \log T_{an}(M, \partial_1 M \cup \partial_3 M, g, \mathcal{F}) + \log T(\mathcal{H}') + \\ &\log T_{an}(\partial_3 M, g_3, \mathcal{F}_3). \end{aligned} \tag{3.22}$$

**Remark.** *For $\mathcal{A} = \mathbb{C}$, the result, as it stands, is implicit in Vishik [Vi3]. Again his proof is very different from ours.*

*Proof.* The first statement follows from Proposition 2.5 (4). To prove (3.22), we apply Theorem 3.2′ (partial gluing theorem) to the systems

$$(M_1, \partial_- M_1, \partial_+ M_1, \mathcal{F}_1) := (M, \partial_1 M, \partial_2 M \cup \partial_3 M, \mathcal{F})$$

and, for $\epsilon > 0$,

$$(M_2, \partial_- M_2, \partial_+ M_2, \mathcal{F}_2) := (\partial_3 M \times [0, \epsilon], \partial_3 M \times \{0, \epsilon\}, \emptyset, \mathcal{F}_{[0,\epsilon]})$$

where $\mathcal{F}_{[0,\epsilon]}$ is the pull back of $\mathcal{F} \restriction_{\partial_3 M}$ by the projection $\partial_3 M \times [0, \epsilon] \to \partial_3 M$. We regard $M_1$ equipped with the metric g and $M_2$ with the metric $g_3 \oplus g_0$ ($g_0$ the Euclidean metric). Further, using the same notation as in subsection 3.3,

$$V_0 = W_0 := \partial_3 M, \ V_1 := \partial_2 M \text{ and } W_1 := \partial_3 M.$$

In view of Proposition 2.3 and (3.1′) we have

$$\begin{aligned} \log T_{an}(M_2, \partial_- M_2, g_2, \mathcal{F}_2) &= -\log T_{an}(\partial_3 M, g_3, \mathcal{F}_3) \\ &+ \frac{1}{2} \chi(\partial_3 M; \mathcal{F}_3) \cdot (\log 2 + \log \epsilon). \end{aligned} \tag{3.23}$$

Denote the system obtained by partial gluing by $(N_\epsilon, \partial_- N_\epsilon, \partial_+ N_\epsilon, \mathcal{F}_\epsilon)$ and the resulting metric by $g_\epsilon$. Then, by Theorem 3.2' and (3.23),

$$\begin{aligned}(3.24)\qquad \log T_{an}(N_\epsilon, \partial_- N_\epsilon, g_\epsilon, \mathcal{F}_\epsilon) &= \log T_{an}(M, \partial_1 M, g, \mathcal{F}) + \log T((\mathcal{H}(\epsilon))\\ &- \log T_{an}(\partial_3 M, g_3, \mathcal{F}_3) + \frac{1}{2}\chi(\partial_3 M, \mathcal{F}_3)\cdot \log\epsilon,\end{aligned}$$

where $\mathcal{H}(\epsilon)$ is the cohomology sequence (3.19) for the partial gluing of $(M_1, \partial_- M_1, \partial_+ M_1, \mathcal{F}_1)$ and $(M_2, \partial_- M_2, \partial_+ M_2, \mathcal{F}_2)$. Notice that there exists a commutative diagram connecting the cohomology sequence $\mathcal{H}'$ (cf (3.21)) and the cohomology sequence $\mathcal{H}(\epsilon)$

$$\begin{array}{ccccccc} H^{q-1}(\partial_3 M, g_3; \mathcal{F}_3) & \rightarrow & H^q(M, \partial_1 M \cup \partial_3 M, g; \mathcal{F}) & \rightarrow & H^q(M, \partial_1 M, g; \mathcal{F}) & \cdots \\ \downarrow (r_{2,\epsilon})_q & & \downarrow (r_\epsilon)_q & & \downarrow Id & \\ H^q(M_2, \partial_- M_2, g_3 \oplus g_0; \mathcal{F}_{[0,\epsilon]}) & \rightarrow & H^q(N_\epsilon, \partial_- N_\epsilon, g_\epsilon; \mathcal{F}_\epsilon) & \rightarrow & H^q(M, \partial_1 M, g; \mathcal{F}) & \cdots . \end{array}$$

In this commutative diagram the vertical arrows are isomorphisms of $\mathcal{A}-$Hilbert modules. The maps $(r_\epsilon)_q$ are induced by a family of smooth retractions $r_\epsilon : (N_\epsilon, \partial_- N_\epsilon) \rightarrow (M, \partial_1 M \cup \partial_3 M)$ with $r_{\epsilon=0} = id$ which depends smoothly on $\epsilon$ and has the property that the restriction of $r_\epsilon$ to $M \setminus U_\epsilon$ is the identity where $U_\epsilon$ is a collar neighborhood of $\partial_3 M$ of size $\epsilon$. The maps $((r_{2,\epsilon})_q)$, $(q \geq 1)$, are isomorphisms and induced by the maps, assigning to $\omega \in \Lambda^{q-1}(\partial_3 M; \mathcal{F}_3)$ the wedge product $dt \wedge \omega$. One can easily see that

$$(3.25)\qquad \lim_{\epsilon\rightarrow 0} \log Vol(r_\epsilon)_q = 0$$

and

$$(3.26)\qquad \log Vol((r_{2,\epsilon})_q) = \frac{1}{2}\log\epsilon \cdot dim_N H^{q-1}(\partial_3 M, g_3; \mathcal{F}_3).$$

From Proposition 1.3 A(iii), using (3.26), we conclude that

$$(3.27)-\log T(\mathcal{H}(\epsilon)) = \log T(\mathcal{H}') + \frac{1}{2}\chi(\partial_3 M; \mathcal{F}_3)\log\epsilon + \sum(-1)^q \log Vol(r_\epsilon)_q.$$

Notice that one can provide a smooth family of diffeomorphisms $\varphi_\epsilon : M_\epsilon \rightarrow M, \varphi_0 = id$ so that $\varphi_\epsilon^{\#}(g_\epsilon) = \tilde{g}_\epsilon$ is a smooth family of Riemannian metrics with $\tilde{g}_0 = g$. This implies that

$$\lim_{\epsilon\rightarrow 0} \log T_{an}(N_\epsilon, \partial_- N_\epsilon, g_\epsilon, \mathcal{F}_\epsilon) = \log T_{an}(M, \partial_1 M \cup \partial_3 M, g, \mathcal{F})$$

The result then follows from (3.24), (3.25) and (3.27) by passing to the limit $\epsilon \rightarrow 0$. □

**Appendix: On manifolds of determinant class.**

To state our result let us introduce the following notation. Let $M$ be a compact, connected manifold (possibly with boundary) with fundamental group $\pi_1(M)$. Let $\Gamma$ be a group and $\theta$ an arbitrary group homomorphism, $\theta : \pi_1(M) \to \Gamma$. Denote by $\mathcal{W}_0$ the $(\mathcal{N}(\Gamma), \pi_1(M)^{op})-$Hilbert module $\ell_2(\Gamma)$ with the $\pi_1(M)^{op}-$action induced by $\theta$ and the right translation of $\Gamma$. Denote by $\mathcal{F}_\theta$ the parallel flat bundle of $\mathcal{N}(\Gamma)-$Hilbert modules of finite type, induced by $\mathcal{W}_0$. Notice that this is the parallel flat bundle induced by the principal covering $\tilde{M} \to M$, defined by $\theta$.

**Theorem A.** *If $\Gamma$ is residually finite then for any bordism $(M, \partial_- M, \partial_+ M)$, the pair $\{(M, \partial_- M, \partial_+ M), \mathcal{F}_\theta\}$ is of determinant class. As a consequence, the system $(M, \partial_- M, \partial_+ M, \mathcal{F}_\theta)$ is of determinant class.*

Although Lück does not use the notion of determinant class, Theorem A is implicit in his paper [Lü3]. Unfortunately, there are a number of misleading misprints and his definition of the $L^2$-determinant [Lü3, Definition, p. 471] is incorrect. For the convenience of the reader we present an outline of Lück's arguments to prove Theorem A.

We recall that a group $\Gamma$ is residually finite if there exists a sequence $(\Gamma_m)_{m\geq 1}$ of nested normal subgroups of $\Gamma$, $\Gamma_0 = \Gamma \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \dots$, such that

(i) $\bigcap_{m\geq 1} \Gamma_m = \{e\}$; (ii) $\gamma_m := \#(\Gamma : \Gamma_m) < \infty$ $(m \geq 1)$.

Further recall that $\big(\mathcal{C}^*(M, \tau, \mathcal{O}_h, \mathcal{F}_\theta), \delta_*\big)$ is of determinant class if, for $0 \leq q \leq d$,

$$-\infty < \int_{0^+}^1 \log \lambda dN_q(\lambda),$$

where $N_q(\lambda) = N_q^{comb}(\lambda)$ denotes the $L_2$-combinatorial spectral distribution function of the combinatorial q-Laplacian $\Delta_q = \Delta_q^{comb}$.

Let $M_m$ be the principal $(\Gamma/\Gamma_m)$- finite cover of $M$ induced by the composition of $\theta$ with the projection $\Gamma \to \Gamma/\Gamma_m$ and denote by $\Delta_{m;q} \equiv \Delta_{m;q}^{comb}$ the combinatorial $q$-Laplacians on $M_m$. Further let $P_{m;q}(\lambda) \equiv P_{m;q}^{comb}(\lambda)$ be the spectral family associated to $\Delta_{m;q}$, defined in such a way that it is right continuous, and introduce the corresponding normalized spectral distribution function

$$N_{m;q}(\lambda) = \frac{1}{\gamma_m} tr P_{m;q}(\lambda).$$

We point out $N_{m;q}(\lambda)$ are step functions. Denote by $det' \Delta_{m;q}$ the modified determinant of $\Delta_{m,q}$, i.e. the product of all <u>non zero</u> eigenvalues of $\Delta_{m;q}$. Let $a_{m;q}$ be the smallest non zero eigenvalue and $b_{m;q}$ the largest eigenvalue of $\Delta_{m;q}$. Then, for any $a$ and $b$, such that $0 < a < a_{m;q}$ and $b > b_{m;q}$,

$$\frac{1}{\gamma_m} \log det' \Delta_{m;q} = \int_a^b \log \lambda dN_{m;q}(\lambda). \tag{A.1}$$

Integrating by parts, the Stieltjes integral $\int_a^b \log \lambda dN_{m;q}(\lambda)$ can be written as

$$
\begin{aligned}
(A.2) \qquad \int_a^b \log \lambda dN_{m;q}(\lambda) =& (\log b)\big(N_{m;q}(b) - N_{m;q}(0)\big) \\
&- \int_a^b \frac{N_{m;q}(\lambda) - N_{m;q}(0)}{\lambda} d\lambda.
\end{aligned}
$$

Recall that the spectral distribution function $N_q(\lambda) \equiv N_q^{comb}(\lambda)$ is given by

$$
N_q(\lambda) := tr_{\mathcal{N}(\Gamma)} P_q(\lambda)
$$

with $tr_{\mathcal{N}(\Gamma)} P_q(\lambda)$ denoting the von Neumann trace where $P_q(\lambda)$ is the spectral family corresponding to $\Delta_q$ defined in such a way that it is right continuous with respect to $\lambda$. Notice that $N_q(\lambda)$ is continuous to the right in $\lambda$. Denote by $det'\Delta_q$ the modified determinant of $\Delta_q$, given by the following Stieltjes integral,

$$
\log\, det'\Delta_q = \int_{0+}^b \log \lambda dN_q(\lambda) := \lim_{\epsilon \to 0^+} \int_\epsilon^b \log \lambda dN_q(\lambda)
$$

with $1 \leq b < \infty$ chosen in such a way that $|||\Delta_q||| < b$ (operator norm).

Notice that $\log\, det'(\Delta_q) \in [-\infty, \infty)$ and recall that $\Delta_q$ is said to be of determinant class if $\int_{0+}^b \log \lambda dN_q(\lambda) \in (-\infty, \infty)$.

Integrating by parts, one obtains

$$
\begin{aligned}
(A.3) \quad \log\, det'(\Delta_q) =& \log b\big(N_q(b) - N_q(0)\big) \\
&+ \lim_{\epsilon \to 0^+} \{(-\log \epsilon)\big(N_q(\epsilon) - N_q(0)\big) - \int_\epsilon^b \frac{N_q(\lambda) - N_q(0)}{\lambda} d\lambda\}.
\end{aligned}
$$

Using that $\liminf\limits_{\epsilon \to 0^+} (-\log \epsilon)\big(N_q(\epsilon) - N_q(0)\big) \geq 0$ (in fact, this limit exists and is zero) and $\frac{N_q(\lambda) - N_q(0)}{\lambda} \geq 0$ for $\lambda > 0$, one sees that

$$
(A.4) \qquad \log\, det'(\Delta_q) \geq (\log b)\big(N_q(b) - N_q(0)\big) - \int_0^b \frac{N_q(\lambda) - N_q(0)}{\lambda} d\lambda.
$$

*Proof of Theorem A.2.*

The main ingredient is Lück's estimate of $\log\, det'(\Delta_q)$ in terms of $\log\, det'\Delta_{m,q}$ combined with the fact that $\log\, det'\Delta_{m,q} \geq 0$ as the determinant $det'\Delta_{m,q}$ is integer valued. By [Lü, Lemma 2.5], there exists $1 \leq b < \infty$ such that, for $m \geq 1$,

$$
|||\Delta_{m,q}||| \leq b; \quad |||\Delta_q||| \leq b.
$$

By [Lü, Lemma 3.3.1],

$$(A.5)\qquad \int_0^b \frac{N_q(\lambda)-N_q(0)}{\lambda}d\lambda \leq \liminf_{m\to\infty}\int_0^b \frac{N_{m;q}(\lambda)-N_{m;q}(0)}{\lambda}d\lambda.$$

Combining (A.1) and (A.2) with the inequalities log $det'\Delta_{m;q} \geq 0$, it follows that

$$(A.6)\qquad \int_0^b \frac{N_{m;q}(\lambda)-N_{m;q}(0)}{\lambda}d\lambda \leq (\log b)\big(N_{m;q}(b)-N_{m;q}(0)\big).$$

¿From (A.4) - (A.6) we then conclude that

$$(A.7)\qquad \begin{aligned}\log\, det'\Delta_q \geq &(\log b)\big(N_q(b)-N_q(0)\big)\\ &-\liminf_{m\to\infty}(\log b)\big(N_{m;q}(b)-N_{m;q}(0)\big).\end{aligned}$$

The estimate (A.7) together with the identities ([Lü3, Theorem 2.3.1])

$$N_q(\lambda) = \lim_{\epsilon\to 0^+}\liminf_{m\to\infty} N_{m;q}(\lambda+\epsilon)\ \ \text{([Lü3, Theorem 2.3.1])}$$

and

$$N_q(0) = \lim_{m\to\infty} N_{m;q}(0)\ \ \text{([Lü3, Theorem 2.3.2])}$$

yield that log $det'\Delta_q \geq 0$ which proves that $\Delta_q$ is of determinant class. Since $q$ is arbitrary this shows that the pair $\{(M,\partial M_-,\partial M_+),\mathcal{F}_\theta\}$ is of determinant class.

$\square$

## References


[BZ] J. P. Bismut, W. Zhang, *An extension of a theorem by Cheeger and Müller*, Astérisque 205 (1992), 1-223.

[BFK] D. Burghelea, L. Friedlander, T. Kappeler, *Asymptotic expansion of the Witten deformation of the analytic torsion*, J. Funct. Anal. 137 (1996), 320-363.

[BFK2] D. Burghelea, L. Friedlander, T. Kappeler, *Relative torsion for representations in finite type Hilbert modules*, in preparation.

[BFKM] D. Burghelea, L. Friedlander, T. Kappeler, P. McDonald, *Analytic and Reidemeister torsion for representations in finite type Hilbert modules*, GAFA 6 (1996), 751-859.

[CMM] A. L. Carey, V. Mathai, A. Mischenko, *On analytic torsion over $C^*$-algebras*, preprint.

[Ch] J. Cheeger, *Analytic torsion and the heat equation*, Ann. of Math. 109 (1979), 259-300.

[Do] J. Dodziuk, *de Rham-Hodge theory for $L^2$ cohomology of infinite coverings*, Topology 16 (1977), 157-165.

[Ef1] A. V. Efremov, *Combinatorial and analytic Novikov-Shubin invariants*, preprint.

[Ef2] A. V. Efremov, *Cellular decomposition and Novikov-Shubin invariants*, Russian Math. Surveys 46 (1991), 219-220.

[FK] B. Fuglede, R. V. Kadison, *Determinant theory in finite factors*, Ann. of Math. 55 (1952), 520-530.

[GS] M. Gromov, M. A. Shubin, *Von Neumann spectra near zero*, Geom. and Funct. Anal. 1 (1991), 375-404.

[LL] J. Lott, W. Lück, *$L^2$-topological invariants of 3-manifolds*, preprint.

[Lü1] W. Lück, *Analytic and algebraic torsion for manifolds with boundary and symmetries*, J. of Diff. Geo. 37 (1993), 263-322.

[Lü2] W. Lück, *$L^2-$torsion and 3-manifolds*, preprint.

[Lü3] W. Lück, *Approximating $L^2$-invariants by their finite dimensional analogues*, GAFA 4 (1994), 455-481.

[Mi1] J. Milnor, *Whitehead torsion*, Bull. AMS 72 (1966), 358-426.

[Mi2] J. Milnor, *Lectures on $h$-cobordism theorem*, Princeton University Press.

[RS] D. B. Ray, I. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Adv. in Math. 7 (1971), 145-210.

[Vi1] S. M. Vishik, *Analytic torsion of boundary value problems*, Soviet Math. Dokl. 36 (1988), 174-179.

[Vi2] S. M. Vishik, *The analytic torsion of finite dimensional local systems*, Soviet Math. Dokl. 37 (1988), 803-807.

[Vi3] S. M. Vishik, *Generalized Ray-Singer conjecture. I: a manifold with a smooth boundary*, preprint.