Complex valued Ray–Singer torsion

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Abstract. In the spirit of Ray and Singer we define a complex valued analytic torsion using non-selfadjoint Laplacians. We establish an anomaly formula which permits to turn this into a topological invariant. Conjecturally this analytically defined invariant computes the complex valued Reidemeister torsion, including its phase. We establish this conjecture in some non-trivial situations.

Mathematics Subject Classification (2000). 57R20, 58J52.

Keywords. Ray–Singer torsion; Reidemeister torsion; analytic torsion; combinatorial torsion; anomaly formula; Bismut–Zhang, Cheeger, Müller theorem; Euler structures; co-Euler structures; asymptotic expansion; heat kernel; Dirac operator;

Contents

Part of this work was done while the second author enjoyed the hospitality of the Max Planck Institute for Mathematics in Bonn. During the final stage of preparation of this manuscript the second author was partially supported by the Austrian Science Fund (FWF), grant P19392-N13.

1. Introduction

Let M be a closed connected smooth manifold with Riemannian metric g. Suppose E is a flat complex vector bundle over M. Let h be a Hermitian metric on E. Recall the deRham differential $d_E : \Omega^*(M;E) \to \Omega^{*+1}(M;E)$ on the space of E-valued differential forms. Let $d^*_{E,g,h} : \Omega^{*+1}(M;E) \to \Omega^*(M;E)$ denote its formal adjoint with respect to the Hermitian scalar product on $\Omega^*(M;E)$ induced by g and h. Consider the Laplacian $\Delta_{E,g,h} = d_E d^*_{E,g,h} + d^*_{E,g,h} d_E : \Omega^*(M;E) \to \Omega^*(M;E)$. Recall the (inverse square of the) Ray–Singer torsion [29]

$$\prod_q \bigl(\det{}'(\Delta_{E,g,h,q})\bigr)^{(-1)^q q} \in \mathbb{R}^+.$$

Here $\det'(\Delta_{E,g,h,q})$ denotes the zeta regularized product of all non-zero eigen values of the Laplacian acting in degree q. This is a positive real number which coincides, up to a computable correction term, with the absolute value of the Reidemeister torsion, see [2].

The aim of this paper is to introduce a complex valued Ray–Singer torsion which, conjecturally, computes the Reidemeister torsion, including its phase. This is accomplished by replacing the Hermitian fiber metric h with a fiber wise non-degenerate symmetric bilinear form b on E. The bilinear form b permits to define a formal transposed $d^\sharp_{E,g,b}$ of d_E, and an in general not selfadjoint Laplacian $\Delta_{E,g,b} := d_E d^\sharp_{E,g,b} + d^\sharp_{E,g,b} d_E$. The (inverse square of the) complex valued Ray–Singer torsion is then defined by

$$\prod_q \bigl(\det{}'(\Delta_{E,g,b,q})\bigr)^{(-1)^q q} \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\}. \tag{1}$$

The main result proved here, see Theorem 4.2 below, is an anomaly formula for the complex valued Ray–Singer torsion, i.e. we compute the variation of the quantity (1) through a variation of g and b. This ultimately permits to define a smooth invariant, the analytic torsion.

The paper is roughly organized as follows. In Section 2 we recall Euler and coEuler structures. These are used to turn the Reidemeister torsion and the complex valued Ray–Singer torsion into topological invariants referred to as combinatorial and analytic torsion, respectively. In Section 3 we discuss some finite dimensional linear algebra and recall the combinatorial torsion which was also called Milnor–Turaev torsion in [11]. Section 4 contains the definition of the proposed complex valued analytic torsion. In Section 5 we formulate a conjecture, see Conjecture 5.1, relating the complex valued analytic torsion with the combinatorial torsion. We establish this conjecture in some non-trivial cases via analytic continuation from a result of Cheeger [16, 17], Müller [28] and Bismut–Zhang [2]. Section 6 contains the derivation of the anomaly formula. This proof is based on the computation of leading and subleading terms in the asymptotic expansion of the heat kernel associated with a certain class of Dirac operators. This asymptotic expansion is formulated and proved in Section 7, see Theorem 7.1. In Section 8 we apply this result to the Laplacians $\Delta_{E,g,b}$ and therewith complete the proof of the anomaly formula.

We restrict the presentation to the case of vanishing Euler–Poincaré characteristics to avoid geometric regularization, see [11] and [12]. With minor modifications everything can easily be extended to the general situation. This is sketched in Section 9. The analytic core of the results, Theorem 7.1 and its corollaries Propositions 6.1 and 6.2, are formulated and proved without any restriction on the Euler–Poincaré characteristics.

Let us also mention the series of recent preprints [3, 4, 5, 6, 7]. In these papers Braverman and Kappeler construct a "refined analytic torsion" based on the odd signature operator on odd dimensional manifolds. Their torsion is closely related to the analytic torsion proposed in this paper. For a comparison result see Theorem 1.4 in [7]. Some of the results below which partially establish Conjecture 5.1, have first appeared in [7], and were not contained in the first version of this paper. The proofs we will provide have been inspired by [7] but do not rely on the results therein.

Recently, in October 2006, two preprints [15] and [33] have been posted on the internet providing the proof of Conjecture 5.1. In [15] Witten–Helffer–Sjöstrand theory has been extended to the non-selfadjoint Laplacians discussed here, and used along the lines of [10], to establish Conjecture 5.1 for odd dimensional manifolds, up to sign. Comments were made how to derive the conjecture in full generality on these lines. A few days earlier, by adapting the methods in [2] to the non-selfadjoint situation, Su and Zhang in [33] provided a proof of the conjecture.

The definition of the complex valued analytic torsion was sketched in [14].

We thank the referees for useful remarks and for pointing out several sign mistakes.

2. Preliminaries

Throughout this section M denotes a closed connected smooth manifold of dimension n. For simplicity we will also assume vanishing Euler–Poincaré characteristics, $\chi(M) = 0$. At the expense of a base point everything can easily be extended to the general situation, see [8], [11], [12] and Section 9.

Euler structures

Let M be a closed connected smooth manifold of dimension n with $\chi(M) = 0$. The set of *Euler structures with integral coefficients* $\mathfrak{Eul}(M;\mathbb{Z})$ is an *affine version* of $H_1(M;\mathbb{Z})$. That is, the homology group $H_1(M;\mathbb{Z})$ acts free and transitively on $\mathfrak{Eul}(M;\mathbb{Z})$ but in general there is no distinguished origin. Euler structures have been introduced by Turaev [34] in order to remove the ambiguities in the definition of the Reidemeister torsion. Below we will briefly recall a possible definition. For more details we refer to [11] and [12].

Recall that a vector field X is called non-degenerate if $X : M \to TM$ is transverse to the zero section. Denote its set of zeros by $\mathcal{X}$. Recall that every

$x \in \mathcal{X}$ has a *Hopf index* $\mathrm{IND}_X(x) \in \{\pm 1\}$. Consider pairs (X, c) where X is a non-degenerate vector field and $c \in C_1^{\mathrm{sing}}(M;\mathbb{Z})$ is a singular 1-chain satisfying

$$\partial c = \mathrm{e}(X) := \sum_{x \in \mathcal{X}} \mathrm{IND}_X(x) x.$$

Every non-degenerate vector field admits such c since we assumed $\chi(M) = 0$.

We call two such pairs (X_1, c_1) and (X_2, c_2) equivalent if

$$c_2 - c_1 = \mathrm{cs}(X_1, X_2) \in C_1^{\mathrm{sing}}(M;\mathbb{Z})/\partial C_2^{\mathrm{sing}}(M;\mathbb{Z}).$$

Here $\mathrm{cs}(X_1, X_2) \in C_1^{\mathrm{sing}}(M;\mathbb{Z})/\partial C_2^{\mathrm{sing}}(M;\mathbb{Z})$ denotes the *Chern–Simons class* which is represented by the zero set of a generic homotopy connecting X_1 with X_2. It follows from $\mathrm{cs}(X_1, X_2) + \mathrm{cs}(X_2, X_3) = \mathrm{cs}(X_1, X_3)$ that this indeed is an equivalence relation.

Define $\mathfrak{Eul}(M;\mathbb{Z})$ as the set of equivalence classes $[X, c]$ of pairs considered above. The action of $[\sigma] \in H_1(M;\mathbb{Z})$ on $[X, c] \in \mathfrak{Eul}(M;\mathbb{Z})$ is simply given by $[X, c] + [\sigma] := [X, c + \sigma]$. Since $\mathrm{cs}(X, X) = 0$ this action is well defined and free. Because of $\partial\, \mathrm{cs}(X_1, X_2) = \mathrm{e}(X_2) - \mathrm{e}(X_1)$ it is transitive.

Replacing singular chains with integral coefficients by singular chains with real or complex coefficients we obtain in exactly the same way *Euler structures with real coefficients* $\mathfrak{Eul}(M;\mathbb{R})$ and *Euler structures with complex coefficients* $\mathfrak{Eul}(M;\mathbb{C})$. These are affine version of $H_1(M;\mathbb{R})$ and $H_1(M;\mathbb{C})$, respectively. There are obvious maps $\mathfrak{Eul}(M;\mathbb{Z}) \to \mathfrak{Eul}(M;\mathbb{R}) \to \mathfrak{Eul}(M;\mathbb{C})$ which are affine over the homomorphisms $H_1(M;\mathbb{Z}) \to H_1(M;\mathbb{R}) \to H_1(M;\mathbb{C})$. We refer to the image of $\mathfrak{Eul}(M;\mathbb{Z})$ in $\mathfrak{Eul}(M;\mathbb{R})$ or $\mathfrak{Eul}(M;\mathbb{C})$ as the *lattice of integral Euler structures*.

Since we have $\mathrm{e}(-X) = (-1)^n \mathrm{e}(X)$ and $\mathrm{cs}(-X_1, -X_2) = (-1)^n \mathrm{cs}(X_1, X_2)$, the assignment $\nu([X, c]) := [-X, (-1)^n c]$ defines *affine involutions* on $\mathfrak{Eul}(M;\mathbb{Z})$, $\mathfrak{Eul}(M;\mathbb{R})$ and $\mathfrak{Eul}(M;\mathbb{C})$. If n is even, then the involutions on $\mathfrak{Eul}(M;\mathbb{R})$ and $\mathfrak{Eul}(M;\mathbb{C})$ are affine over the identity and so we must have $\nu = \mathrm{id}$. If n is odd the involutions on $\mathfrak{Eul}(M;\mathbb{R})$ and $\mathfrak{Eul}(M;\mathbb{C})$ are affine over $-\mathrm{id}$ and thus must have a unique fixed point $\mathfrak{e}_{\mathrm{can}} \in \mathfrak{Eul}(M;\mathbb{R}) \subseteq \mathfrak{Eul}(M;\mathbb{C})$. This *canonic Euler structure* permits to naturally identify $\mathfrak{Eul}(M;\mathbb{R})$ resp. $\mathfrak{Eul}(M;\mathbb{C})$ with $H_1(M;\mathbb{R})$ resp. $H_1(M;\mathbb{C})$, provided n is odd. Note that in general none of these statements is true for the involution on $\mathfrak{Eul}(M;\mathbb{Z})$. This is due to the fact that in general $H_1(M;\mathbb{Z})$ contains non-trivial elements of order 2, and elements which are not divisible by 2.

Finally, observe that the assignment $[X, c] \mapsto [X, \bar{c}]$ defines a *conjugation* $\mathfrak{e} \mapsto \bar{\mathfrak{e}}$ on $\mathfrak{Eul}(M;\mathbb{C})$ which is affine over the complex conjugation $H_1(M;\mathbb{C}) \to H_1(M;\mathbb{C})$, $[\sigma] \mapsto [\bar{\sigma}]$. Clearly, the set of fixed points of this conjugation coincides with $\mathfrak{Eul}(M;\mathbb{R}) \subseteq \mathfrak{Eul}(M;\mathbb{C})$.

Lemma 2.1. *Let M be a closed connected smooth manifold with $\chi(M) = 0$, let $\mathfrak{e} \in \mathfrak{Eul}(M;\mathbb{Z})$ be an Euler structure, and let $x_0 \in M$ be a base point. Suppose X is a non-degenerate vector field on M with zero set $\mathcal{X} \neq \emptyset$. Then there exists a collection of paths σ_x, $\sigma_x(0) = x_0$, $\sigma_x(1) = x$, $x \in \mathcal{X}$, so that $\mathfrak{e} = \big[X, \sum_{x \in \mathcal{X}} \mathrm{IND}_X(x) \sigma_x\big]$.*

Proof. For every zero $x \in \mathcal{X}$ choose a path $\tilde\sigma_x$ with $\tilde\sigma_x(0) = x_0$ and $\tilde\sigma_x(1) = x$. Set $\tilde c := \sum_{x\in\mathcal{X}} \mathrm{IND}_X(x)\tilde\sigma_x$. Since $\chi(M) = 0$ we clearly have $\partial\tilde c = \mathrm{e}(X)$. So the pair $(X, \tilde c)$ represents an Euler structure $\tilde{\mathfrak{e}} := [X, \tilde c] \in \mathfrak{Eul}(M;\mathbb{Z})$. Because $H_1(M;\mathbb{Z})$ acts transitively on $\mathfrak{Eul}(M;\mathbb{Z})$ we find $a \in H_1(M;\mathbb{Z})$ with $\tilde{\mathfrak{e}} + a = \mathfrak{e}$. Since the Huréwicz homomorphism is onto we can represent a by a closed path π with $\pi(0) = \pi(1) = x_0$. Choose $y \in \mathcal{X}$. Define σ_y as the concatenation of $\tilde\sigma_y$ with $\pi^{\mathrm{IND}_X(y)}$, and set $\sigma_x := \tilde\sigma_x$ for $x \neq y$. Then the pair $(x, \sum_{x\in\mathcal{X}} \mathrm{IND}_X(x)\sigma_x)$ represents $\tilde{\mathfrak{e}} + a = \mathfrak{e}$. □

CoEuler structures

Let M be a closed connected smooth manifold of dimension n with $\chi(M) = 0$. The set of *coEuler structures* $\mathfrak{Eul}^*(M;\mathbb{C})$ is an affine version of $H^{n-1}(M;\mathcal{O}_M^{\mathbb{C}})$. That is the cohomology group $H^{n-1}(M;\mathcal{O}_M^{\mathbb{C}})$ with values in the complexified orientation bundle $\mathcal{O}_M^{\mathbb{C}}$ acts free and transitively on $\mathfrak{Eul}^*(M;\mathbb{C})$. CoEuler structures are well suited to remove the metric dependence from the Ray–Singer torsion. Below we will briefly recall their definition, and discuss an affine version of Poincaré duality relating Euler with coEuler structures. For more details and the general situation we refer to [11] or [12].

Consider pairs (g, α), g a Riemannian metric on M, $\alpha \in \Omega^{n-1}(M;\mathcal{O}_M^{\mathbb{C}})$, which satisfy

$$d\alpha = \mathrm{e}(g).$$

Here $\mathrm{e}(g) \in \Omega^n(M;\mathcal{O}_M^{\mathbb{C}})$ denotes the *Euler form* associated with g. In view of the Gauss–Bonnet theorem every g admits such α for we assumed $\chi(M) = 0$.

Two pairs (g_1, α_1) and (g_2, α_2) as above are called equivalent if

$$\alpha_2 - \alpha_1 = \mathrm{cs}(g_1, g_2) \in \Omega^{n-1}(M;\mathcal{O}_M^{\mathbb{C}})/d\Omega^{n-2}(M;\mathcal{O}_M^{\mathbb{C}}).$$

Here $\mathrm{cs}(g_1, g_2) \in \Omega^{n-1}(M;\mathcal{O}_M^{\mathbb{C}})/d\Omega^{n-2}(M;\mathcal{O}_M^{\mathbb{C}})$ denotes the *Chern–Simons class* [18] associated with g_1 and g_2. Since $\mathrm{cs}(g_1, g_2)+\mathrm{cs}(g_2, g_3) = \mathrm{cs}(g_1, g_3)$ this is indeed an equivalence relation.

Define the set of *coEuler structures with complex coefficients* $\mathfrak{Eul}^*(M;\mathbb{C})$ as the set of equivalence classes $[g, \alpha]$ of pairs considered above. The action of $[\beta] \in H^{n-1}(M;\mathcal{O}_M^{\mathbb{C}})$ on $[g, \alpha] \in \mathfrak{Eul}^*(M;\mathbb{C})$ is defined by $[g, \alpha] + [\beta] := [g, \alpha - \beta]$. Since $\mathrm{cs}(g, g) = 0$ this action is well defined and free. Because of $d\,\mathrm{cs}(g_1, g_2) = \mathrm{e}(g_2) - \mathrm{e}(g_1)$ it is transitive too.

Replacing forms with values in $\mathcal{O}_M^{\mathbb{C}}$ by forms with values in the real orientation bundle $\mathcal{O}_M^{\mathbb{R}}$ we obtain in exactly the same way *coEuler structures with real coefficients* $\mathfrak{Eul}^*(M;\mathbb{R})$, an affine version of $H^{n-1}(M;\mathcal{O}_M^{\mathbb{R}})$. There is an obvious map $\mathfrak{Eul}^*(M;\mathbb{R}) \to \mathfrak{Eul}^*(M;\mathbb{C})$ which is affine over the homomorphism $H^{n-1}(M;\mathcal{O}_M^{\mathbb{R}}) \to H^{n-1}(M;\mathcal{O}_M^{\mathbb{C}})$.

In view of $(-1)^n \mathrm{e}(g) = \mathrm{e}(g)$ and $(-1)^n \mathrm{cs}(g_1, g_2) = \mathrm{cs}(g_1, g_2)$ the assignment $\nu([g, \alpha]) := [g, (-1)^n\alpha]$ defines *affine involutions* on $\mathfrak{Eul}^*(M;\mathbb{R})$ and $\mathfrak{Eul}^*(M;\mathbb{C})$. For even n these involutions are affine over the identity and so we must have $\nu = \mathrm{id}$. For odd n they are affine over $-\mathrm{id}$ and thus must have a unique fixed point $\mathfrak{e}^*_{\mathrm{can}} \in \mathfrak{Eul}^*(M;\mathbb{R}) \subseteq \mathfrak{Eul}^*(M;\mathbb{C})$. Since $\mathrm{e}(g) = 0$ in this case, we have $\mathfrak{e}^*_{\mathrm{can}} = [g, 0]$

where g is any Riemannian metric. This *canonic coEuler structure* provides a natural identification of $\mathfrak{Eul}^*(M;\mathbb{R})$ resp. $\mathfrak{Eul}^*(M;\mathbb{C})$ with $H^{n-1}(M;\mathcal{O}_M^{\mathbb{R}})$ resp. $H^{n-1}(M;\mathcal{O}_M^{\mathbb{C}})$, provided the dimension is odd.

Finally, observe that the assignment $[g,\alpha] \mapsto [g,\bar\alpha]$ defines a *complex conjugation* $\mathfrak{e}^* \mapsto \bar{\mathfrak{e}}^*$ on $\mathfrak{Eul}^*(M;\mathbb{C})$ which is affine over the complex conjugation $H^{n-1}(M;\mathcal{O}_M^{\mathbb{C}}) \to H^{n-1}(M;\mathcal{O}_M^{\mathbb{C}})$, $[\beta] \mapsto [\bar\beta]$. Clearly, the set of fixed points of this conjugation coincides with the image of $\mathfrak{Eul}^*(M;\mathbb{R}) \subseteq \mathfrak{Eul}^*(M;\mathbb{C})$.

Poincaré duality for Euler structures

Let M be a closed connected smooth manifold of dimension n with $\chi(M) = 0$. There is a canonic isomorphism

$$P : \mathfrak{Eul}(M;\mathbb{C}) \to \mathfrak{Eul}^*(M;\mathbb{C}) \tag{2}$$

which is affine over the Poincaré duality $H_1(M;\mathbb{C}) \to H^{n-1}(M;\mathcal{O}_M^{\mathbb{C}})$. If $[X,c] \in \mathfrak{Eul}(M;\mathbb{C})$ and $[g,\alpha] \in \mathfrak{Eul}^*(M;\mathbb{C})$ then $P([X,c]) = [g,\alpha]$ iff we have

$$\int_{M\setminus\mathcal{X}} \omega \wedge (X^*\Psi(g) - \alpha) = \int_c \omega \tag{3}$$

for all closed one forms ω which vanish in a neighborhood of $\mathcal{X}$, the zero set of X. Here $\Psi(g) \in \Omega^{n-1}(TM \setminus M; \pi^*\mathcal{O}_M^{\mathbb{C}})$ denotes the *Mathai–Quillen form* [26] associated with g, and $\pi : TM \to M$ denotes the projection. With a little work one can show that (3) does indeed define an assignment as in (2). Once this is established (2) is obviously affine over the Poincaré duality and hence an isomorphism. It follows immediately from $(-X)^*\Psi(g) = (-1)^n X^*\Psi(g)$ that P intertwines the involution on $\mathfrak{Eul}(M;\mathbb{C})$ with the involution on $\mathfrak{Eul}^*(M;\mathbb{C})$. Moreover, P obviously intertwines the complex conjugations on $\mathfrak{Eul}(M;\mathbb{C})$ and $\mathfrak{Eul}^*(M;\mathbb{C})$. Particularly, (2) restricts to an isomorphism

$$P : \mathfrak{Eul}(M;\mathbb{R}) \to \mathfrak{Eul}^*(M;\mathbb{R})$$

affine over the Poincaré duality $H_1(M;\mathbb{R}) \to H^{n-1}(M;\mathcal{O}_M^{\mathbb{R}})$.

Kamber–Tondeur form

Suppose E is a flat complex vector bundle over a smooth manifold M. Let ∇^E denote the flat connection on E. Suppose b is a fiber wise non-degenerate symmetric bilinear form on E. The *Kamber–Tondeur form* is the one form

$$\omega_{E,b} := -\tfrac{1}{2}\operatorname{tr}(b^{-1}\nabla^E b) \in \Omega^1(M;\mathbb{C}). \tag{4}$$

More precisely, for a vector field Y on M we have $\omega_{E,b}(Y) := \operatorname{tr}(b^{-1}\nabla^E_Y b)$. Here the derivative of b with respect to the induced flat connection on $(E\otimes E)'$ is considered as $\nabla^E_Y b : E \to E'$. Then $b^{-1}\nabla^E_Y b : E \to E$ and $\omega_{E,b}(Y)$ is obtained by taking the fiber wise trace.

The bilinear form b induces a non-degenerate bilinear form $\det b$ on $\det E := \Lambda^{\operatorname{rk}(E)}E$. From $\det b^{-1}\nabla^{\det E}(\det b) = \operatorname{tr}(b^{-1}\nabla^E b)$ we obtain

$$\omega_{\det E,\det b} = \omega_{E,b}. \tag{5}$$

Particularly, $\omega_{E,b}$ depends on the flat line bundle $\det E$ and the induced bilinear form $\det b$ only. Since ∇^E is flat, $\omega_{E,b}$ is a closed 1-form, cf. (5).

Suppose b_1 and b_2 are two fiber wise non-degenerate symmetric bilinear forms on E. Set $A := b_1^{-1}b_2 \in \mathrm{Aut}(E)$, i.e. $b_2(v,w) = b_1(Av,w)$ for all v, w in the same fiber of E. Then $\det b_2 = \det b_1 \det A$, hence

$$\nabla^{\det E}(\det b_2) = \nabla^{\det E}(\det b_1)\det A + (\det b_1) d\det A$$

and therefore

$$\omega_{E,b_2} = \omega_{E,b_1} + \det A^{-1} d \det A. \tag{6}$$

If $\det b_1$ and $\det b_2$ are homotopic as fiber wise non-degenerate bilinear forms on $\det E$, then the function $\det A : M \to \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ is homotopic to the constant function 1. So we find a function $\log\det A : M \to \mathbb{C}$ with $d\log\det A = \det A^{-1} d\det A$, and in view of (6) the cohomology classes of ω_{E,b_1} and ω_{E,b_2} coincide. We conclude that the cohomology class $[\omega_{E,b}] \in H^1(M;\mathbb{C})$ depends on the flat line bundle $\det E$ and the homotopy class $[\det b]$ of the induced non-degenerate bilinear form $\det b$ on $\det E$ only.

If E_1 and E_2 are two flat vector bundles with fiber wise non-degenerate symmetric bilinear forms b_1 and b_2 then

$$\omega_{E_1\oplus E_2, b_1\oplus b_2} = \omega_{E_1,b_1} + \omega_{E_2,b_2}. \tag{7}$$

If E' denotes the dual of a flat vector bundle E, and if b' denotes the bilinear form on E' induced from a fiber wise non-degenerate symmetric bilinear form b on E then clearly

$$\omega_{E',b'} = -\omega_{E,b}. \tag{8}$$

If $\bar{E}$ denotes the complex conjugate of a flat complex vector bundle E, and if $\bar{b}$ denotes the complex conjugate bilinear form of a fiber wise non-degenerate symmetric bilinear form b on E, then obviously

$$\omega_{\bar{E},\bar{b}} = \overline{\omega_{E,b}}. \tag{9}$$

Finally, if F is a real flat vector bundle and h is a fiber wise non-degenerate symmetric bilinear form on F one defines in exactly the same way a real Kamber–Tondeur form $\omega_{F,h} := -\frac{1}{2}\operatorname{tr}(h^{-1}\nabla^F h)$ which is closed too. If $F^{\mathbb{C}} := F \otimes \mathbb{C}$ denotes the complexification of F and $h^{\mathbb{C}}$ denotes the complexification of h then clearly

$$\omega_{F^{\mathbb{C}},h^{\mathbb{C}}} = \omega_{F,h} \tag{10}$$

in $\Omega^1(M;\mathbb{R}) \subseteq \Omega^1(M;\mathbb{C})$. Note that all such h give rise to the same cohomology class $[\omega_{F,h}] \in H^1(M;\mathbb{R})$, see (5) and (6). To see this also note that the induced fiber wise non-degenerate bilinear form $\det h$ on $\det F$ has to be positive definite or negative definite, but $\omega_{\det F, -\det h} = \omega_{\det F, \det h}$.

Holonomy

Suppose E is a flat complex vector bundle over a connected smooth manifold M. Let $x_0 \in M$ be a base point. Parallel transport along closed loops provides an anti homomorphism $\pi_1(M, x_0) \to \mathrm{GL}(E_{x_0})$, where E_{x_0} denotes the fiber of E over x_0. Composing with the inversion in $\mathrm{GL}(E_{x_0})$ we obtain the *holonomy representation* of E at x_0

$$\mathrm{hol}^E_{x_0} : \pi_1(M, x_0) \to \mathrm{GL}(E_{x_0}).$$

Applying this to the flat line bundle $\det E := \Lambda^{\mathrm{rk}(E)} E$ we obtain a homomorphism $\mathrm{hol}^{\det E}_{x_0} : \pi_1(M, x_0) \to \mathrm{GL}(\det E_{x_0}) = \mathbb{C}^\times$ which factors to a homomorphism

$$\theta_E : H_1(M; \mathbb{Z}) \to \mathbb{C}^\times. \tag{11}$$

Lemma 2.2. *Suppose b is a non-degenerate symmetric bilinear form on E. Then*

$$\theta_E(\sigma) = \pm e^{\langle [\omega_{E,b}], \sigma\rangle}, \qquad \sigma \in H_1(M; \mathbb{Z}).$$

Here $\langle [\omega_{E,b}], \sigma\rangle \in \mathbb{C}$ denotes the natural pairing of the cohomology class $[\omega_{E,b}] \in H^1(M; \mathbb{C})$ and $\sigma \in H_1(M; \mathbb{Z})$.

Proof. Let $\tau : [0, 1] \to M$ be a smooth path with $\tau(0) = \tau(1) = x_0$. Consider the flat vector bundle $(\det E)^{-2} := (\det E \otimes \det E)'$. Let $\beta : [0, 1] \to (\det E)^{-2}$ be a section over τ which is parallel. Since $\det b$ defines a global nowhere vanishing section of $(\det E)^{-2}$ we find $\lambda : [0, 1] \to \mathbb{C}$ so that $\beta = \lambda \det b$. Clearly,

$$\lambda(1)\, \mathrm{hol}^{(\det E)^{-2}}_{x_0}([\tau]) = \lambda(0). \tag{12}$$

Differentiating $\beta = \lambda \det b$ we obtain $0 = \lambda' \det b + \lambda \nabla^{(\det E)^{-2}}_{\tau'}(\det b)$. Using (5) this yields $0 = \lambda' - 2\lambda\omega_{E,b}(\tau')$. Integrating we get

$$\lambda(1) = \lambda(0) \exp\Big(\int_0^1 2\omega_{E,b}(\tau'(t))dt\Big) = \lambda(0) e^{2\langle [\omega_{E,b}], [\tau]\rangle}.$$

Taking (12) into account we obtain $\mathrm{hol}^{(\det E)^{-2}}_{x_0}([\tau]) = e^{-2\langle [\omega_{E,b}], [\tau]\rangle}$, and this gives $\mathrm{hol}^{\det E}_{x_0}([\tau]) = \pm e^{\langle [\omega_{E,b}], [\tau]\rangle}$. □

3. Reidemeister torsion

The combinatorial torsion is an invariant associated to a closed connected smooth manifold M, an Euler structure with integral coefficients $\mathfrak{e}$, and a flat complex vector bundle E over M. In the way we consider it here this invariant is a non-degenerate bilinear form $\tau^{\mathrm{comb}}_{E,\mathfrak{e}}$ on the complex line $\det H^*(M; E)$ — the graded determinant line of the cohomology with values in (the local system of coefficients provided by) E. If $H^*(M; E)$ vanishes, then $\tau^{\mathrm{comb}}_{E,\mathfrak{e}}$ becomes a non-vanishing complex number. The aim of this section is to recall these definitions, and to provide some linear algebra which will be used in the analytic approach to this invariant in Section 4.

Throughout this section M denotes a closed connected smooth manifold of dimension n. For simplicity we will also assume vanishing Euler–Poincaré characteristics, $\chi(M) = 0$. At the expense of a base point everything can easily be extended to the general situation, see [8], [11], [12] and Section 9.

Finite dimensional Hodge theory

Suppose C^* is a finite dimensional graded complex over $\mathbb{C}$ with differential $d : C^* \to C^{*+1}$. Its cohomology is a finite dimensional graded vector space and will be denoted by $H(C^*)$. Recall that there is a canonic isomorphism of complex lines

$$\det C^* = \det H(C^*). \tag{13}$$

Let us explain the terms appearing in (13) in more details. If V is a finite dimensional vector space its *determinant line* is defined to be the top exterior product $\det V := \Lambda^{\dim(V)} V$. If V^* is a finite dimensional graded vector space its *graded determinant line* is defined by $\det V^* := \det V^{\text{even}} \otimes (\det V^{\text{odd}})'$. Here $V^{\text{even}} := \bigoplus_q V^{2q}$ and $V^{\text{odd}} := \bigoplus_q V^{2q+1}$ are considered as ungraded vector spaces and $V' := L(V; \mathbb{C})$ denotes the dual space. For more details on determinant lines consult for instance [24]. Let us only mention that every short exact sequence of graded vector spaces $0 \to U^* \to V^* \to W^* \to 0$ provides a canonic isomorphism of determinant lines $\det U^* \otimes \det W^* = \det V^*$. The complex C^* gives rise to two short exact sequences

$$0 \to B^* \to Z^* \to H(C^*) \to 0 \quad \text{and} \quad 0 \to Z^* \to C^* \xrightarrow{d} B^{*+1} \to 0 \tag{14}$$

where B^* and Z^* denote the boundaries and cycles in C^*, respectively. The isomorphism (13) is then obtained from the isomorphisms of determinant lines induced by (14) together with the canonic isomorphism $\det B^* \otimes \det B^{*+1} = \det B^* \otimes (\det B^*)' = \mathbb{C}$.

Suppose our complex C^* is equipped with a *graded non-degenerate symmetric bilinear form* b. That is, we have a non-degenerate symmetric bilinear form on every homogeneous component C^q, and different homogeneous components are b-orthogonal. The bilinear form b will induce a non-degenerate bilinear form on $\det C^*$. Using (13) we obtain a non-degenerate bilinear form on $\det H(C^*)$ which is called the *torsion associated with* C^* *and* b. It will be denoted by $\tau_{C^*,b}$.

Remark 3.1. Note that a non-degenerate bilinear form on a complex line essentially is a non-vanishing complex number. If C^* happens to be acyclic, i.e. $H(C^*) = 0$, then canonically $\det H(C^*) = \mathbb{C}$ and $\tau_{C^*,b} \in \mathbb{C}^\times$ is a genuine non-vanishing complex number — the entry in the 1×1-matrix representing this bilinear form.

Example 3.2. Suppose $q \in \mathbb{Z}$, $n \in \mathbb{N}$ and $A \in \mathrm{GL}_n(\mathbb{C})$. Let C^* denote the acyclic complex $\mathbb{C}^n \xrightarrow{d=A} \mathbb{C}^n$ concentrated in degrees q and $q+1$. Let b denote the standard non-degenerate symmetric bilinear form on C^*. In this situation we have $\tau_{C^*,b} = (\det A)^{(-1)^{q+1}2} = (\det AA^t)^{(-1)^{q+1}}$.

The bilinear form b permits to define the *transposed* $d_b^\sharp$ of d

$$d_b^\sharp : C^{*+1} \to C^*, \quad b(dv, w) = b(v, d_b^\sharp w), \quad v, w \in C^*.$$

Define the *Laplacian* $\Delta_b := dd_b^\sharp + d_b^\sharp d : C^* \to C^*$. Let us write $C_b^*(\lambda)$ for the generalized λ-eigen space of Δ_b. Clearly,

$$C^* = \bigoplus_\lambda C_b^*(\lambda). \tag{15}$$

Since Δ_b is symmetric with respect to b, different generalized eigen spaces of Δ are b-orthogonal. It follows that the restriction of b to $C_b^*(\lambda)$ is non-degenerate.

Since Δ_b commutes with d and $d_b^\sharp$ the latter two will preserve the decomposition (15). Hence every eigen space $C_b^*(\lambda)$ is a subcomplex of C^*. The inclusion $C_b^*(0) \to C^*$ induces an isomorphism in cohomology. Indeed, the Laplacian factors to an invertible map on $C^*/C_b^*(0)$ and thus induces an isomorphism on $H(C^*/C_b^*(0))$. On the other hand, the equation $\Delta_b = dd_b^\sharp + d_b^\sharp d$ tells that the Laplacian will induce the zero map on cohomology. Therefore $H(C^*/C_b^*(0))$ must vanish and $C_b^*(0) \to C^*$ is indeed a quasi isomorphism. Particularly, we obtain a canonic isomorphism of complex lines

$$\det H(C_b^*(0)) = \det H(C^*). \tag{16}$$

Lemma 3.3. *Suppose C^* is a finite dimensional graded complex over $\mathbb{C}$ which is equipped with a graded non-degenerate symmetric bilinear form b. Then via* (16) *we have*

$$\tau_{C^*,b} = \tau_{C_b^*(0), b|_{C_b^*(0)}} \cdot \prod_q (\det{}'(\Delta_{b,q}))^{(-1)^q q}$$

where $\det'(\Delta_{b,q})$ *denotes the product over all non-vanishing eigen values of the Laplacian acting in degree q, $\Delta_{b,q} := \Delta_b|_{C^q} : C^q \to C^q$.*

Proof. Suppose (C_1^*, b_1) and (C_2^*, b_2) are finite dimensional complexes equipped with graded non-degenerate symmetric bilinear forms. Clearly, $H(C_1^* \oplus C_2^*) = H(C_1^*) \oplus H(C_2^*)$ and we obtain a canonic isomorphism of determinant lines

$$\det H(C_1^* \oplus C_2^*) = \det H(C_1^*) \otimes \det H(C_2^*).$$

It is not hard to see that via this identification we have

$$\tau_{C_1^* \oplus C_2^*, b_1 \oplus b_2} = \tau_{C_1^*, b_1} \otimes \tau_{C_2, b_2}. \tag{17}$$

In view of the b-orthogonal decomposition (15) we may therefore w.l.o.g. assume $\ker \Delta_b = 0$. Particularly, C^* is acyclic.

Then $\operatorname{img} d \cap \ker d_b^\sharp \subseteq \ker d \cap \ker d_b^\sharp \subseteq \ker \Delta_b = 0$. Since $\operatorname{img} d$ and $\ker d_b^\sharp$ are of complementary dimension we conclude $\operatorname{img} d \oplus \ker d_b^\sharp = C^*$. The acyclicity of C^* implies $\ker d_b^\sharp = \operatorname{img} d_b^\sharp$ and hence $\operatorname{img} d \oplus \operatorname{img} d_b^\sharp = C^*$. This decomposition is b-orthogonal and invariant under Δ_b. We obtain

$$\det{}'(\Delta_{b,q}) = \det(\Delta_{b,q}) = \det(\Delta_b|_{C^q \cap \operatorname{img} d}) \cdot \det(\Delta_b|_{C^q \cap \operatorname{img} d_b^\sharp}).$$

Since $d : C^q \cap \operatorname{img} d_b^\sharp \to C^{q+1} \cap \operatorname{img} d$ is an isomorphism commuting with Δ

$$\det(\Delta|_{C^q \cap \operatorname{img} d_b^\sharp}) = \det(\Delta|_{C^{q+1} \cap \operatorname{img} d}).$$

A telescoping argument then shows

$$\prod_q (\det{}'(\Delta_{b,q}))^{(-1)^q q} = \prod_q \det(\Delta_b|_{C^q \cap \operatorname{img} d})^{(-1)^q}. \tag{18}$$

On the other hand, the b-orthogonal decomposition of complexes

$$C^* = \bigoplus_q \Big(C^q \cap \operatorname{img} d_b^\sharp \xrightarrow{d} C^{q+1} \cap \operatorname{img} d \Big)$$

together with (17) and the computation in Example 3.2 imply

$$\tau_{C^*,b} = \prod_q \det\big(dd_b^\sharp|_{C^{q+1} \cap \operatorname{img} d} \big)^{(-1)^{q+1}}$$

which clearly coincides with (18) since $\Delta_b|_{\operatorname{img} d} = dd_b^\sharp|_{\operatorname{img} d}$. □

Example 3.4. Suppose $0 \neq v \in \mathbb{C}^2$ satisfies $v^t v = 0$. Moreover, suppose $0 \neq z \in \mathbb{C}$ and set $w := zv^t$. Let C^* denote the acyclic complex $\mathbb{C} \xrightarrow{v} \mathbb{C}^2 \xrightarrow{w} \mathbb{C}$ concentrated in degrees 0, 1 and 2. Equip this complex with the standard symmetric bilinear form b. Then $\Delta_{b,0} = v^t v = 0$, $\Delta_{b,2} = ww^t = 0$, $\Delta_{b,1} = (1 + z^2) vv^t$, $(\Delta_{b,1})^2 = 0$. Thus all of this complex is contained in the generalized 0-eigen space of Δ_b. The torsion of the complex computes to $\tau_{C^*,b} = -z^2$. Observe that the kernel of Δ_b does not compute the cohomology; that the bilinear form becomes degenerate when restricted to the kernel of Δ_b; and that the torsion cannot be computed from the spectrum of Δ_b.

Morse complex

Let E be a flat complex vector bundle over a closed connected smooth manifold M of dimension n. Suppose $X = -\operatorname{grad}_g(f)$ is a *Morse–Smale vector field* on M, see [30]. Let $\mathcal{X}$ denote the zero set of X. Elements in $\mathcal{X}$ are called *critical points* of f. Every $x \in \mathcal{X}$ has a *Morse index* $\operatorname{ind}(x) \in \mathbb{N}$ which coincides with the dimension of the unstable manifold of x with respect to X. We will write $\mathcal{X}_q := \{x \in \mathcal{X} \mid \operatorname{ind}(x) = q\}$ for the set of critical points of index q.

Recall that the Morse–Smale vector field provides a *Morse complex* $C^*(X; E)$ with underlying finite dimensional graded vector space

$$C^q(X; E) = \bigoplus_{x \in \mathcal{X}_q} E_x \otimes_{\{\pm 1\}} \mathcal{O}_x.$$

Here E_x denotes the fiber of E over x, and $\mathcal{O}_x$ denotes the set of orientations of the unstable manifold of x. The Smale condition tells that stable and unstable manifolds intersect transversally. It follows that for two critical points of index difference one there is only a finite number of unparametrized trajectories connecting them. The differential in $C^*(X; E)$ is defined with the help of these isolated trajectories and parallel transport in E along them.

Integration over unstable manifolds provides a homomorphism of complexes

$$\mathrm{Int} : \Omega^*(M;E) \to C^*(X;E) \tag{19}$$

where $\Omega^*(M;E)$ denotes the deRham complex with values in E. It is a folklore fact that (19) induces an isomorphism on cohomology, see [30]. Particularly, we obtain a canonic isomorphism of complex lines

$$\det H^*(M;E) = \det H(C^*(X;E)). \tag{20}$$

Suppose $\chi(M) = 0$ and let $\mathfrak{e} \in \mathfrak{Eul}(M;\mathbb{Z})$ be an Euler structure. Choose a base point $x_0 \in M$. For every critical point $x \in \mathcal{X}$ choose a path σ_x with $\sigma(0) = x_0$ and $\sigma_x(1) = x$ so that $\mathfrak{e} = \big[-X, \sum_{x\in\mathcal{X}}(-1)^{\mathrm{ind}(x)}\sigma_x\big]$. This is possible in view of Lemma 2.1. Also note that $\mathrm{IND}_{-X}(x) = (-1)^{\mathrm{ind}(x)}$. Choose a non-degenerate symmetric bilinear form b_{x_0} on the fiber E_{x_0} over x_0. For $x \in \mathcal{X}$ define a bilinear form b_x on E_x by parallel transport of b_{x_0} along σ_x. The collection of bilinear forms $\{b_x\}_{x\in\mathcal{X}}$ defines a non-degenerate symmetric bilinear form on the Morse complex $C^*(X;E)$. It is elementary to check that the induced bilinear form on $\det C^*(X;E)$ does not depend on the choice of $\{\sigma_x\}_{x\in\mathcal{X}}$, and because $\chi(M) = 0$ it does not depend on x_0 or b_{x_0} either. Hence the corresponding torsion is a non-degenerate bilinear form on $\det H(C^*(X;E))$ depending on E, $\mathfrak{e}$ and X only. Using (20) we obtain a non-degenerate bilinear form on $\det H^*(M;E)$ which we will denote by $\tau^{\mathrm{comb}}_{E,\mathfrak{e},X}$. For the following non-trivial statement we refer to [27], [34] or [25].

Theorem 3.5 (Milnor, Turaev). *The bilinear form $\tau^{\mathrm{comb}}_{E,\mathfrak{e},X}$ does not depend on X.*

In view of Theorem 3.5 we will denote $\tau^{\mathrm{comb}}_{E,\mathfrak{e},X}$ by $\tau^{\mathrm{comb}}_{E,\mathfrak{e}}$ from now on.

Definition 3.6 (Combinatorial torsion). The non-degenerate bilinear form $\tau^{\mathrm{comb}}_{E,\mathfrak{e}}$ on $\det H^*(M;E)$ is called the *combinatorial torsion* associated with the flat complex vector bundle E and the Euler structure $\mathfrak{e} \in \mathfrak{Eul}(M;\mathbb{Z})$.

Remark 3.7. The combinatorial torsion's dependence on the Euler structure is very simple. For $\mathfrak{e} \in \mathfrak{Eul}(M;\mathbb{Z})$ and $\sigma \in H_1(M;\mathbb{Z})$ we obviously have, see (11)

$$\tau^{\mathrm{comb}}_{E,\mathfrak{e}+\sigma} = \tau^{\mathrm{comb}}_{E,\mathfrak{e}} \cdot \theta_E(\sigma)^2.$$

The dependence on E, i.e. the dependence on the flat connection, is subtle and interesting. Let us only mention the following

Example 3.8 (Torsion of mapping tori). Consider a mapping torus

$$M = N \times [0,1]/_{(x,1)\sim(\varphi(x),0)}$$

where $\varphi : N \to N$ is a diffeomorphism. Let $\pi : M \to S^1 = [0,1]/_{0\sim 1}$ denote the canonic projection. The set of vector fields which project to the vector field $-\frac{\partial}{\partial\theta}$ on S^1 is contractible and thus defines an Euler structure $\mathfrak{e} \in \mathfrak{Eul}(M;\mathbb{Z})$ represented by $[X, 0]$ where X is any of these vector fields. Let $\tilde{E}^z$ denote the flat line bundle over S^1 with holonomy $z \in \mathbb{C}^\times$, i.e. $\theta_{\tilde{E}^z} : H_1(S^1;\mathbb{Z}) = \mathbb{Z} \to \mathbb{C}^\times$, $\theta_{\tilde{E}^z}(k) = z^k$. Consider the flat line bundle $E^z := \pi^* \tilde{E}^z$ over M. It follows from the Wang sequence of the fibration $\pi : M \to S^1$ that for generic z we will have $H^*(M; E^z) = 0$. In this case

$$\tau^{\mathrm{comb}}_{E^z,\mathfrak{e}} = (\zeta_\varphi(z))^2$$

where

$$\begin{aligned}\zeta_\varphi(z) &= \exp\left(\sum_{k\geq 1} \operatorname{str}\Big(H^*(N;\mathbb{Q}) \xrightarrow{(\varphi^k)^*} H^*(N;\mathbb{Q})\Big)\frac{z^k}{k}\right)\\ &= \operatorname{sdet}\Big(H^*(N;\mathbb{C}) \xrightarrow{1-z\varphi^*} H^*(N;\mathbb{C})\Big)^{-1}\end{aligned}$$

denotes the Lefschetz zeta function of φ. Here we wrote str and sdet for the super trace and the super determinant, respectively. For more details and proofs we refer to [20] and [11].

Remark 3.9. Often the combinatorial torsion is considered as an element in (rather than a bilinear form on) $\det H^*(M;E)$. This element is one of the two unit vectors of $\tau^{\mathrm{comb}}_{E,\mathfrak{e}}$. It is a non-trivial task (and requires the choice of a homology orientation) to fix the sign, i.e. to describe which of the two unite vectors actually is the torsion [19]. Considering bilinear forms this sign issue disappears.

Basic properties of the combinatorial torsion

If E_1 and E_2 are two flat vector bundles over M then we have a canonic isomorphism $H^*(M;E_1 \oplus E_2) = H^*(M;E_1) \oplus H^*(M;E_2)$ which induces a canonic isomorphism of complex lines $\det H^*(M;E_1 \oplus E_2) = \det H^*(M;E_1) \otimes \det H^*(M;E_2)$. Via this identification we have

$$\tau^{\mathrm{comb}}_{E_1\oplus E_2,\mathfrak{e}} = \tau^{\mathrm{comb}}_{E_1,\mathfrak{e}} \otimes \tau^{\mathrm{comb}}_{E_2,\mathfrak{e}}. \tag{21}$$

This follows from $C^*(X;E_1 \oplus E_2) = C^*(X;E_1) \oplus C^*(X;E_2)$ and (17).

If E' denotes the dual of a flat vector bundle E then Poincaré duality induces an isomorphism $H^*(M;E' \otimes \mathcal{O}_M) = H^{n-*}(M;E)'$ which induces a canonic isomorphism $\det H^*(M;E' \otimes \mathcal{O}_M) = (\det H^*(M;E))^{(-1)^{n+1}}$. Via this identification we have

$$\tau^{\mathrm{comb}}_{E'\otimes\mathcal{O}_M,\nu(\mathfrak{e})} = (\tau^{\mathrm{comb}}_{E,\mathfrak{e}})^{(-1)^{n+1}} \tag{22}$$

where ν denotes the involution on $\mathfrak{Eul}(M;\mathbb{Z})$ discussed in Section 2. To see that use a Morse–Smale vector field X to compute $\tau^{\mathrm{comb}}_{E,\mathfrak{e}}$ and use the Morse–Smale vector field $-X$ to compute $\tau^{\mathrm{comb}}_{E'\otimes\mathcal{O}_M,\nu(\mathfrak{e})}$. Then there is an obvious isomorphism of complexes $C^*(-X;E'\otimes\mathcal{O}_M) = C^{n-*}(X;E)'$ which induces Poincaré duality on cohomology.

If V is a complex vector space let $\bar{V}$ denote the complex conjugate vector space. If b is a bilinear form on V let $\bar{b}$ denote the complex conjugate bilinear form on $\bar{V}$, that is $\bar{b}(v,w) = \overline{b(v,w)}$. Let $\bar{E}$ denote the complex conjugate of a flat vector bundle E. Then we have a canonic isomorphism $H^*(M;\bar{E}) = \overline{H^*(M;E)}$ which induces a canonic isomorphism of complex lines $\det H^*(M;\bar{E}) = \overline{\det H^*(M;E)}$. Via this identification we have

$$\tau^{\mathrm{comb}}_{\bar{E},\mathfrak{e}} = \overline{\tau^{\mathrm{comb}}_{E,\mathfrak{e}}}. \tag{23}$$

This follows from $C^*(X;\bar{E}) = \overline{C^*(X;E)}$.

If V is a real vector space we let $V^{\mathbb{C}} := V \otimes \mathbb{C}$ denote its complexification. If h is a real bilinear form on V we let $h^{\mathbb{C}}$ denote its complexification, more explicitly $h^{\mathbb{C}}(v_1 \otimes z_1, v_2 \otimes z_2) = h(v_1, v_2) z_1 z_2$. If F is real flat vector bundle its torsion, defined analogously to the complex case, is a real non-degenerate bilinear form on $\det H^*(M;F)$. Let $F^{\mathbb{C}} = F \otimes \mathbb{C}$ denote the complexification of the flat vector bundle F. We have a canonic isomorphism $H^*(M;F^{\mathbb{C}}) = H^*(M;F)^{\mathbb{C}}$ which induces a canonic isomorphism of complex lines $\det H^*(M;F^{\mathbb{C}}) = (\det H^*(M;F))^{\mathbb{C}}$. Via this identification we have

$$\tau^{\mathrm{comb}}_{F^{\mathbb{C}},\mathfrak{e}} = (\tau^{\mathrm{comb}}_{F,\mathfrak{e}})^{\mathbb{C}}. \tag{24}$$

This follows from $C^*(X;F^{\mathbb{C}}) = C^*(X;F)^{\mathbb{C}}$. Note that $\tau^{\mathrm{comb}}_{F,\mathfrak{e}}$ is positive definite.

4. Ray–Singer torsion

The analytic torsion defined below is an invariant associated to a closed connected smooth manifold M, a complex flat vector bundle E over M, a coEuler structure $\mathfrak{e}^*$ and a homotopy class $[b]$ of fiber wise non-degenerate symmetric bilinear forms on E. In the way considered below, this invariant is a non-degenerate symmetric bilinear form $\tau^{\mathrm{an}}_{E,\mathfrak{e}^*,[b]}$ on the complex line $\det H^*(M;E)$. If $H^*(M;E)$ vanishes, then $\tau^{\mathrm{an}}_{E,\mathfrak{e}^*,[b]}$ becomes a non-vanishing complex number.

Throughout this section M denotes a closed connected smooth manifold of dimension n. For simplicity we will also assume vanishing Euler–Poincaré characteristics, $\chi(M) = 0$. At the expense of a base point everything can easily be extended to the general situation, see [8], [11], [12] and Section 9.

Laplacians and spectral theory

Suppose M is a closed connected smooth manifold of dimension n. Let E be a flat vector bundle over M. We will denote the flat connection of E by ∇^E. Suppose there exists a *fiber wise non-degenerate symmetric bilinear* form b on E. Moreover, let g be a Riemannian metric on M. This permits to define a symmetric bilinear form $\beta_{g,b}$ on the space of E-valued differential forms $\Omega^*(M;E)$,

$$\beta_{g,b}(v,w) := \int_M v \wedge (\star_g \otimes b) w, \qquad v, w \in \Omega^*(M;E).$$

Here $\star_g \otimes b : \Omega^*(M;E) \to \Omega^{n-*}(M;E' \otimes \mathcal{O}_M)$ denotes the isomorphism induced by the Hodge star operator[1] $\star_g : \Omega^*(M;\mathbb{R}) \to \Omega^{n-*}(M;\mathcal{O}_M)$ and the isomorphism

[1]The normalization of the Hodge star operator we are using is $\alpha_1 \wedge \star_g \alpha_2 = \langle \alpha_1, \alpha_2 \rangle_g \Omega_g$, where $\alpha_1, \alpha_2 \in \Omega(M;\mathbb{R})$, $\Omega_g \in \Omega^n(M;\mathcal{O}_M)$ denotes the volume density associated with g, and $\langle \alpha_1, \alpha_2 \rangle_g$ denotes the inner product on $\Lambda^* T^* M$ induced by g, see [23, Section 2.1]. Although we will

of vector bundles $b : E \to E'$. The wedge product is computed with respect to the canonic pairing of $E \otimes E' \to \mathbb{C}$.

Let $d_E : \Omega^*(M; E) \to \Omega^{*+1}(M; E)$ denote the deRham differential. Let

$$d^\sharp_{E,g,b} : \Omega^{*+1}(M; E) \to \Omega^*(M; E)$$

denote its formal transposed with respect to $\beta_{g,b}$. A straight forward computation shows that $d^\sharp_{E,g,b} : \Omega^q(M; E) \to \Omega^{q-1}(M; E)$ is given by

$$d^\sharp_{E,g,b} = (-1)^q(\star_g \otimes b)^{-1} \circ d_{E'\otimes\mathcal{O}_M} \circ (\star_g \otimes b). \tag{25}$$

Define the *Laplacian* by

$$\Delta_{E,g,b} := d_E \circ d^\sharp_{E,g,b} + d^\sharp_{E,g,b} \circ d_E. \tag{26}$$

These are generalized Laplacians in the sense that their principal symbol coincides with the symbol of the Laplace–Beltrami operator.

In the next proposition we collect some well known facts concerning the spectral theory of $\Delta_{E,g,b}$. For details we refer to [32], particularly Theorems 8.4 and 9.3 therein.

Proposition 4.1. *For the Laplacian $\Delta_{E,g,b}$ constructed above the following hold:*

(i) *The spectrum of $\Delta_{E,g,b}$ is discrete. For every $\theta > 0$ all but finitely many points of the spectrum are contained in the angle $\{z \in \mathbb{C} \mid -\theta < \arg(z) < \theta\}$.*

(ii) *If λ is in the spectrum of $\Delta_{E,g,b}$ then the image of the associated spectral projection is finite dimensional and contains smooth forms only. We will refer to this image as the (generalized) λ-eigen space of $\Delta_{E,g,b}$ and denote it by $\Omega^*_{g,b}(M; E)(\lambda)$. There exists $N_\lambda \in \mathbb{N}$ such that*

$$(\Delta_{E,g,b} - \lambda)^{N_\lambda}|_{\Omega^*_{g,b}(M;E)(\lambda)} = 0.$$

We have a $\Delta_{E,g,b}$-invariant $\beta_{g,b}$-orthogonal decomposition

$$\Omega^*_{g,b}(M; E) = \Omega^*_{g,b}(M; E)(\lambda) \oplus \Omega^*_{g,b}(M; E)(\lambda)^{\perp_{\beta_{g,b}}}. \tag{27}$$

The restriction of $\Delta_{E,g,b} - \lambda$ to $\Omega^*_{g,b}(M; E)(\lambda)^{\perp_{\beta_{g,b}}}$ is invertible.

(iii) *The decomposition (27) is invariant under d_E and $d^\sharp_{E,g,b}$.*

(iv) *For $\lambda \neq \mu$ the eigen spaces $\Omega^*_{g,b}(M; E)(\lambda)$ and $\Omega^*_{g,b}(M; E)(\mu)$ are orthogonal with respect to $\beta_{g,b}$.*

In view of Proposition 4.1 the generalized 0-eigen space $\Omega^*_{g,b}(M; E)(0)$ is a finite dimensional subcomplex of $\Omega^*(M; E)$. The inclusion

$$\Omega^*_{g,b}(M; E)(0) \to \Omega^*(M; E) \tag{28}$$

induces an isomorphism in cohomology. Indeed, in view of Proposition 4.1(ii) the Laplacian $\Delta_{E,g,b}$ induces an isomorphism on $\Omega^*_{g,b}(M; E)/\Omega^*_{g,b}(M; E)(0)$ and thus an isomorphism on $H(\Omega^*_{g,b}(M; E)/\Omega^*_{g,b}(M; E)(0))$. On the other hand (26) tells

frequently refer to [1] in the subsequent sections, the convention for the Hodge star operator we are using differs from the one in [1].

that $\Delta_{E,g,b}$ induces 0 on cohomology, hence $H(\Omega^*_{g,b}(M;E)/\Omega^*_{g,b}(M;E)(0))$ must vanish and (28) is indeed a quasi isomorphism. We obtain a canonic isomorphism of complex lines

$$\det H(\Omega^*_{g,b}(M;E)(0)) = \det H^*(M;E). \tag{29}$$

In view of Proposition 4.1(ii) the bilinear form $\beta_{g,b}$ restricts to a non-degenerate bilinear form on $\Omega^*_{g,b}(M;E)(0)$. Using the linear algebra discussed in Section 3 we obtain a non-degenerate bilinear form on $\det H(\Omega^*_{g,b}(M;E)(0))$. Via (29) this gives rise to a non-degenerate bilinear form on $\det H^*(M;E)$ which will be denoted by $\tau^{\mathrm{an}}_{E,g,b}(0)$.

Let $\Delta_{E,g,b,q}$ denote the Laplacian acting in degree q. Define the zeta regularized product of its non-vanishing eigen values, as

$$\det{}'(\Delta_{E,g,b,q}) := \exp\left(-\frac{\partial}{\partial s}\Big|_{s=0} \operatorname{tr}\Big(\big(\Delta_{E,g,b,q}|\Omega^q_{g,b}(M;E)(0)^{\perp_{\beta_{g,b}}}\big)^{-s}\Big)\right).$$

Here the complex powers are defined with respect to any non-zero Agmon angle which avoids the spectrum of $\Delta_{E,g,b,q}|\Omega^q_{g,b}(M;E)(0)^{\perp_{\beta_{g,b}}}$, see Proposition 4.1(i). Recall that for $\Re(s) > n/2$ the operator $(\Delta_{E,g,b,q}|\Omega^q_{g,b}(M;E)(0)^{\perp_{\beta_{g,b}}})^{-s}$ is trace class. As a function in s this trace extends to a meromorphic function on the complex plane which is holomorphic at 0, see [31] or [32, Theorem 13.1]. It is clear from Proposition 4.1(i) that $\det'(\Delta_{E,g,b,q})$ does not depend on the Agmon angle used to define the complex powers.

Assume $\chi(M) = 0$ and suppose $\alpha \in \Omega^{n-1}(M;\mathcal{O}^{\mathbb{C}}_M)$ such that $d\alpha = \mathrm{e}(g)$. Consider the non-degenerate bilinear form on $\det H^*(M;E)$ defined by, cf. (4),

$$\tau^{\mathrm{an}}_{E,g,b,\alpha} := \tau^{\mathrm{an}}_{E,g,b}(0) \cdot \prod_q \big(\det{}'(\Delta_{E,g,b,q})\big)^{(-1)^q q} \cdot \exp\Big(-2\int_M \omega_{E,b} \wedge \alpha\Big).$$

In Section 6 we will provide a proof of the following result which can be interpreted as an anomaly formula for the complex valued Ray–Singer torsion (1).

Theorem 4.2 (Anomaly formula). *Let M be a closed connected smooth manifold with vanishing Euler–Poincaré characteristics. Let E be a flat complex vector bundle over M. Suppose g_u is a smooth one-parameter family of Riemannian metrics on M, and $\alpha_u \in \Omega^{n-1}(M;\mathcal{O}^{\mathbb{C}}_M)$ is a smooth one-parameter family so that $[g_u, \alpha_u]$ represent the same coEuler structure in $\mathfrak{Eul}^*(M;\mathbb{C})$. Moreover, suppose b_u is a smooth one-parameter family of fiber wise non-degenerate symmetric bilinear forms on E. Then, as bilinear forms on* $\det H^*(M;E)$, *we have* $\frac{\partial}{\partial u}\tau^{\mathrm{an}}_{E,g_u,b_u,\alpha_u} = 0$.

In view of Theorem 4.2 the bilinear form $\tau^{\mathrm{an}}_{E,g,b,\alpha}$ does only depend on the flat vector bundle E, the coEuler structure $\mathfrak{e}^* \in \mathfrak{Eul}^*(M;\mathbb{C})$ represented by (g,α), and the homotopy class $[b]$ of b. We will denote it by $\tau^{\mathrm{an}}_{E,\mathfrak{e}^*,[b]}$ from now on.

Definition 4.3 (Analytic torsion). The non-degenerate bilinear form $\tau^{\mathrm{an}}_{E,\mathfrak{e}^*,[b]}$ on $\det H^*(M;E)$ is called the *analytic torsion* associated to the flat complex vector bundle E, the coEuler structure $\mathfrak{e}^* \in \mathfrak{Eul}^*(M;\mathbb{C})$ and the homotopy class $[b]$ of fiber wise non-degenerate symmetric bilinear forms on E.

Remark 4.4. The analytic torsion's dependence on the coEuler structure is very simple. For $\mathfrak{e}^* \in \mathfrak{Eul}^*(M;\mathbb{C})$ and $\beta \in H^{n-1}(M;\mathcal{O}_M^{\mathbb{C}})$ we obviously have:

$$\tau^{\mathrm{an}}_{E,\mathfrak{e}^*+\beta,[b]} = \tau^{\mathrm{an}}_{E,\mathfrak{e}^*,[b]} \cdot \left(e^{\langle[\omega_{E,b}]\cup\beta,[M]\rangle}\right)^2$$

Here $\langle[\omega_{E,b}]\cup\beta,[M]\rangle \in \mathbb{C}$ denotes the evaluation of $[\omega_{E,b}]\cup\beta \in H^n(M;\mathcal{O}_M^{\mathbb{C}})$ on the fundamental class $[M] \in H_n(M;\mathcal{O}_M)$.

Remark 4.5. Recall from Section 2 that for odd n there is a canonic coEuler structure $\mathfrak{e}^*_{\mathrm{can}} \in \mathfrak{Eul}^*(M;\mathbb{C})$ given by $\mathfrak{e}^*_{\mathrm{can}} = [g,0]$. The corresponding analytic torsion is:

$$\tau^{\mathrm{an}}_{E,\mathfrak{e}^*_{\mathrm{can}},[b]} = \tau^{\mathrm{an}}_{E,g,b}(0) \cdot \prod_q \bigl(\det{}'(\Delta_{E,g,b,q})\bigr)^{(-1)^q q}$$

Note however that in general this does depend on the homotopy class $[b]$, see for instance the computation for the circle in Section 5 below. This is related to the fact that $\mathfrak{e}^*_{\mathrm{can}}$ in general is not integral, cf. Remark 5.3 below.

Basic properties of the analytic torsion

Suppose E_1 and E_2 are two flat vector bundles with fiber wise non-degenerate symmetric bilinear forms b_1 and b_2. Via the canonic isomorphism of complex lines $\det H^*(M;E_1\oplus E_2) = \det H^*(M;E_1)\otimes \det H^*(M;E_2)$ we have:

$$\tau^{\mathrm{an}}_{E_1\oplus E_2,\mathfrak{e}^*,[b_1\oplus b_2]} = \tau^{\mathrm{an}}_{E_1,\mathfrak{e}^*,[b_1]} \otimes \tau^{\mathrm{an}}_{E_2,\mathfrak{e}^*,[b_2]} \tag{30}$$

For this note that via the identification $\Omega^*(M;E_1\oplus E_2) = \Omega^*(M;E_1)\oplus\Omega^*(M;E_2)$ we have $\Delta_{E_1\oplus E_2,g,b_1\oplus b_2} = \Delta_{E_1,g,b_1} \oplus \Delta_{E_2,g,b_2}$, hence $\det{}'(\Delta_{E_1\oplus E_2,g,b_1\oplus b_2,q}) = \det{}'(\Delta_{E_1,g,b_1,q})\det{}'(\Delta_{E_2,g,b_2,q})$. Moreover, recall (7) for the correction terms.

Suppose E' is the dual of a flat vector bundle E. Let b' denote the bilinear form on E' dual to the non-degenerate symmetric bilinear form b on E. The bilinear form b' induces a fiber wise non-degenerate symmetric bilinear form on the flat vector bundle $E'\otimes\mathcal{O}_M$ which will be denoted by b' too. Via the canonic isomorphism of complex lines $\det H^*(M;E'\otimes\mathcal{O}_M) = (\det H^*(M;E))^{(-1)^{n+1}}$ induced by Poincaré duality we have

$$\tau^{\mathrm{an}}_{E'\otimes\mathcal{O}_M,\nu(\mathfrak{e}^*),[b']} = \bigl(\tau^{\mathrm{an}}_{E,\mathfrak{e}^*,[b]}\bigr)^{(-1)^{n+1}} \tag{31}$$

where ν denotes the involution introduced in Section 2. This follows from the fact that $\star_g \otimes b : \Omega^q(M;E) \to \Omega^{n-q}(M;E'\otimes\mathcal{O}_M)$ intertwines the Laplacians $\Delta_{E,g,b,q}$ and $\Delta_{E'\otimes\mathcal{O}_M,g,b',n-q}$, see (25). Therefore $\Delta_{E,g,b,q}$ and $\Delta_{E'\otimes\mathcal{O}_M,g,b',n-q}$ are isospectral and thus $\det{}'(\Delta_{E,g,b,q}) = \det{}'(\Delta_{E'\otimes\mathcal{O}_M,g,b',n-q})$. Here one also has to use $\prod_q\bigl(\det{}'(\Delta_{E,g,b,q})\bigr)^{(-1)^q} = 1$, and (8).

Let $\bar E$ denote the complex conjugate of a flat vector bundle E. Let $\bar b$ denote the complex conjugate of a fiber wise non-degenerate symmetric bilinear form on E. Via the canonic isomorphism of complex lines $\det H^*(M;\bar E) = \overline{\det H^*(M;E)}$ we obviously have

$$\tau^{\mathrm{an}}_{\bar E,\bar{\mathfrak{e}}^*,[\bar b]} = \overline{\tau^{\mathrm{an}}_{E,\mathfrak{e}^*,[b]}} \tag{32}$$

where $\mathfrak{e}^* \mapsto \bar{\mathfrak{e}}^*$ denotes the complex conjugation of coEuler structures introduced in Section 2. For this note that $\Delta_{\bar{E},g,\bar{b}} = \Delta_{E,g,b}$ but the spectrum of $\Delta_{\bar{E},g,\bar{b}}$ is complex conjugate to the spectrum of $\Delta_{E,g,b}$ and thus $\det'(\Delta_{\bar{E},g,\bar{b},q}) = \overline{\det'(\Delta_{E,g,b,q})}$. Also recall (9).

Suppose F is a flat real vector bundle over M. Let $\mathfrak{e}^* \in \mathfrak{Eul}(M;\mathbb{R})$ be a coEuler structure with real coefficients. Let h be a fiber wise non-degenerate symmetric bilinear form on F. Proceeding exactly as in the complex case we obtain a non-degenerate bilinear form $\tau^{\mathrm{an}}_{F,\mathfrak{e},[h]}$ on the real line $\det H^*(M;F)$. Note that although the Laplacians $\Delta_{F,g,h}$ need not be selfadjoint their spectra are invariant under complex conjugation and hence $\det'(\Delta_{F,g,h,q})$ will be real. Let $F^{\mathbb{C}}$ denote the complexification of the flat bundle F, and let $h^{\mathbb{C}}$ denote the complexification of h, a non-degenerate symmetric bilinear form on $F^{\mathbb{C}}$. Via the canonic isomorphism of complex lines $\det H^*(M;F^{\mathbb{C}}) = (\det H^*(M;F))^{\mathbb{C}}$ we have:

$$\tau^{\mathrm{an}}_{F^{\mathbb{C}},\mathfrak{e}^*,[h^{\mathbb{C}}]} = \left(\tau^{\mathrm{an}}_{F,\mathfrak{e}^*,[h]}\right)^{\mathbb{C}} \tag{33}$$

For this note that via $\Omega^*(M;F^{\mathbb{C}}) = \Omega^*(M;F)^{\mathbb{C}}$ we have $\Delta_{F^{\mathbb{C}},g,h^{\mathbb{C}}} = (\Delta_{F,g,h})^{\mathbb{C}}$ and thus $\det'(\Delta_{F^{\mathbb{C}},g,h^{\mathbb{C}},q}) = \det'(\Delta_{F,g,h,q})$, and also recall (10). If n is odd, $H^*(M;F) = 0$, and if h is positive definite, then $\tau^{\mathrm{an}}_{F,\mathfrak{e}^*_{\mathrm{can}},[h]}$ is the square of the analytic torsion considered in [29], see Remark 4.5.

Remark 4.6. Not every flat complex vector bundle E admits a fiber wise non-degenerate symmetric bilinear form b. However, since E is flat all rational Chern classes of E must vanish. Since M is compact, the Chern character induces an isomorphism on rational K-theory, and hence E is trivial in rational K-theory. Thus there exists $N \in \mathbb{N}$ so that $E^N = E \oplus \cdots \oplus E$ is a trivial vector bundle. Particularly, there exists a fiber wise non-degenerate bilinear form b on E^N. In view of (30) the non-degenerate bilinear form $\left(\tau^{\mathrm{an}}_{E^N,\mathfrak{e}^*,[b]}\right)^{1/N}$ on $\det H^*(M;E)$ is a reasonable candidate for the analytic torsion of E. Note however, that this is only defined up to a root of unity.

Rewriting the analytic torsion

Instead of just treating the 0-eigen space by means of finite dimensional linear algebra one can equally well do this with finitely many eigen spaces of $\Delta_{E,g,b}$. Proposition 4.7 below makes this precise. We will make use of this formula when computing the variation of the analytic torsion through a variation of g and b. This is necessary since the dimension of the 0-eigen space need not be locally constant through such a variation. Note that this kind of problem does not occur in the selfadjoint situation, i.e. when instead of a non-degenerate symmetric bilinear form we have a hermitian structure.

Suppose γ is a simple closed curve around 0, avoiding the spectrum of $\Delta_{E,g,b}$. Let $\Omega^*_{g,b}(M;E)(\gamma)$ denote the sum of eigen spaces corresponding to eigen values in the interior of γ. Using Proposition 4.1 we see that the inclusion $\Omega^*_{g,b}(M;E)(\gamma) \to \Omega^*(M;E)$ is a quasi isomorphism. We obtain a canonic isomorphism of determinant lines

$$\det H(\Omega^*_{g,b}(M;E)(\gamma)) = \det H^*(M;E). \tag{34}$$

Moreover, the restriction of $\beta_{g,b}$ to $\Omega^*_{g,b}(M;E)(\gamma)$ is non-degenerate. Hence the torsion provides us with a non-degenerate bilinear form on $\det H(\Omega^*_{g,b}(M;E)(\gamma))$ and via (34) we get a non-degenerate bilinear form $\tau^{\mathrm{an}}_{E,g,b}(\gamma)$ on $\det H^*(M;E)$. Moreover, introduce

$$\det{}^\gamma(\Delta_{E,g,b,q}) := \exp\left(-\frac{\partial}{\partial s}\Big|_{s=0} \operatorname{tr}\Big(\big(\Delta_{E,g,b,q}|\Omega^q_{g,b}(M;E)(\gamma)^{\perp_{\beta_{g,b}}}\big)^{-s}\Big)\right),$$

the zeta regularized product of eigen values in the exterior of γ.

Proposition 4.7. *In this situation, as bilinear forms on* $\det H^*(M;E)$*, we have:*

$$\tau^{\mathrm{an}}_{E,g,b}(0)\cdot\prod_q\big(\det{}'(\Delta_{E,g,b,q})\big)^{(-1)^q q} = \tau^{\mathrm{an}}_{E,g,b}(\gamma)\cdot\prod_q\big(\det{}^\gamma(\Delta_{E,g,b,q})\big)^{(-1)^q q}$$

Proof. Let $C^* \subseteq \Omega^*_{g,b}(M;E)(\gamma)$ denote the sum of the eigen spaces of $\Delta_{E,g,b}$ corresponding to non-zero eigen values in the interior of γ. Clearly, for every q we have

$$\det{}'(\Delta_{E,g,b,q}) = \det(\Delta_{E,g,b,q}|_{C^q})\cdot\det{}^\gamma(\Delta_{E,g,b,q}).$$

Particularly,

$$\prod_q\big(\det{}'(\Delta_{E,g,b,q})\big)^{(-1)^q q} = \prod_q\big(\det(\Delta_{E,g,b,q}|_{C^q})\big)^{(-1)^q q}\cdot\prod_q\big(\det{}^\gamma(\Delta_{E,g,b,q})\big)^{(-1)^q q}. \tag{35}$$

Applying Lemma 3.3 to the finite dimensional complex $\Omega^*_{g,b}(M;E)(\gamma)$ we obtain

$$\tau^{\mathrm{an}}_{E,g,b}(\gamma) = \tau^{\mathrm{an}}_{E,g,b}(0)\cdot\prod_q\big(\det(\Delta_{E,g,b,q}|_{C^q})\big)^{(-1)^q q} \tag{36}$$

Multiplying (35) with $\tau^{\mathrm{an}}_{E,g,b}(0)$ and using (36) we obtain the statement. □

5. A Bismut–Zhang, Cheeger, Müller type formula

The conjecture below asserts that the complex valued analytical torsion defined in Section 4 coincides with the combinatorial torsion from Section 3. It should be considered as a complex valued version of a theorem of Cheeger [16, 17], Müller [28] and Bismut–Zhang [2].

Conjecture 5.1. *Let M be a closed connected smooth manifold with vanishing Euler–Poincaré characteristics. Let E be a flat complex vector bundle over M, and suppose b is a fiber wise non-degenerate symmetric bilinear form on E. Let $\mathfrak{e} \in \mathfrak{Eul}(M;\mathbb{Z})$ be an Euler structure. Then, as bilinear forms on the complex line $\det H^*(M;E)$, we have:*

$$\tau^{\mathrm{comb}}_{E,\mathfrak{e}} = \tau^{\mathrm{an}}_{E,P(\mathfrak{e}),[b]}$$

Here we slightly abuse notation and let P also denote the composition $\mathfrak{Eul}(M;\mathbb{Z}) \to \mathfrak{Eul}(M;\mathbb{C}) \xrightarrow{P} \mathfrak{Eul}^*(M;\mathbb{C})$, see Section 2.

We will establish this conjecture in several special cases, see Remark 5.8, Theorem 5.10, Corollary 5.13, Corollary 5.14 and the discussion for the circle below. Some of these results have been established by Braverman–Kappeler [7] and were not contained in the first version of this manuscript. The proofs we provide below have been inspired by a trick used in [7] but do not rely on the results therein.

Remark 5.2. If Conjecture 5.1 holds for one Euler structure $\mathfrak{e} \in \mathfrak{Eul}(M;\mathbb{Z})$ then it will hold for all Euler structures. This follows immediately from Remark 4.4, Remark 3.7 and Lemma 2.2.

Remark 5.3. If Conjecture 5.1 holds, and if $\mathfrak{e}^* \in \mathfrak{Eul}^*(M;\mathbb{C})$ is integral, then $\tau^{\text{an}}_{E,\mathfrak{e}^*,[b]}$ is independent of $[b]$. This is not obvious from the definition of the analytic torsion.

Remark 5.4. If Conjecture 5.1 holds, $\mathfrak{e} \in \mathfrak{Eul}(M;\mathbb{Z})$ and $\mathfrak{e}^* \in \mathfrak{Eul}^*(M;\mathbb{C})$ then:

$$\tau^{\text{comb}}_{E,\mathfrak{e}} = \tau^{\text{an}}_{E,\mathfrak{e}^*,[b]} \cdot \left(e^{\langle[\omega_{E,b}]\cup(P(\mathfrak{e})-\mathfrak{e}^*),[M]\rangle}\right)^2$$

This follows from Remark 4.4.

Remark 5.5. If Conjecture 5.1 holds, and $\mathfrak{e}^* \in \mathfrak{Eul}^*(M;\mathbb{C})$, then $\tau^{\text{an}}_{E,\mathfrak{e}^*,[b]}$ does only depend on E, $\mathfrak{e}^*$ and the induced homotopy class $[\det b]$ of non-degenerate bilinear forms on $\det E$. This follows from Remark 5.4 and the fact that the cohomology class $[\omega_{E,b}]$ does depend on $\det E$ and the homotopy class $[\det b]$ on $\det E$ only, see Section 2.

Relative torsion

In the situation above, consider the non-vanishing complex number

$$\mathcal{S}_{E,\mathfrak{e},[b]} := \frac{\tau^{\text{an}}_{E,P(\mathfrak{e}),[b]}}{\tau^{\text{comb}}_{E,\mathfrak{e}}} \in \mathbb{C}^\times.$$

It follows from Remark 4.4, Remark 3.7 and Lemma 2.2 that this does not depend on $\mathfrak{e} \in \mathfrak{Eul}(M;\mathbb{Z})$. We will thus denote it by $\mathcal{S}_{E,[b]}$. The number $\mathcal{S}_{E,[b]}$ will be referred to as the *relative torsion* associated with the flat complex vector bundle E and the homotopy class $[b]$. Conjecture 5.1 asserts that $\mathcal{S}_{E,[b]} = 1$.

Similarly, if F is a real flat vector bundle over M equipped with a fiber wise non-degenerate symmetric bilinear form h, we set

$$\mathcal{S}_{F,[h]} := \frac{\tau^{\text{an}}_{F,P(\mathfrak{e}),[h]}}{\tau^{\text{comb}}_{F,\mathfrak{e}}} \in \mathbb{R}^\times := \mathbb{R} \setminus \{0\}$$

where $\mathfrak{e} \in \mathfrak{Eul}(M;\mathbb{Z})$ is any Euler structure. The combinatorial torsion $\tau^{\text{comb}}_{F,\mathfrak{e}}$ and the analytic torsion $\tau^{\text{an}}_{F,P(\mathfrak{e}),[h]}$ on $\det H^*(M;F)$ have been introduced in Sections 3 and 4, respectively. It follows via complexification from the corresponding statements for complex vector bundles that this does indeed only depend on F and the homotopy class of h, see (24) and (33).

Remark 5.6. If F is a flat real vector bundle equipped with a positive definite symmetric bilinear form h, then the Bismut–Zhang theorem [2, Theorem 0.2] asserts that $\mathcal{S}_{F,[h]} = 1$. This follows from the formula in Proposition 5.11 below (applied to a simple closed curve whose interior contains the eigen value 0 only) which, via complexification, provides an analogous formula for flat real vector bundles. For the relation of the first factor in this formula with the statement in [2, Theorem 0.2] see (45).

Proposition 5.7. *The following properties hold:*

(i) $\mathcal{S}_{E_1\oplus E_2,[b_1\oplus b_2]} = \mathcal{S}_{E_1,[b_1]} \cdot \mathcal{S}_{E_2,[b_2]}$
(ii) $\mathcal{S}_{E'\otimes\mathcal{O}_M,[b']} = (\mathcal{S}_{E,[b]})^{(-1)^{n+1}}$
(iii) $\mathcal{S}_{\bar{E},[\bar{b}]} = \overline{\mathcal{S}_{E,[b]}}$
(iv) $\mathcal{S}_{F^{\mathbb{C}},[h^{\mathbb{C}}]} = \mathcal{S}_{F,[h]}$

Proof. This follows immediately from the basic properties of analytic and combinatorial torsion discussed in Sections 3 and 4. For (ii) and (iii) one also has to use $P(\nu(\mathfrak{e})) = \nu(P(\mathfrak{e}))$ and $\overline{P(\mathfrak{e})} = P(\bar{\mathfrak{e}})$, see Section 2. □

Remark 5.8. Proposition 5.7(ii) permits to verify Conjecture 5.1, up to sign, for even dimensional orientable manifolds and parallel bilinear forms. More precisely, let M be an even dimensional closed connected orientable smooth manifold with vanishing Euler–Poincaré characteristics. Let E be a flat complex vector bundle over M and suppose b is a parallel fiber wise non-degenerate symmetric bilinear form on E. Let $\mathfrak{e} \in \mathfrak{Eul}(M;\mathbb{Z})$ be an Euler structure. Then

$$\tau^{\mathrm{comb}}_{E,\mathfrak{e}} = \pm\tau^{\mathrm{an}}_{E,P(\mathfrak{e}),[b]} \tag{37}$$

i.e. in this situation Conjecture 5.1 holds up to sign. To see this, note that the parallel bilinear form b and the choice of an orientation provides an isomorphism of flat vector bundles $b : E \to E' \otimes \mathcal{O}_M$ which maps b to b'. Thus $\mathcal{S}_{E'\otimes\mathcal{O}_M,[b']} = \mathcal{S}_{E,[b]}$. Combining this with Proposition 5.7(ii) we obtain $(\mathcal{S}_{E,[b]})^2 = 1$, and hence (37). Note, however, that in this situation the arguments used to establish (31) immediately yield

$$\prod_q \big(\det{}'(\Delta_{E,g,b,q})\big)^{(-1)^q q} = 1.$$

Corollary 5.9 below has been established by Braverman and Kappeler see [7, Theorem 5.3] by comparing $\tau^{\mathrm{an}}_{E,P(\mathfrak{e}),[b]}$ with their refined analytic torsion, see [7, Theorem 1.4]. We will give an elementary proof relying on Proposition 5.7 and a trick similar to the one used in the proof of Theorem 1.4 in [7].

Corollary 5.9. *Let M be a closed connected smooth orientable manifold of odd dimension. Suppose E is a flat complex vector bundle over M equipped with a non-degenerate symmetric bilinear form b. Let $\mathfrak{e}^* \in \mathfrak{Eul}^*(M;\mathbb{C})$ be an integral coEuler structure. Then, up to sign, $\tau^{\mathrm{an}}_{E,\mathfrak{e}^*,[b]}$ is independent of $[b]$, cf. Remark 5.3.*

Proof. It suffices to show $(\mathcal{S}_{E,[b]})^2$ is independent of $[b]$. The choice of an orientation provides an isomorphism of flat vector bundles $E' \cong E' \otimes \mathcal{O}_M$ from which we obtain

$$\mathcal{S}_{E',[b']} = \mathcal{S}_{E'\otimes\mathcal{O}_M,[b']} = \mathcal{S}_{E,[b]}$$

where the latter equality follows from Proposition 5.7(ii). Together with Proposition 5.7(i) we thus obtain

$$(\mathcal{S}_{E,[b]})^2 = \mathcal{S}_{E,[b]} \cdot \mathcal{S}_{E',[b']} = \mathcal{S}_{E\oplus E',[b\oplus b']}. \tag{38}$$

Observe that on $E \oplus E'$ there exists a canonic (independent of b) symmetric non-degenerate bilinear form b_{can} defined by

$$b_{\text{can}}\big((x_1,\alpha_1),(x_2,\alpha_2)\big) := \alpha_1(x_2) + \alpha_2(x_1), \qquad x_1, x_2 \in E,\ \alpha_1,\alpha_2 \in E'.$$

This bilinear form b_{can} is homotopic to $b \oplus b'$, and thus

$$\mathcal{S}_{E\oplus E',[b\oplus b']} = \mathcal{S}_{E\oplus E',[b_{\text{can}}]}.$$

Hence $\mathcal{S}_{E\oplus E',[b\oplus b']}$ does not depend on $[b]$. In view of (38) the same holds for $(\mathcal{S}_{E,[b]})^2$, and the proof is complete.

To see that $b \oplus b'$ is indeed homotopic to b_{can} let us consider b as an isomorphism $b : E \to E'$. For $t \in \mathbb{R}$ consider the endomorphisms

$$\Phi_t : \text{end}(E \oplus E'), \qquad \Phi_t := \begin{pmatrix} \text{id}_E \cos t & -b^{-1}\sin t \\ b \sin t & \text{id}_{E'}\cos t \end{pmatrix}$$

From $\Phi_{t+s} = \Phi_t \Phi_s$ we conclude that every Φ_t is invertible. Consider the curve of non-degenerate symmetric bilinear forms

$$b_t := \Phi_t^* b_{\text{can}}, \qquad b_t(X_1,X_2) = b_{\text{can}}(\Phi_t X_1, \Phi_t X_2), \qquad X_1, X_2 \in E \oplus E'.$$

Then clearly $b_0 = b_{\text{can}}$. An easy calculation shows $b_{\pi/4} = b \oplus (-b')$. Clearly, $b \oplus (-b')$ is homotopic to $b \oplus b'$. So we see that b_{can} is homotopic to $b \oplus b'$. □

Using a result of Cheeger [16, 17], Müller [28] and Bismut–Zhang [2] we will next show that the absolute value of the relative torsion is always one. In odd dimensions this has been established by Braverman and Kappeler, see Theorem 1.10 in [7]. We will again use a trick similar to the one in [7].

Theorem 5.10. *Suppose M is a closed connected smooth manifold with vanishing Euler–Poincaré characteristics. Let E be a flat complex vector bundle over M equipped with a non-degenerate symmetric bilinear form b. Then $|\mathcal{S}_{E,[b]}| = 1$.*

Proof. Note first that in view of Proposition 5.7(iii) and (i) we have

$$|\mathcal{S}_{E,[b]}|^2 = \mathcal{S}_{E,[b]} \cdot \overline{\mathcal{S}_{E,[b]}} = \mathcal{S}_{E\oplus\bar{E},[b\oplus\bar{b}]}. \tag{39}$$

Set $k := \operatorname{rank} E$, and observe that b provides a reduction of the structure group of E to $O_k(\mathbb{C})$. Since the inclusion $O_k(\mathbb{R}) \subseteq O_k(\mathbb{C})$ is a homotopy equivalence, the structure group can thus be further reduced to $O_k(\mathbb{R})$. In other words, there exists a complex anti-linear involution $\nu : E \to E$ such that

$$\nu^2 = \text{id}_E, \qquad b(\nu x, y) = \overline{b(x, \nu y)}, \qquad b(x, \nu x) \geq 0, \qquad x, y \in E.$$

Then

$$\mu : E \otimes E \to \mathbb{C}, \qquad \mu(x,y) := b(x, \nu y)$$

is a fiber wise positive definite Hermitian structure on E, anti-linear in the second variable. Define a non-degenerate symmetric bilinear form b^μ on $E \oplus \bar{E}$ by

$$b^\mu\big((x_1,y_1),(x_2,y_2)\big) := \mu(x_1,y_2) + \mu(x_2,y_1).$$

We claim that the symmetric bilinear form b^μ is homotopic to $b \oplus \bar{b}$. To see this, consider $\nu : E \to \bar{E}$ as a complex linear isomorphism. For $t \in \mathbb{R}$, define

$$\Phi_t \in \operatorname{end}(E \oplus \bar{E}), \qquad \Phi_t := \left(\begin{smallmatrix} \mathrm{id}_E \cos t & -\nu^{-1}\sin t \\ \nu \sin t & \mathrm{id}_{\bar{E}} \cos t \end{smallmatrix}\right)$$

From $\Phi_{t+s} = \Phi_t \Phi_s$ we conclude that every Φ_t is invertible. Consider the curve of non-degenerate symmetric bilinear forms

$$b_t := \Phi_t^* b^\mu, \qquad b_t(X_1,X_2) = b^\mu(\Phi_t X_1, \Phi_t X_2), \qquad X_1, X_2 \in E \oplus \bar{E}.$$

Clearly, $b_0 = b^\mu$. An easy computation shows $b_{\pi/4} = b \oplus (-\bar{b})$. Since $b \oplus (-\bar{b})$ is homotopic to $b \oplus \bar{b}$ we see that b^μ is indeed homotopic to $b \oplus \bar{b}$. Together with (39) we conclude

$$|\mathcal{S}_{E,[b]}|^2 = \mathcal{S}_{E\oplus\bar{E},b^\mu}. \tag{40}$$

Next, recall that there is a canonic isomorphism of flat vector bundles

$$\psi : E^{\mathbb{C}} \cong E \oplus \bar{E}, \qquad \psi(x + \mathbf{i}y) := (x + \mathbf{i}y, x - \mathbf{i}y), \qquad x, y \in E.$$

Consider the fiber wise positive definite symmetric real bilinear form $h := \Re\mu$ on $E^{\mathbb{R}}$, the underlying real vector bundle. Its complexification $h^{\mathbb{C}}$ is a non-degenerate symmetric bilinear form on $E^{\mathbb{C}}$. A simple computations shows $\psi^* b^\mu = 2h^{\mathbb{C}}$. Together with (40) we obtain

$$|\mathcal{S}_{E,[b]}|^2 = \mathcal{S}_{E^{\mathbb{C}},2h^{\mathbb{C}}} = \mathcal{S}_{E^{\mathbb{R}},2h}$$

where the last equation follows from Proposition 5.7(iv). The Bismut–Zhang theorem [2, Theorem 0.2] asserts that $\mathcal{S}_{E^{\mathbb{R}},2h} = 1$, see Remark 5.6, and the proof is complete. □

Analyticity of the relative torsion

In this section we will show that the relative torsion $\mathcal{S}_{E,[b]}$ depends holomorphically on the flat connection, see Proposition 5.12 below. Combined with Theorem 5.10 this implies that $\mathcal{S}_{E,[b]}$ is locally constant on the space of flat connections on a fixed vector bundle, see Corollary 5.13 below. We start by establishing an explicit formula for the relative torsion, see Proposition 5.11.

Suppose $f : C_1 \to C_2$ is a homomorphism of finite dimensional complexes. Consider the mapping cone $C_2^{*-1} \oplus C_1^*$ with differential $\left(\begin{smallmatrix} d & f \\ 0 & -d \end{smallmatrix}\right)$. If C_1^* and C_2^* are equipped with graded non-degenerate symmetric bilinear forms b_1 and b_2 we equip the mapping cylinder with the bilinear form $b_2 \oplus b_1$. The resulting torsion $\tau(f, b_1, b_2) := \tau_{C_2^{*-1}\oplus C_1^*, b_2 \oplus b_1}$ is called the relative torsion of f. It is a non-degenerate bilinear form on the determinant line $\det H(C_2^{*-1} \oplus C_1^*)$. Recall that if

f is a quasi isomorphism then $C_2^{*-1} \oplus C_1^*$ is acyclic and

$$\tau(f, b_1, b_2) = \frac{(\det H(f))(\tau_{C_1^*, b_1})}{\tau_{C_2^*, b_2}} \tag{41}$$

where $\det H(f) : \det H(C_1^*) \to \det H(C_2^*)$ denotes the isomorphism of complex lines induces from the isomorphism in cohomology $H(f) : H(C_1^*) \to H(C_2^*)$.

Let us apply this to the integration homomorphism

$$\mathrm{Int} : \Omega^*_{g,b}(M; E)(\gamma) \to C^*(X; E) \tag{42}$$

where the notation is as in Proposition 4.7. Equip $\Omega^*_{g,b}(M; E)(\gamma)$ with the restriction of $\beta_{g,b}$, and equip $C^*(X; E)$ with the bilinear form $b|_{\mathcal{X}}$ obtained by restricting b to the fibers over $\mathcal{X}$. Since (42) is a quasi isomorphism the mapping cylinder is acyclic and the corresponding relative torsion is a non-vanishing complex number we will denote by

$$\tau\Big(\Omega^*_{g,b}(M; E)(\gamma) \xrightarrow{\mathrm{Int}} C^*_b(X; E)\Big) \in \mathbb{C}^\times.$$

Proposition 5.11. *Let M be a closed connected smooth manifold with vanishing Euler–Poincaré characteristics. Let E be a flat complex vector bundle over M. Let g be a Riemannian metric, and let X be a Morse–Smale vector field on M. Suppose b is a fiber wise non-degenerate symmetric bilinear form on E which is parallel in a neighborhood of the critical points $\mathcal{X}$. Moreover, let γ be a simple closed curve around 0 which avoids the spectrum of $\Delta_{E,g,b}$. Then:*

$$\mathcal{S}_{E,[b]} = \tau\Big(\Omega^*_{g,b}(M; E)(\gamma) \xrightarrow{\mathrm{Int}} C^*_b(X; E)\Big) \cdot \prod_q \big(\det{}^\gamma(\Delta_{E,g,b,q})\big)^{(-1)^q q} \cdot \exp\Big(-2\int_{M\setminus\mathcal{X}} \omega_{E,b} \wedge (-X)^*\Psi(g)\Big)$$

The integral is absolutely convergent since $\omega_{E,b}$ vanishes in a neighborhood of $\mathcal{X}$.

Proof. Let $x_0 \in M$ be a base point. For every critical point $x \in \mathcal{X}$ choose a path σ_x with $\sigma_x(0) = x_0$ and $\sigma_x(1) = x$. Set $c := \sum_{x\in\mathcal{X}} (-1)^{\mathrm{ind}(x)} \sigma_x$ and consider the Euler structure $\mathfrak{e} := [-X, c] \in \mathfrak{Eul}(M; \mathbb{Z})$. For the dual coEuler structure $P(\mathfrak{e}) = [g, \alpha]$ we have, see (3),

$$\int_{M\setminus\mathcal{X}} \omega_{E,b} \wedge \big((-X)^*\Psi(g) - \alpha\big) = \int_c \omega_{E,b}. \tag{43}$$

Let b_{x_0} denote the bilinear form on the fiber E_{x_0} obtained by restricting b. For $x \in \mathcal{X}$ let $\tilde{b}_x$ denote the bilinear form obtained from b_{x_0} by parallel transport along σ_x. Let $\tilde{b}_{\det C^*(X;E)}$ denote the induced bilinear form on $\det C^*(X; E)$. This is the bilinear form used in the definition of the combinatorial torsion. We want to compare it with the bilinear form $b_{\det C^*(X;E)}$ on $\det C^*(X; E)$ induced by the restriction $b|_{\mathcal{X}}$ of b to the fibers over $\mathcal{X}$. A simple computation similar to the proof of Lemma 2.2 yields

$$\tilde{b}_{\det C^*(X;E)} = \exp\Big(2\int_c \omega_{E,b}\Big)\cdot b_{\det C^*(X;E)}. \tag{44}$$

Let $\tau_{C^*(X;E),b|_{\mathcal{X}}}$ denote the non-degenerate bilinear form on $\det H^*(M;E)$ obtained from the torsion of the complex $C^*(X;E)$ equipped with the bilinear form $b|_{\mathcal{X}}$ via the isomorphism $\det H^*(M;E) = \det H(C^*(X;E))$, see (19) and (20). Then, using (41),

$$\frac{\tau^{\mathrm{an}}_{E,g,b}(\gamma)}{\tau_{C^*(X;E),b|_{\mathcal{X}}}} = \tau\Big(\Omega^*_{g,b}(M;E)(\gamma)\xrightarrow{\mathrm{Int}} C^*_b(X;E)\Big). \tag{45}$$

Moreover, (44) implies

$$\tau^{\mathrm{comb}}_{E,\mathfrak{e}} = \tau_{C^*(X;E),b|_{\mathcal{X}}}\cdot\exp\Big(2\int_c\omega_{E,b}\Big). \tag{46}$$

From Proposition 4.7 we obtain

$$\tau^{\mathrm{an}}_{E,P(\mathfrak{e}),[b]} = \tau^{\mathrm{an}}_{E,g,b}(\gamma)\cdot\prod_q\big(\det{}^\gamma(\Delta_{E,g,b,q})\big)^{(-1)^q q}\cdot\exp\Big(-2\int_M\omega_{E,b}\wedge\alpha\Big). \tag{47}$$

Combining (43), (45), (46) and (47) we obtain the statement of the proposition. □

Consider an open subset $U\subseteq\mathbb{C}$ and a family of flat complex vector bundles $\{E^z\}_{z\in U}$. Such a family is called *holomorphic* if the underlying vector bundles are the same for all $z\in U$ and the mapping $z\mapsto\nabla^{E^z}$ is holomorphic into the affine Fréchet space of linear connections equipped with the C^∞-topology.

Proposition 5.12. *Let M be a closed connected smooth manifold with vanishing Euler–Poincaré characteristics. Let $\{E^z\}_{z\in U}$ be a holomorphic family of flat complex vector bundles over M, and let b^z be a holomorphic family of fiber wise non-degenerate symmetric bilinear forms on E^z. Then $\mathcal{S}_{E^z,[b^z]}$ depends holomorphically on z.*

Proof. Let X be a Morse–Smale vector field on M. Let g be a Riemannian metric on M. In view of Theorem 4.2 we may w.l.o.g. assume $\nabla^{E^z}b^z = 0$ in a neighborhood of $\mathcal{X}$. W.l.o.g. we may assume that there exists a simple closed curve γ around 0 so that the spectrum of Δ_{E^z,g,b^z} avoids γ for all $z\in U$. From Proposition 5.11 we know:

$$\mathcal{S}_{E^z,[b^z]} = \tau\Big(\Omega^*_{g,b^z}(M;E^z)(\gamma)\xrightarrow{\mathrm{Int}} C^*_{b^z}(X;E^z)\Big)$$
$$\cdot\prod_q\big(\det{}^\gamma(\Delta_{E^z,g,b^z,q})\big)^{(-1)^q q}\cdot\exp\Big(-2\int_{M\setminus\mathcal{X}}\omega_{E^z,b^z}\wedge(-X)^*\Psi(g)\Big)$$

Since Δ_{E^z,g,b^z} depends holomorphically on z, each of the three factors in this expression for $S_{E^z,[b^z]}$ will depend holomorphically on z too. □

In odd dimensions the following result has been established by Braverman and Kappeler, see Theorem 1.10 in [7].

Corollary 5.13. *Let M be a closed connected smooth manifold with vanishing Euler–Poincaré characteristics. Let E be a complex vector bundle over M, and let b be a fiber wise non-degenerate symmetric bilinear form on E. Then the assignment $\nabla \mapsto \mathcal{S}_{(E,\nabla),[b]}$ is locally constant, and of absolute value one, on the space of flat connections on E.*

Proof. Note that in view of Theorem 5.10 and Proposition 5.12 the relative torsion $\mathcal{S}_{(E,\nabla^z),[b]}$ is constant along every holomorphic path of flat connections $z \mapsto \nabla^z$ on E. Moreover, note that two flat connections, contained in the same connected component, can always be joined by a piecewise holomorphic path of flat connections. □

Using the Bismut–Zhang, Cheeger, Müller theorem again, we are able to verify Conjecture 5.1 for flat connections contained in particular connected components of the space of flat connections on a fixed complex vector bundle. More precisely, we have[2]

Corollary 5.14. *Let M be a closed connected smooth manifold with vanishing Euler–Poincaré characteristics. Let (F,∇^F) be a flat real vector bundle over M equipped with a fiber wise Hermitian structure h. Let (E,∇^E) denote the flat complex vector bundle obtained by complexifying (F,∇^F), and let b denote the fiber wise non-degenerate symmetric bilinear form on E obtained by complexifying h. Then, for every flat connection ∇ on E which is contained in the connected component of ∇^E, we have $\mathcal{S}_{(E,\nabla),[b]} = 1$.*

Proof. In view of Corollary 5.13 it suffices to show $\mathcal{S}_{(E,\nabla^E),[b]} = 1$. From Proposition 5.7(iv) we have $\mathcal{S}_{(E,\nabla^E),[b]} = \mathcal{S}_{(F,\nabla^F),[h]}$. In view of [2, Theorem 0.2], see Remark 5.6, we indeed have $\mathcal{S}_{(F,\nabla^F),[h]} = 1$, and the statement follows. □

The circle, a simple explicit example

Consider $M := S^1$. In this case it is possible to explicitly compute the combinatorial and analytic torsion, see below. It turns out that Conjecture 5.1 holds true for every flat vector bundle over the circle.

We think of S^1 as $\{z \in \mathbb{C} \mid |z| = 1\}$. Equip S^1 with the standard Riemannian metric g of circumference 2π. Orient S^1 in the standard way. Let θ denote the angular 'coordinate'. Let $\frac{\partial}{\partial\theta}$ denote the corresponding vector field which is of length 1 and induces the orientation. For the dual 1-form we write $d\theta$.

Let $k \in \mathbb{N}$ and suppose $a \in C^\infty(S^1, \mathfrak{gl}_k(\mathbb{C}))$. Let E^a denote the trivial vector bundle $S^1 \times \mathbb{C}^k$ equipped with the flat connection $\nabla = \frac{\partial}{\partial\theta} + a$. Here and in what follows we use the identifications $\Omega^0(M;E^a) = C^\infty(S^1;\mathbb{C}^k) = \Omega^1(M;E^a)$ where the latter stems from the global coframe $d\theta$.

Let $b \in C^\infty(S^1, \mathrm{Sym}_k^\times(\mathbb{C}))$ where $\mathrm{Sym}_k^\times(\mathbb{C})$ denotes the space of complex non-degenerate symmetric $k \times k$-matrices. We consider b as a fiber wise non-degenerate

[2] In a recent preprint [22] R.-T. Huang verified a similar statement for flat connections whose connected component contains a flat connection which admits a parallel Hermitian structure.

symmetric bilinear form on E^a. For the induced bilinear form on $\Omega^*(S^1;E^a)$ we have:

$$\beta_{g,b}(v,w)=\int_{S^1} v^t bw\, d\theta, \qquad v,w\in\Omega^0(S^1;E^a)=C^\infty(S^1,\mathbb{C}^k)$$
$$\beta_{g,b}(v,w)=\int_{S^1} v^t bw\, d\theta, \qquad v,w\in\Omega^1(S^1;E^a)=C^\infty(S^1,\mathbb{C}^k)$$

A straight forward computations yields:

$$\begin{aligned}
d_{E^a} &= \tfrac{\partial}{\partial\theta}+a\\
d^\sharp_{E^a,g,b} &= -\tfrac{\partial}{\partial\theta}-b^{-1}b'+b^{-1}a^tb\\
\Delta_{E^a,g,b,0} &= -\big(\tfrac{\partial}{\partial\theta}\big)^2+\big(b^{-1}a^tb-b^{-1}b'-a\big)\tfrac{\partial}{\partial\theta}+\big(b^{-1}a^tba-b^{-1}b'a-a'\big)\\
\Delta_{E^a,g,b,1} &= -\big(\tfrac{\partial}{\partial\theta}\big)^2+\big(b^{-1}a^tb-b^{-1}b'-a\big)\tfrac{\partial}{\partial\theta}\\
&\quad+\big((b^{-1}a^tb)'-(b^{-1}b')'-ab^{-1}b'+ab^{-1}a^tb\big)\\
b^{-1}\nabla_{\frac{\partial}{\partial\theta}}b &= b^{-1}b'-b^{-1}a^tb-a\\
\omega_{E^a,b} &= -\tfrac12\operatorname{tr}\big(b^{-1}b'-b^{-1}a^tb-a\big)d\theta=-\tfrac12\big(\operatorname{tr}(b^{-1}b')-2\operatorname{tr}(a)\big)d\theta
\end{aligned}$$

Here $b':=\frac{\partial}{\partial\theta}b$ and $a':=\frac{\partial}{\partial\theta}a$.

Let us write $A\in\mathrm{GL}_k(\mathbb{C})$ for the holonomy in E^a along the standard generator of $\pi_1(S^1)$. Recall that $\det A=\exp\big(\int_{S^1}\operatorname{tr}(a)d\theta\big)$. Using the explicit formula in [9, Theorem 1] we get:

$$\begin{aligned}
\det(\Delta_{E^a,g,b,1}) &= \mathbf{i}^{2k}\exp\Big(\frac{\mathbf{i}}{2}\int_{S^1}\operatorname{tr}\big(\mathbf{i}(b^{-1}a^tb-b^{-1}b'-a)\big)d\theta\Big)\det\Big(1-\left(\begin{smallmatrix}A^{-1}&*\\0&A^t\end{smallmatrix}\right)\Big)\\
&=\exp\Big(\frac12\int_{S^1}\operatorname{tr}(b^{-1}b')d\theta\Big)\det(A-1)^2\det A^{-1}\\
&=\exp\Big(\frac12\int_{S^1}\big(\operatorname{tr}(b^{-1}b')-2\operatorname{tr}(a)\big)d\theta\Big)\det(A-1)^2
\end{aligned}$$

Consider the Euler structure $\mathfrak{e}:=[-\frac{\partial}{\partial\theta},0]\in\mathfrak{Eul}(S^1;\mathbb{Z})$, and the coEuler structure $\mathfrak{e}^*:=[g,\frac12]\in\mathfrak{Eul}^*(S^1;\mathbb{C})$. Then $P(\mathfrak{e})=\mathfrak{e}^*$, see (3). Assuming acyclicity, i.e. 1 is not an eigen value of A, we conclude:

$$\tau^{\mathrm{an}}_{E^a,\mathfrak{e}^*,[b]}=\det(A-1)^{-2}. \tag{48}$$

Observe that this is independent of $[b]$, cf. Remark 5.3.

Considering a Morse–Smale vector field X with two critical points and the Euler structure $\mathfrak{e}$ we obtain a Morse complex $C^*(X;E^a)$ isomorphic to

$$\mathbb{C}^k\xrightarrow{A-1}\mathbb{C}^k$$

equipped with the standard bilinear form. From Example 3.2 we obtain

$$\tau^{\mathrm{comb}}_{E^a,\mathfrak{e}}=\det(A-1)^{-2}$$

which coincides with (48). So we see that $\tau^{\text{comb}}_{E^a,\mathfrak{e}} = \tau^{\text{an}}_{E^a,\mathfrak{e}^*,[b]}$, i.e. $\mathcal{S}_{E^a,[b]} = 1$, whenever E^a is acyclic. From Proposition 5.12 we conclude $\mathcal{S}_{E^a,[b]} = 1$ for all, not necessarily acyclic, E^a. Thus Conjecture 5.1 holds for $M = S^1$.

Remark 5.15. Recall the canonic coEuler structure $\mathfrak{e}^*_{\text{can}} = [g, 0]$ defined as the unique fixed point of the involution on $\mathfrak{Eul}^*(S^1;\mathbb{C})$, see Section 2. Note that $\mathfrak{e}^*_{\text{can}}$ is not integral. The computations above show that for the analytic torsion we have

$$\tau^{\text{an}}_{E^a,\mathfrak{e}^*_{\text{can}},[b]} = s_{[b]} \det A \det(A-1)^{-2}$$

where

$$s_{[b]} = \exp\Big(-\frac{1}{2}\int_{S^1} \operatorname{tr}(b^{-1}b')\Big) \in \{\pm 1\}$$

does depend on b. Note that this sign $s_{[b]}$ appears, although we consider the torsion as a bilinear form, i.e. we essentially consider the square of what is traditionally called the torsion.

On odd dimensional manifolds one often considers the analytic torsion without a correction term, i.e. one considers $\tau^{\text{an}}_{E,\mathfrak{e}^*_{\text{can}},[b]}$. Let us give two reasons why this is not such a natural choice as it might seem. First, the celebrated fact that the Ray–Singer torsion on odd dimensional manifolds does only depend on the flat connection, is no longer true in the complex setting as the appearance of the sign $s_{[b]}$ shows. Of course a different definition of complex valued analytic torsion might circumvent this problem. More serious is the second point. One would expect that the analytic torsion as considered above is the square of a rational function on the space of acyclic representations of the fundamental group. As the computation for the circle shows, this cannot be true for $\tau^{\text{an}}_{E,\mathfrak{e}^*_{\text{can}},[b]}$, simply because $\sqrt{\det A}$ cannot be rational in $A \in \mathrm{GL}_k(\mathbb{C})$. Any reasonable definition of complex valued analytic torsion will have to face this problem.

If one is willing to consider $\tau^{\text{an}}_{E,\mathfrak{e}^*,[b]}$ where $\mathfrak{e}^*$ is an integral coEuler structure both problems disappear, assuming E admits a non-degenerate symmetric bilinear form and Conjecture 5.1 is true. Then $\tau^{\text{an}}_{E,\mathfrak{e}^*,[b]}$ is indeed independent of $[b]$, see Remark 5.3, and the dependence on $\mathfrak{e}^*$ is very simple, see Remark 4.4. More importantly, $\tau^{\text{an}}_{E,\mathfrak{e}^*,[b]}$ is the square of a rational function on the space of acyclic representations of the fundamental group. This follows from the fact that $\tau^{\text{comb}}_{E,\mathfrak{e}}$ with $P(\mathfrak{e}) = \mathfrak{e}^*$ is the square of such a rational function, see [11].

6. Proof of the anomaly formula

We continue to use the notation of Section 4. The proof of Theorem 4.2 is based on the following two results whose proof we postpone till Section 8.

Proposition 6.1. *Suppose* $\phi \in \Gamma(\operatorname{end}(E))$. *Then*

$$\operatorname*{LIM}_{t\to 0} \operatorname{str}\big(\phi e^{-t\Delta_{E,g,b}}\big) = \int_M \operatorname{tr}(\phi)\, \mathrm{e}(g).$$

Here LIM *denotes the renormalized limit, see* [1, Section 9.6]*, which in this case is actually an ordinary limit.*

Proposition 6.2. *Suppose* $\xi \in \Gamma(\mathrm{end}(TM))$ *is symmetric with respect to* g*, and let* $\Lambda^*\xi \in \mathrm{end}(\Lambda^*T^*M)$ *denote its extension to a derivation on* Λ^*T^*M*. Then*

$$\operatorname*{LIM}_{t\to 0} \operatorname{str}\Big(\big(\Lambda^*\xi - \tfrac{1}{2}\operatorname{tr}(\xi)\big)e^{-t\Delta_{E,g,b}}\Big) = \int_M \operatorname{tr}(b^{-1}\nabla^E b) \wedge (\partial_2 \operatorname{cs})(g, g\xi).$$

Again LIM *denotes the renormalized limit, which in this case is just the constant term of the asymptotic expansion for* $t \to 0$*. Moreover, we use the notation* $(\partial_2 \operatorname{cs})(g, g\xi) := \frac{\partial}{\partial t}|_0 \operatorname{cs}(g, g + tg\xi)$*.*

Let us now give a proof of Theorem 4.2. Suppose g_u and b_u depend smoothly on a real parameter u. Let γ be a simple closed curve around 0 and assume w.l.o.g. that u varies in an open set U so that the spectrum of $\Delta_u := \Delta_{E,g_u,b_u}$ avoids the curve γ for all $u \in U$. Let Q_u denote the spectral projection onto the eigen spaces corresponding to eigen values in the exterior of γ, and $Q_{u,q}$ the part acting in degree q. Let us write $\dot\Delta_u := \frac{\partial}{\partial u}\Delta_u$, and $\dot\Delta_{u,q}$ for the part acting in degree q. From the variation formula for the determinant of generalized Laplacians, see for instance [1, Proposition 9.38], we obtain

$$\begin{aligned}\frac{\partial}{\partial u} \log \prod_q \big(\det{}^\gamma(\Delta_{u,q})\big)^{(-1)^q q} &= \sum_q (-1)^q q\Big(\frac{\partial}{\partial u} \log \det{}^\gamma(\Delta_{u,q})\Big) \\ &= \sum_q (-1)^q q\Big(\operatorname*{LIM}_{t\to 0} \operatorname{tr}\big(\dot\Delta_{u,q}(\Delta_{u,q})^{-1}Q_{u,q}e^{-t\Delta_{u,q}}\big)\Big) \\ &= \operatorname*{LIM}_{t\to 0} \operatorname{str}\big(N\dot\Delta_u\Delta_u^{-1}Q_u e^{-t\Delta_u}\big) \qquad (49)\end{aligned}$$

where N denotes the grading operator which acts by multiplication with q on $\Omega^q(M;E)$.

Choose $u_0 \in U$ and define $G_u \in \Gamma(\mathrm{Aut}(TM))$ by

$$g_u(a,b) = g_{u_0}(G_u a, b) = g_{u_0}(a, G_u b)$$

and similarly $B_u \in \Gamma(\mathrm{Aut}(E))$ by

$$b_u(e,f) = b_{u_0}(B_u e, f) = b_{u_0}(e, B_u f).$$

Let $\Lambda^*G_u^{-1}$ denote the natural extension of G_u^{-1} to $\Gamma(\mathrm{Aut}(\Lambda^*T^*M))$ and define

$$A_u = \det(G_u)^{1/2} \cdot \Lambda^*G_u^{-1} \otimes B_u \in \Gamma(\mathrm{Aut}(\Lambda^*T^*M \otimes E)).$$

Then

$$\beta_{g_u,b_u}(v,w) = \beta_{g_{u_0},b_{u_0}}(A_u v, w) = \beta_{g_{u_0},b_{u_0}}(v, A_u w), \qquad v,w \in \Omega(M;E). \quad (50)$$

Abbreviating $d_u^\sharp := d^\sharp_{E,g_u,b_u}$ we immediately get

$$d_u^\sharp := A_u^{-1} d_{u_0}^\sharp A_u.$$

Writing $\dot{A}_u := \frac{\partial}{\partial u}A_u$, $\dot{g}_u := \frac{\partial}{\partial u}g_u$ and $\dot{b}_u := \frac{\partial}{\partial u}b_u$ we have

$$A_u^{-1}\dot{A}_u = \big(-\Lambda^*(g_u^{-1}\dot{g}_u) + \tfrac{1}{2}\operatorname{tr}(g_u^{-1}\dot{g}_u)\big)\otimes 1 + 1\otimes(b_u^{-1}\dot{b}_u) \in \Gamma(\operatorname{end}(\Lambda^*T^*M\otimes E)) \tag{51}$$

where $\Lambda^*(g_u^{-1}\dot{g}_u)$ denotes the extension of $g_u^{-1}\dot{g}_u \in \Gamma(\operatorname{end}(TM))$ to a derivation on Λ^*T^*M.

Let us write $d := d_E$ and $\dot{d}_u^\sharp := \frac{\partial}{\partial u}d_u^\sharp$. Using the obvious relations $\dot{\Delta}_u = [d, \dot{d}_u^\sharp]$, $[N, d] = d$, $[d, \Delta_u] = 0$, $[d, Q_u] = 0$, $\dot{d}_u^\sharp = [d_u^\sharp, A_u^{-1}\dot{A}_u]$ and the fact that the super trace vanishes on super commutators we get:

$$\begin{aligned}
\operatorname{str}\big(N\dot{\Delta}_u\Delta_u^{-1}Q_ue^{-t\Delta_u}\big) &= \operatorname{str}\big(Nd\dot{d}_u^\sharp\Delta_u^{-1}Q_ue^{-t\Delta_u}\big) + \operatorname{str}\big(N\dot{d}_u^\sharp d\Delta_u^{-1}Q_ue^{-t\Delta_u}\big)\\
&= \operatorname{str}\big(d\dot{d}_u^\sharp\Delta_u^{-1}Q_ue^{-t\Delta_u}\big)\\
&= \operatorname{str}\big(dd_u^\sharp A_u^{-1}\dot{A}_u\Delta_u^{-1}Q_ue^{-t\Delta_u}\big)\\
&\qquad - \operatorname{str}\big(dA_u^{-1}\dot{A}_ud_u^\sharp\Delta_u^{-1}Q_ue^{-t\Delta_u}\big)\\
&= \operatorname{str}\big(A_u^{-1}\dot{A}_u(dd_u^\sharp + d_u^\sharp d)\Delta_u^{-1}Q_ue^{-t\Delta_u}\big)\\
&= \operatorname{str}\big(A_u^{-1}\dot{A}_uQ_ue^{-t\Delta_u}\big)
\end{aligned}$$

Together with (49) this gives

$$\frac{\partial}{\partial u}\log\prod_q\big(\det{}^\gamma(\Delta_{u,q})\big)^{(-1)^qq} = \operatorname*{LIM}_{t\to 0}\operatorname{str}\big(A_u^{-1}\dot{A}_uQ_ue^{-t\Delta_u}\big) \tag{52}$$

Let us write $\Omega_u^* := \Omega_{E,g_u,b_u}^*(M;E)(\gamma)$. Note that this is a family of finite dimensional complexes smoothly parametrized by $u \in U$. Let $P_u = 1 - Q_u$ denote the spectral projection of Δ_u onto Ω_u^*. Note that since $\operatorname{str}P_uP_u = \text{const}$ we have $\operatorname{str}P_u\dot{P}_u = 0$. For sufficiently small $w - u$ the restriction of the spectral projection $P_w|_{\Omega_u^*} : \Omega_u^* \to \Omega_w^*$ is an isomorphism of complexes. We get a commutative diagram of determinant lines:

$$\begin{array}{ccccc}
\det\Omega_u^* & \longrightarrow & \det H(\Omega_u^*) & \longrightarrow & \det H^*(M;E)\\
\Big\downarrow{\scriptstyle \det(P_w|_{\Omega_u^*})} & & \Big\downarrow{\scriptstyle \det H(P_w|_{\Omega_u^*})} & & \Big\downarrow{\scriptstyle \det H(P_w)=1}\\
\det\Omega_w^* & \longrightarrow & \det H(\Omega_w^*) & \longrightarrow & \det H^*(M;E)
\end{array}$$

Writing $\beta_u := \beta_{E,g_u,b_u}$ and $\tau_u^{\text{an}}(\gamma) := \tau_{E,g_u,b_u}^{\text{an}}(\gamma)$, we obtain, for sufficiently small $w - u$,

$$\frac{\tau_w^{\text{an}}(\gamma)}{\tau_u^{\text{an}}(\gamma)} = \operatorname{sdet}\Big((\beta_u|_{\Omega_u^*})^{-1}(P_w|_{\Omega_u^*})^*\beta_w\Big). \tag{53}$$

Here the two non-degenerate bilinear forms $\beta_u|_{\Omega_u^*}$ and $(P_w|_{\Omega_u^*})^*\beta_w$ on Ω_u^* are considered as isomorphisms from Ω_u^* to its dual, hence $(\beta_u|_{\Omega_u^*})^{-1}(P_w|_{\Omega_u^*})^*\beta_w$ is an automorphism of Ω_u^*. Using (50) we find

$$(\beta_u|_{\Omega_u^*})^{-1}(P_w|_{\Omega_u^*})^*\beta_w = P_uA_u^{-1}A_wP_w|_{\Omega_u^*}.$$

Using (53) we thus obtain

$$\frac{\tau_w^{\mathrm{an}}(\gamma)}{\tau_u^{\mathrm{an}}(\gamma)} = \operatorname{sdet}\bigl(P_u A_u^{-1} A_w P_w|_{\Omega_u^*}\bigr).$$

In view of $\operatorname{str}(P_u \dot P_u) = 0$ we get

$$\frac{\partial}{\partial w}\Big|_u \left(\frac{\tau_w^{\mathrm{an}}(\gamma)}{\tau_u^{\mathrm{an}}(\gamma)}\right) = \operatorname{str}\bigl(P_u A_u^{-1}\dot A_u P_u + P_u A_u^{-1} A_u \dot P_u\bigr) = \operatorname{str}\bigl(A_u^{-1}\dot A_u P_u\bigr).$$

Combining this with (52) and Proposition 4.7 we obtain

$$\frac{\partial}{\partial w}\Big|_u \left(\frac{\tau_w^{\mathrm{an}}(0)\cdot\prod_q(\det{}'\Delta_{w,q})^{(-1)^q q}}{\tau_u^{\mathrm{an}}(0)\cdot\prod_q(\det{}'\Delta_{u,q})^{(-1)^q q}}\right) = \operatorname*{LIM}_{t\to 0}\operatorname{str}\bigl(A_u^{-1}\dot A_u e^{-t\Delta_u}\bigr). \tag{54}$$

Applying Proposition 6.1 to $\phi = b_u^{-1}\dot b_u$ we obtain

$$\operatorname*{LIM}_{t\to 0}\operatorname{str}\bigl(b_u^{-1}\dot b_u e^{-t\Delta_u}\bigr) = \int_M \operatorname{tr}(b_u^{-1}\dot b_u)\,\mathrm{e}(g_u).$$

Using Proposition 6.2 with $\xi = g_u^{-1}\dot g_u$ we get

$$\operatorname*{LIM}_{t\to 0}\operatorname{str}\Bigl(\bigl(\Lambda^*(g_u^{-1}\dot g_u) - \tfrac{1}{2}\operatorname{tr}(g_u^{-1}\dot g_u)\bigr)e^{-t\Delta_u}\Bigr) = \int_M \operatorname{tr}(b_u^{-1}\nabla^E b_u)\wedge(\partial_2\,\mathrm{cs})(g_u,\dot g_u).$$

Using (51) we conclude

$$\operatorname*{LIM}_{t\to 0}\operatorname{str}\bigl(A_u^{-1}\dot A_u e^{-t\Delta_u}\bigr) = \int_M \operatorname{tr}(b_u^{-1}\dot b_u)\,\mathrm{e}(g_u) - \int_M \operatorname{tr}(b_u^{-1}\nabla^E b_u)\wedge(\partial_2\,\mathrm{cs})(g_u,\dot g_u). \tag{55}$$

Let us finally turn to the correction term. If $[g_u,\alpha_u]\in\mathfrak{Eul}^*(M;\mathbb{C})$ represent the same coEuler structure then $\alpha_w - \alpha_u = \mathrm{cs}(g_u,g_w)$ and thus

$$\frac{\partial}{\partial u}\alpha_u = \frac{\partial}{\partial w}\Big|_u \mathrm{cs}(g_u,g_w) = (\partial_2\,\mathrm{cs})(g_u,\dot g_u).$$

Moreover, we have

$$\begin{aligned}
\frac{\partial}{\partial u}\operatorname{tr}(b_u^{-1}\nabla^E b_u) &= \operatorname{tr}\bigl(-b_u^{-1}\dot b_u b_u^{-1}\nabla^E b_u\bigr) + \operatorname{tr}\bigl(b_u^{-1}\nabla^E \dot b_u\bigr)\\
&= \operatorname{tr}\bigl(-b_u^{-1}(\nabla^E b_u)b_u^{-1}\dot b_u\bigr) + \operatorname{tr}\bigl(b_u^{-1}\nabla^E \dot b_u\bigr)\\
&= \operatorname{tr}\bigl((\nabla^E b_u^{-1})\dot b_u\bigr) + \operatorname{tr}\bigl(b_u^{-1}\nabla^E \dot b_u\bigr)\\
&= \operatorname{tr}\bigl(\nabla^E(b_u^{-1}\dot b_u)\bigr)\\
&= d\operatorname{tr}(b_u^{-1}\dot b_u).
\end{aligned}$$

Using $-2\omega_{E,b_u} = \operatorname{tr}(b_u^{-1}\nabla^E b_u)$, $d\alpha_u = \mathrm{e}(g_u)$ and Stokes' theorem we get

$$\begin{aligned}\frac{\partial}{\partial u}\int_M -2\omega_{E,b_u}\wedge\alpha_u &= \int_M d\operatorname{tr}(b_u^{-1}\dot b_u)\wedge\alpha_u + \int_M \operatorname{tr}(b_u^{-1}\nabla^E b_u)\wedge(\partial_2\operatorname{cs})(g_u,\dot g_u)\\ &= -\int_M \operatorname{tr}(b_u^{-1}\dot b_u)\,\mathrm{e}(g_u) + \int_M \operatorname{tr}(b_u^{-1}\nabla^E b_u)\wedge(\partial_2\operatorname{cs})(g_u,\dot g_u). \quad (56)\end{aligned}$$

Combining (54), (55) and (56) we obtain

$$\frac{\partial}{\partial w}\Big|_u \frac{\tau^{\mathrm{an}}_{E,g_w,b_w,\alpha_w}}{\tau^{\mathrm{an}}_{E,g_u,b_u,\alpha_u}} = 0.$$

This completes the proof of Theorem 4.2.

7. Asymptotic expansion of the heat kernel

In this section we will consider Dirac operators associated to a class of Clifford super connections. The main result Theorem 7.1 below computes the leading and subleading terms of the asymptotic expansion of the corresponding heat kernels. In Section 8 we will apply these results to the Laplacians introduced in Section 4 which are squares of such Dirac operators. We refer to [1] for background on the Clifford super connection formalism.

Let (M,g) be a closed Riemannian manifold of dimension n. Let $\mathrm{Cl} = \mathrm{Cl}(T^*M,g)$ denote the corresponding Clifford bundle. Recall that $\mathrm{Cl} = \mathrm{Cl}^+\oplus\mathrm{Cl}^-$ is a bundle of $\mathbb{Z}_2$-graded filtered algebras, and let us write Cl_k for the subbundle of filtration degree k. Recall that we have the symbol map

$$\sigma:\mathrm{Cl}\to\Lambda^*T^*M, \qquad \sigma(a) := c(a)\cdot 1$$

where c denotes the usual Clifford action on Λ^*T^*M. Explicitly, for $a\in T_x^*M\subseteq \mathrm{Cl}_x$ and $\alpha\in\Lambda^*T_x^*M$ we have $c(a)\cdot\alpha = a\wedge\alpha - i_{\sharp a}\alpha$, where $\sharp a = g^{-1}a\in T_xM$ and $i_{\sharp a}$ denotes contraction with $\sharp a$. Here the metric is considered as an isomorphism $g: TM\to T^*M$ and g^{-1} denotes its inverse. Recall that σ is an isomorphism of filtered $\mathbb{Z}_2$-graded vector bundles inducing an isomorphism on the associated graded bundles of algebras.

Let $\mathcal{E} = \mathcal{E}^+\oplus\mathcal{E}^-$ be a $\mathbb{Z}_2$-graded complex Clifford module over M. The forms with values in $\mathcal{E}$ inherit a $\mathbb{Z}_2$-grading which will be denoted by:

$$\Omega(M;\mathcal{E}) = \Omega(M;\mathcal{E})^+\oplus\Omega(M;\mathcal{E})^-$$

We have $\Omega(M;\mathcal{E})^+ = \Omega^{\mathrm{even}}(M;\mathcal{E}^+)\oplus\Omega^{\mathrm{odd}}(M;\mathcal{E}^-)$ and similarly for $\Omega(M;\mathcal{E})^-$. Let us write $\mathrm{end}_{\mathrm{Cl}}(\mathcal{E})$ for the bundle of algebras of endomorphisms of $\mathcal{E}$ which (super) commute with the Clifford action, and let us indicate its $\mathbb{Z}_2$-grading by:

$$\mathrm{end}_{\mathrm{Cl}}(\mathcal{E}) = \mathrm{end}^+_{\mathrm{Cl}}(\mathcal{E})\oplus\mathrm{end}^-_{\mathrm{Cl}}(\mathcal{E})$$

Recall that we have a canonic isomorphism of bundles of $\mathbb{Z}_2$-graded algebras

$$\mathrm{end}(\mathcal{E}) = \mathrm{Cl}\otimes\mathrm{end}_{\mathrm{Cl}}(\mathcal{E}). \quad (57)$$

Suppose $\mathbb{A} : \Omega(M;\mathcal{E})^{\pm} \to \Omega(M;\mathcal{E})^{\mp}$ is a Clifford super connection, see [1, Definition 3.39]. Recall that with respect to (57) its curvature $\mathbb{A}^2 \in \Omega(M;\mathrm{end}(\mathcal{E}))^+$ decomposes as

$$\mathbb{A}^2 = R^{\mathrm{Cl}} \otimes 1 + 1 \otimes F_{\mathbb{A}}^{\mathcal{E}/S} \tag{58}$$

where $R^{\mathrm{Cl}} \in \Omega^2(M;\mathrm{Cl}^2)$ with $\mathrm{Cl}^2 := \sigma^{-1}(\Lambda^2 T^*M) \subseteq \mathrm{Cl}^+$ is a variant of the Riemannian curvature

$$R^{\mathrm{Cl}}(X,Y) = \tfrac{1}{4}\sum_{i,j} g(R_{X,Y}e_i, e_j)c^i c^j \tag{59}$$

and $F_{\mathbb{A}}^{\mathcal{E}/S} \in \Omega(M;\mathrm{end}_{\mathrm{Cl}}(\mathcal{E}))^+$ is called the twisting curvature, see [1, Proposition 3.43]. Here e_i is a local orthonormal frame of TM, $e^i := ge_i$ denotes its dual local coframe and $c^i = c(e^i)$ denotes Clifford multiplication with e^i.

Recall that the Dirac operator $D_{\mathbb{A}}$ associated to the Clifford super connection $\mathbb{A}$ is given by the composition

$$\Gamma(\mathcal{E}) \xrightarrow{\mathbb{A}} \Omega(M;\mathcal{E}) = \Gamma(\Lambda^* T^*M \otimes \mathcal{E}) \xrightarrow{\sigma^{-1}\otimes 1_{\mathcal{E}}} \Gamma(\mathrm{Cl}\otimes\mathcal{E}) \xrightarrow{c} \Gamma(\mathcal{E})$$

where $c : \mathrm{Cl}\otimes\mathcal{E} \to \mathcal{E}$ denotes Clifford multiplication.

We will from now on restrict to very special Clifford super connections on $\mathcal{E}$ which are of the form

$$\mathbb{A} = \nabla + A$$

where $\nabla : \Omega^*(M;\mathcal{E}^{\pm}) \to \Omega^{*+1}(M;\mathcal{E}^{\pm})$ is a Clifford connection on $\mathcal{E}$, and $A \in \Omega^0(M;\mathrm{end}^-_{\mathrm{Cl}}(\mathcal{E}))$. For the associated Dirac operator acting on $\Gamma(\mathcal{E})$ we have

$$D_{\mathbb{A}} = D_{\nabla} + A.$$

Consider the induced connection $\nabla : \Omega^*(M;\mathrm{end}^{\pm}(\mathcal{E})) \to \Omega^{*+1}(M;\mathrm{end}^{\pm}(\mathcal{E}))$. Since ∇ is a Clifford connection this induced connection preserves the subbundle $\mathrm{end}_{\mathrm{Cl}}(\mathcal{E})$. Moreover, we have $[D_{\nabla}, A] = c(\nabla A)$ and thus

$$D^2_{\mathbb{A}} = D^2_{\nabla} + c(\nabla A) + A^2. \tag{60}$$

Here $\nabla A \in \Omega^1(M;\mathrm{end}^-_{\mathrm{Cl}}(\mathcal{E}))$, $A^2 \in \Omega^0(M;\mathrm{end}^+_{\mathrm{Cl}}(\mathcal{E}))$, and the Clifford action $c(B)$ of $B \in \Omega(M;\mathrm{end}(\mathcal{E}))$ on $\Gamma(\mathcal{E})$ is given by the composition:

$$\Gamma(\mathcal{E}) \xrightarrow{B} \Omega(M;\mathcal{E}) = \Gamma(\Lambda^* T^*M \otimes \mathcal{E}) \xrightarrow{\sigma^{-1}\otimes 1_{\mathcal{E}}} \Gamma(\mathrm{Cl}\otimes\mathcal{E}) \xrightarrow{c} \Gamma(\mathcal{E})$$

Note that for $B \in \Omega^0(M;\mathrm{end}(\mathcal{E}))$ the Clifford action coincides with the usual action $c(B) = B$.

Theorem 7.1. *Let $\mathcal{E}$ be a $\mathbb{Z}_2$-graded complex Clifford bundle over a closed Riemannian manifold (M,g) of dimension n. Suppose ∇ is a Clifford connection on $\mathcal{E}$ and $A \in \Omega^0(M;\mathrm{end}^-_{\mathrm{Cl}}(\mathcal{E}))$. Consider the Clifford super connection $\mathbb{A} = \nabla + A$ and the associated Dirac operator $D_{\mathbb{A}}$ acting on $\Gamma(\mathcal{E})$. Let $\Omega_g \in \Omega^n(M;\mathcal{O}_M^{\mathbb{C}})$ denote the volume density associated with the Riemannian metric g. Let $k_t \in \Gamma(\mathrm{end}(\mathcal{E}))$*

so that $k_t\Omega_g$ is the restriction of the kernel of $e^{-tD^2_{\mathbb{A}}}$ to the diagonal in $M\times M$. Consider its asymptotic expansion

$$k_t \sim (4\pi t)^{-n/2}\sum_{i\geq 0} t^i\tilde{k}_i \qquad \textit{as } t\to 0 \tag{61}$$

with $\tilde{k}_i\in\Gamma(\operatorname{end}(\mathcal{E}))$, see [1, Theorem 2.30]. *Then*

$$\tilde{k}_i\in\Gamma(\mathrm{Cl}_{2i}\otimes\operatorname{end}_{\mathrm{Cl}}(\mathcal{E}))\subseteq\Gamma(\mathrm{Cl}\otimes\operatorname{end}_{\mathrm{Cl}}(\mathcal{E}))=\Gamma(\operatorname{end}(\mathcal{E})). \tag{62}$$

Moreover, with the help of the symbol map

$$\sigma:\Gamma(\operatorname{end}(\mathcal{E}))=\Gamma(\mathrm{Cl}\otimes\operatorname{end}_{\mathrm{Cl}}(\mathcal{E}))\xrightarrow{\sigma\otimes 1}\Omega^*(M;\operatorname{end}_{\mathrm{Cl}}(\mathcal{E}))$$

and writing $\alpha_{[j]}$ for the j-form piece of α we have

$$\sum_{i\geq 0}\sigma(\tilde{k}_i)_{[2i]}=\hat{A}_g\wedge\exp(-F^{\mathcal{E}/S}_{\nabla}). \tag{63}$$

Here $\hat{A}_g\in\Omega^{4*}(M;\mathbb{R})$ denotes the $\hat{A}$-genus

$$\hat{A}_g=\det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)$$

and $R\in\Omega^2(M;\operatorname{end}(TM))$ the Riemannian curvature. Moreover, we have

$$\sum_{i\geq 0}\sigma(\tilde{k}_i)_{[2i-1]}=-\nabla\left(\hat{A}_g\wedge\left(\frac{e^{\operatorname{ad}(-F^{\mathcal{E}/S}_{\nabla})}-1}{\operatorname{ad}(-F^{\mathcal{E}/S}_{\nabla})}A\right)\wedge\exp\bigl(-F^{\mathcal{E}/S}_{\nabla}\bigr)\right) \tag{64}$$

where $\operatorname{ad}(F^{\mathcal{E}/S}_{\nabla}):\Omega^*(M;\operatorname{end}^{\pm}_{\mathrm{Cl}}(\mathcal{E}))\to\Omega^{*+2}(M;\operatorname{end}^{\pm}_{\mathrm{Cl}}(\mathcal{E}))$, is given by

$$\operatorname{ad}(F^{\mathcal{E}/S}_{\nabla})\phi:=F^{\mathcal{E}/S}_{\nabla}\wedge\phi-\phi\wedge F^{\mathcal{E}/S}_{\nabla}.$$

Remark 7.2. Note that (62) and (63) tell that on this level the asymptotic expansions for $e^{-tD^2_{\mathbb{A}}}$ and $e^{-tD^2_{\nabla}}$ are the same.

Proof. The proof below parallels the one of Theorem 4.1 in [1] where the case $A=0$ is treated. It too is based on Getzler's scaling techniques, see [21]. In order to prove Theorem 7.1 we need to compute one more term in the asymptotic expansion of the rescaled operator.

The calculation is local. Let $x_0\in M$. Use normal coordinates, i.e. the exponential mapping of g, to identify a convex neighborhood U of $0\in T_{x_0}M$ with a neighborhood of x_0. Choose an orthonormal basis $\{\partial_i\}$ of $T_{x_0}M$ and linear coordinates $\mathbf{x}=(x^1,\dots,x^n)$ on $T_{x_0}M$ such that $\{dx^i\}$ is dual to $\{\partial_i\}$. Let $\mathcal{R}:=\sum_i x^i\partial_i$ denote the radial vector field. Note that every affinely parametrized line through the origin in $T_{x_0}M$ is a geodesic. Let $\{e_i\}$ denote the local orthonormal frame of TM obtained from $\{\partial_i\}$ by parallel transport along $\mathcal{R}$, i.e. $\nabla^g_{\mathcal{R}}e_i=0$ and $e_i(x_0)=\partial_i$. Let $\{e^i\}$ denote the dual local coframe.

Trivialize $\mathcal{E}$ with the help of radial parallel transport by ∇. Use this trivialization to identify $\Gamma(\mathcal{E}|_U)$ with $C^\infty(U,\mathcal{E}_0)$, where $\mathcal{E}_0:=\mathcal{E}_{x_0}$. Let $\omega\in\Omega^1(U;\operatorname{end}(\mathcal{E}_0))$ denote the connection one form of this trivialization, i.e. $\nabla_{\partial_i}=\partial_i+\omega(\partial_i)$. For the curvature F of ∇ we then have $F = d\omega + \omega\wedge\omega \in \Omega^2(M;\mathrm{end}^+(\mathcal{E}_0))$. By the choice of trivialization of $\mathcal{E}|_U$ we have $i_{\mathcal{R}}\omega = 0$ and thus $i_{\mathcal{R}}F = i_{\mathcal{R}}(d\omega+\omega\wedge\omega) = i_{\mathcal{R}}d\omega$. Contracting this with ∂_i and using $[\partial_i,\mathcal{R}] = \partial_i$ we obtain

$$-F(\partial_i,\mathcal{R}) = F(\mathcal{R},\partial_i) = (d\omega)(\mathcal{R},\partial_i) = (L_{\mathcal{R}}+1)(\omega(\partial_i)) \tag{65}$$

where $L_{\mathcal{R}}$ denotes Lie derivative with respect to the vector field $\mathcal{R}$. Let $\omega(\partial_i) \sim \sum_\alpha \frac{\partial_\alpha\omega(\partial_i)_{x_0}}{\alpha!}x^\alpha$ denote the Taylor expansion of $\omega(\partial_i)$ at x_0, written with the help of multi index notation. Using $L_{\mathcal{R}}x^\alpha = |\alpha|x^\alpha$ we obtain the following Taylor expansion $(L_{\mathcal{R}}+1)(\omega(\partial_i)) \sim \sum_\alpha\bigl(|\alpha|+1\bigr)\frac{\partial_\alpha\omega(\partial_i)_{x_0}}{\alpha!}x^\alpha$. If $F(\partial_i,\partial_j) \sim \sum_\alpha \frac{\partial_\alpha F(\partial_i,\partial_j)_{x_0}}{\alpha!}x^\alpha$ denotes the Taylor expansion of $F(\partial_i,\partial_j)$ at x_0 then we obtain the Taylor expansion $F(\partial_i,\mathcal{R}) \sim \sum_{j,\alpha}\frac{\partial_\alpha F(\partial_i,\partial_j)_{x_0}}{\alpha!}x^jx^\alpha$. Comparing the Taylor expansions of both sides of (65) we obtain the Taylor expansion, cf. Proposition 1.18 in [1],

$$\nabla_{\partial_i} - \partial_i = \omega(\partial_i) \sim -\sum_{j,\alpha}\frac{\partial_\alpha F(\partial_i,\partial_j)_{x_0}}{(|\alpha|+2)\alpha!}x^jx^\alpha.$$

For the first few terms this gives:

$$\nabla_{\partial_i} = \partial_i - \tfrac12\sum_j F(\partial_i,\partial_j)_{x_0}x^j - \tfrac13\sum_{j,k}\partial_k F(\partial_i,\partial_j)_{x_0}x^jx^k + O(|\mathbf{x}|^3) \tag{66}$$

Let $c^i := c(e^i) \in \Gamma(\mathcal{E}|_U) = C^\infty(U,\mathrm{end}(\mathcal{E}_0))$ denote Clifford multiplication with e^i. Since $\nabla^g_{\mathcal{R}}e^i = 0$ and since ∇ is a Clifford connection we have $\nabla_{\mathcal{R}}c^i = c(\nabla^g_{\mathcal{R}}e^i) = 0$. So we see that c^i is actually a constant in $\mathrm{end}(\mathcal{E}_0)$, cf. [1, Lemma 4.14]. Particularly, our trivialization of $\mathcal{E}|_U$ identifies $\Gamma(\mathrm{end}_{\mathrm{Cl}}(\mathcal{E}|_U))$ with $C^\infty(U,\mathrm{end}_{\mathrm{Cl}}(\mathcal{E}_0))$. Recall that

$$F(\partial_i,\partial_j) = \tfrac14\sum_{l,m} g(R_{\partial_i,\partial_j}e_l,e_m)c^lc^m + F^{\mathcal{E}/S}_\nabla(\partial_i,\partial_j)$$

with $F^{\mathcal{E}/S}_\nabla \in \Omega^2(U;\mathrm{end}^+_{\mathrm{Cl}}(\mathcal{E}_0))$. From (66) we thus obtain, cf. [1, Lemma 4.15],

$$\begin{aligned}\nabla_{\partial_i} = \partial_i &- \tfrac18\sum_{j,l,m} g(R_{\partial_i,\partial_j}e_l,e_m)_{x_0}x^jc^lc^m\\ &- \tfrac1{12}\sum_{j,k,l,m}\partial_k g(R_{\partial_i,\partial_j}e_l,e_m)_{x_0}x^jx^kc^lc^m\\ &+ \sum_{l,m}u_{ilm}(\mathbf{x})c^lc^m + v_i(\mathbf{x})\end{aligned} \tag{67}$$

with $u_{ilm}(\mathbf{x}) = O(|\mathbf{x}|^3) \in C^\infty(U)$ and $v_i(\mathbf{x}) = O(|\mathbf{x}|) \in C^\infty(U,\mathrm{end}_{\mathrm{Cl}}(\mathcal{E}_0))$.

Let Δ denote the connection Laplacian given by the composition

$$\Gamma(\mathcal{E}) \xrightarrow{\nabla} \Gamma(T^*M\otimes\mathcal{E}) \xrightarrow{\nabla^g\otimes 1+1\otimes\nabla} \Gamma(T^*M\otimes T^*M\otimes\mathcal{E}) \xrightarrow{-\mathrm{tr}_g} \Gamma(\mathcal{E})$$

Let r denote the scalar curvature of g and recall Lichnerowicz' formula [1, Theorem 3.52]

$$D^2_\nabla = \Delta + c(F^{\mathcal{E}/S}_\nabla) + \tfrac{r}{4}.$$

Recall our Clifford super connection $\mathbb{A} = \nabla + A$ with $A \in \Omega^0(M; \mathrm{end}^-_{\mathrm{Cl}}(\mathcal{E}))$. Since $D^2_{\mathbb{A}} = D^2_\nabla + c(\nabla A) + A^2$ we obtain

$$D^2_{\mathbb{A}} = \Delta + c\big(F^{\mathcal{E}/S}_\nabla + \nabla A\big) + A^2 + \tfrac{r}{4}. \tag{68}$$

Use the symbol map to identify $\mathrm{end}(\mathcal{E}_0) = \Lambda^* T^*_{x_0} M \otimes \mathrm{end}_{\mathrm{Cl}}(\mathcal{E}_0)$. For $0 < u \leq 1$ and $\alpha \in C^\infty\big(\mathbb{R}^+ \times U, \Lambda^* T^*_{x_0} M \otimes \mathrm{end}_{\mathrm{Cl}}(\mathcal{E}_0)\big)$ define Getzler's rescaling

$$(\delta_u \alpha)(t, \mathbf{x}) := \sum_i u^{-i/2} \alpha(ut, u^{1/2}\mathbf{x})_{[i]}.$$

Consider the kernel $p \in C^\infty\big(\mathbb{R}^+ \times U, \Lambda^* T^*_{x_0} M \otimes \mathrm{end}_{\mathrm{Cl}}(\mathcal{E}_0)\big)$ of $e^{-tD^2_{\mathbb{A}}}$, $p(t, \mathbf{x}) = p_t(\mathbf{x}, x_0)$. Note that $p(t, 0) = k_t(x_0)$. Define the rescaled kernel $r_u := u^{n/2}\delta_u p$ and the rescaled operator $L_u := u\delta_u D^2_{\mathbb{A}} \delta_u^{-1}$. Since $(\partial_t + D^2_{\mathbb{A}})p = 0$ and $\delta_u \partial_t \delta_u^{-1} = u^{-1}\partial_t$ we have

$$(\partial_t + L_u) r_u = 0. \tag{69}$$

Note that setting $t = 1$ and $x = 0$ and using (61) we get an asymptotic expansion

$$r_u(1, 0) \sim (4\pi)^{-n/2} \sum_{j \geq -n} u^{j/2} \sum_{i \geq 0} \sigma\big(\tilde{k}_i(x_0)\big)_{[2i-j]} \qquad \text{as } u \to 0. \tag{70}$$

The claim (62) just states that the terms for $-n \leq j < 0$ vanish, i.e. there are no Laurent terms in (70). Statements (63) and (64) are explicit expressions for the term $j = 0$ and $j = 1$ in (70).

Let us compute the first terms in the asymptotic expansion of L_u in powers of $u^{1/2}$. Let us write ε^j for the exterior multiplication with e^j, and ι^j for the contraction with e_j. Note that

$$\delta_u \varepsilon^j \delta_u^{-1} = u^{-1/2}\varepsilon^j, \qquad \delta_u \iota^j \delta_u^{-1} = u^{1/2}\iota^j, \qquad \delta_u \partial_i \delta_u^{-1} = u^{-1/2}\partial_i$$

and recall that $c^j = \varepsilon^j - \iota^j$. Let us look at the simplest part first. Clearly,

$$u\delta_u(A^2 + \tfrac{r}{4})\delta_u^{-1} = O(u) \qquad \text{as } u \to 0. \tag{71}$$

Next we have $\nabla A = \sum_i (\nabla_{e_i} A) e^i$, hence $c(\nabla A) = \sum_i (\nabla_{e_i} A) c^i$ and therefore

$$u\delta_u c(\nabla A)\delta_u^{-1} = u^{1/2}\mathsf{A}' + O(u^{3/2}) \qquad \text{as } u \to 0, \tag{72}$$

where $\mathsf{A}' := \sum_i (\nabla_{\partial_i} A)_{x_0} \varepsilon^i$. Moreover, $F^{\mathcal{E}/S}_\nabla = \frac{1}{2}\sum_{i,j} F^{\mathcal{E}/S}_\nabla(e_i, e_j) e^i \wedge e^j$, hence $c(F^{\mathcal{E}/S}_\nabla) = \frac{1}{2}\sum_{i,j} F^{\mathcal{E}/S}_\nabla(e_i, e_j) c^i c^j$, and thus

$$u\delta_u c(F^{\mathcal{E}/S}_\nabla)\delta_u^{-1} = \mathsf{F} + O(u) \qquad \text{as } u \to 0, \tag{73}$$

where $\mathsf{F} := \frac{1}{2}\sum_{i,j} F^{\mathcal{E}/S}_\nabla(\partial_i, \partial_j)_{x_0} \varepsilon^i \varepsilon^j$. From (67) we easily get

$$u^{1/2}\delta_u \nabla_{\partial_i} \delta_u^{-1} = \partial_i - \tfrac{1}{4}\sum_j \mathsf{R}_{ij} x^j + u^{1/2} R'_i + O(u) \qquad \text{as } u \to 0,$$

where $\mathsf{R}_{ij} := \frac{1}{2}\sum_{l,m} g(R_{\partial_i,\partial_j} e_l, e_m)_{x_0} \varepsilon^l \varepsilon^m$ and R'_i is an operator which acts on $C^\infty\big(U, \Lambda^{\mathrm{even/odd}} T^*_{x_0} M \otimes \mathrm{end}_{\mathrm{Cl}}(\mathcal{E}_0)\big)$ in a way which preserves the parity of the form degree. Using the formula $\Delta = -\sum_i \big((\nabla_{e_i})^2 - \nabla_{\nabla^g_{e_i} e_i}\big)$ and the fact that $\nabla^g_{e_i} e_i$ vanishes at x_0 we obtain

$$u\delta_u \Delta \delta_u^{-1} = -\sum_i \Big(\partial_i - \tfrac{1}{4}\sum_j \mathsf{R}_{ij} x^j\Big)^2 + u^{1/2} K^{\mathrm{even}} + O(u) \qquad \text{as } u \to 0, \tag{74}$$

where K^{even} acts on $C^\infty\big(U, \Lambda^{\mathrm{even/odd}} T^*_{x_0} M \otimes \mathrm{end}_{\mathrm{Cl}}(\mathcal{E}_0)\big)$ in a parity preserving way. Let us write

$$\mathsf{K} := -\sum_i \Big(\partial_i - \tfrac{1}{4}\sum_j \mathsf{R}_{ij} x^j\Big)^2 + \mathsf{F}.$$

Then (71), (72), (73), (74) together with (68) finally give

$$L_u = \mathsf{K} + u^{1/2}\big(\mathsf{A}' + K^{\mathrm{even}}\big) + O(u) \qquad \text{as } u \to 0. \tag{75}$$

Recall, see [1, Lemma 4.19], that there exist $\Lambda^* T^* M \otimes \mathrm{end}_{\mathrm{Cl}}(\mathcal{E}_0)$ valued polynomials $\tilde{r}_i$ on $\mathbb{R} \times U$ so that we have an asymptotic expansion

$$r_u(t, \mathbf{x}) \sim q_t(\mathbf{x}) \sum_{j \geq -n} u^{j/2} \tilde{r}_j(t, \mathbf{x}) \qquad \text{as } u \to 0, \tag{76}$$

where $q_t(\mathbf{x}) = (4\pi t)^{-n/2} e^{-|\mathbf{x}|^2/4t}$. Moreover, the initial condition for the heat kernel translates to

$$\tilde{r}_j(0,0) = \delta_{j,0}. \tag{77}$$

Setting $t = 1$, $\mathbf{x} = 0$ in (76) we get

$$r_u(1,0) \sim (4\pi)^{-n/2} \sum_{j \geq -n} u^{j/2} \tilde{r}_j(1,0) \qquad \text{as } u \to 0. \tag{78}$$

Comparing this with (70) we obtain

$$\tilde{r}_j(1,0) = \sum_{i \geq 0} \sigma(\tilde{k}_i)_{[2i-j]}(x_0). \tag{79}$$

Expanding the equation $(\partial_t + L_u) r_u = 0$ in a power series in $u^{1/2}$ with the help of (76) and (75) the leading term $q\tilde{r}_l$ satisfies $(\partial_t + \mathsf{K})(q\tilde{r}_l) = 0$. Because of the initial condition (77) and the uniqueness of formal solutions [1, Theorem 4.13] we must have $l \geq 0$ and thus $\tilde{r}_j = 0$ for $j < 0$. In view of (79) this proves (62).

So $q\tilde{r}_0$ satisfies $(\partial_t + \mathsf{K})(q\tilde{r}_0) = 0$ with initial condition $\tilde{r}_0(0,0) = 1$, see (77). Mehler's formula [1, Theorem 4.13] provides an explicit solution:

$$\begin{aligned} &q_t(\mathbf{x})\tilde{r}_0(t,\mathbf{x}) \\ &= (4\pi t)^{-n/2} \det{}^{1/2}\Big(\frac{t\mathsf{R}/2}{\sinh(t\mathsf{R}/2)}\Big) \wedge \exp\Big(-\tfrac{1}{4t}\big\langle \mathbf{x} \,\big|\, \tfrac{t\mathsf{R}}{2} \coth(\tfrac{t\mathsf{R}}{2}) \,\big|\, \mathbf{x}\big\rangle\Big) \wedge \exp(-t\mathsf{F}) \end{aligned}$$

Setting $t = 1$, $\mathbf{x} = 0$ we obtain

$$\tilde{r}_0(1,0) = \det{}^{1/2}\Big(\frac{\mathsf{R}/2}{\sinh(\mathsf{R}/2)}\Big) \wedge \exp(-\mathsf{F}). \tag{80}$$

In view of (79) we thus have established (63).

The term $q\tilde r_1$ satisfies $(\partial_t+\mathsf{K})(q\tilde r_1)=-(\mathsf{A}'+K^{\text{even}})(\tilde q r_0)$. Let us write

$$\tilde r_1(t,\mathbf{x})=\tilde r_1^{\text{even}}(t,\mathbf{x})+\tilde r_1^{\text{odd}}(t,\mathbf{x})$$

with $\tilde r_1^{\text{even}}(t,\mathbf{x})\in\Lambda^{\text{even}}T^*_{x_0}M\otimes\text{end}_{\text{Cl}}(\mathcal{E}_0)$ and $\tilde r_1^{\text{odd}}(t,\mathbf{x})\in\Lambda^{\text{odd}}T^*_{x_0}M\otimes\text{end}_{\text{Cl}}(\mathcal{E}_0)$. Note that in view of (79) we have

$$\tilde r_1(1,0)=\sum_{i\geq 0}\sigma(\tilde k_i)_{[2i-1]}(x_0)=\tilde r_1^{\text{odd}}(1,0). \tag{81}$$

It thus suffices to determine $\tilde r_1^{\text{odd}}$. Since

$$\begin{aligned}(K^{\text{even}}(q\tilde r_0))(t,\mathbf{x})&\in\Lambda^{\text{even}}T^*_{x_0}M\otimes\text{end}_{\text{Cl}}(\mathcal{E}_0)\\ (\mathsf{A}'(q\tilde r_0))(t,\mathbf{x})&\in\Lambda^{\text{odd}}T^*_{x_0}M\otimes\text{end}_{\text{Cl}}(\mathcal{E}_0)\end{aligned}$$

we must have $(\partial_t+\mathsf{K})(q\tilde r_1^{\text{odd}})=-\mathsf{A}'(q\tilde r_0)$. We make the following ansatz, we suppose that $\tilde r_1^{\text{odd}}=B\tilde r_0$ with $B\in C^\infty(\mathbb{R},\Lambda^{\text{odd}}T^*_{x_0}M\otimes\text{end}_{\text{Cl}}(\mathcal{E}_0))$. Then

$$\begin{aligned}(\partial_t+\mathsf{K})(q\tilde r_1^{\text{odd}})&=(\partial_tB)q\tilde r_0+B\partial_t(q\tilde r_0)+\mathsf{K}(Bq\tilde r_0)\\ &=(\partial_tB)q\tilde r_0-B\mathsf{K}(q\tilde r_0)+\mathsf{K}(Bq\tilde r_0)\\ &=(\partial_tB)q\tilde r_0-B\mathsf{F}q\tilde r_0+\mathsf{F}Bq\tilde r_0\\ &=\big(\partial_tB+\text{ad}(\mathsf{F})B\big)q\tilde r_0\end{aligned}$$

Hence we have to solve $\partial_tB=\text{ad}(-\mathsf{F})B-\mathsf{A}'$ with initial condition $B(0)=0$. This is easily carried out and we find the solution:

$$B(t)=-\frac{e^{\text{ad}(-t\mathsf{F})}-1}{\text{ad}(-\mathsf{F})}\mathsf{A}'$$

Thus $qB\tilde r_0$ satisfies $(\partial_t+\mathsf{K})(qB\tilde r_0)=-\mathsf{A}'(q\tilde r_0)$ with initial condition $(B\tilde r_0)(0,0)=0$. Again, the uniqueness of formal solutions of the heat equation guarantees that we actually have $\tilde r_1^{\text{odd}}=B\tilde r_0$. Setting $t=1$, $\mathbf{x}=0$ and using (80) we get

$$\tilde r_1^{\text{odd}}(1,0)=B(1)\tilde r_0(1,0)=-\det{}^{1/2}\left(\frac{\mathsf{R}/2}{\sinh(\mathsf{R}/2)}\right)\wedge\left(\frac{e^{\text{ad}(-\mathsf{F})}-1}{\text{ad}(-\mathsf{F})}\mathsf{A}'\right)\wedge\exp(-\mathsf{F}).$$

Using (81) we conclude

$$\sum_{i\geq 0}\sigma(\tilde k_i)_{[2i-1]}=-\hat A_g\wedge\left(\frac{e^{\text{ad}(-F_\nabla^{\mathcal{E}/S})}-1}{\text{ad}(-F_\nabla^{\mathcal{E}/S})}\nabla A\right)\wedge\exp\big(-F_\nabla^{\mathcal{E}/S}\big).$$

The Bianchi identity $\nabla F_\nabla^{\mathcal{E}/S}=0$ implies $\nabla\exp(-F_\nabla^{\mathcal{E}/S})=0$, $\nabla\,\text{ad}(-F_\nabla^{\mathcal{E}/S})=0$, and similarly $d\hat A_g=0$, from which we finally obtain (64). □

Certain heat traces

Since $\mathcal{E}$ is $\mathbb{Z}_2$-graded we have a *super trace* $\operatorname{str}_{\mathcal{E}} : \Gamma(\operatorname{end}(\mathcal{E})) \to \Omega^0(M;\mathbb{C})$. If n is even we will also make use of the so called *relative super trace,* see [1, Definition 3.28], $\operatorname{str}_{\mathcal{E}/S} : \Gamma(\operatorname{end}_{\mathrm{Cl}}(\mathcal{E})) \to \Omega^0(M;\mathcal{O}_M^{\mathbb{C}})$

$$\operatorname{str}_{\mathcal{E}/S}(b) := 2^{-n/2}\operatorname{str}_{\mathcal{E}}(c(\Gamma)b).$$

Here $\Gamma \in \Gamma(\mathrm{Cl}\otimes\mathcal{O}_M^{\mathbb{C}})$ denotes the chirality element, see [1, Lemma 3.17]. With respect to a local orthonormal frame $\{e_i\}$ of TM and its dual local coframe $\{e^i\}$ the chirality element Γ is given as $\mathbf{i}^{n/2}e^1\cdots e^n$ times the orientation of $(e_1,\dots,e_n)$. This relative super trace gives rise to

$$\operatorname{str}_{\mathcal{E}/S} : \Omega^*(M;\operatorname{end}_{\mathrm{Cl}}(\mathcal{E})) \to \Omega^*(M;\mathcal{O}_M^{\mathbb{C}})$$

which will be denoted by the same symbol. For every $\phi \in \Gamma(\operatorname{end}(\mathcal{E}))$ we have

$$(\operatorname{str}_{\mathcal{E}}(\phi))\cdot\Omega_g = (\mathbf{i}/2)^{-n/2}\operatorname{str}_{\mathcal{E}/S}\bigl(\sigma(\phi)_{[n]}\bigr), \tag{82}$$

where $\Omega_g \in \Omega^n(M;\mathcal{O}_M^{\mathbb{C}})$ denotes the volume density associated with g. To see (82) note first that

$$\mathrm{Cl}_{n-1} = [\mathrm{Cl},\mathrm{Cl}] \tag{83}$$

where Cl_k denotes the filtration on Cl, see [1, Proof of Proposition 3.21]. Hence both sides of (82) vanish on $\Gamma(\mathrm{Cl}_{n-1}\otimes\operatorname{end}_{\mathrm{Cl}}(\mathcal{E}))$. It remains to check (82) on sections of $\mathrm{Cl}/\mathrm{Cl}_{n-1}\otimes\operatorname{end}_{\mathrm{Cl}}(\mathcal{E})$, but for these the desired equality follows immediately from the definition of the relative super trace.

Lemma 7.3. *Let $D_{\mathbb{A}}$ be a Dirac operator and $\tilde{k}_i \in \Gamma(\operatorname{end}(\mathcal{E}))$ as in Theorem 7.1. Moreover, let $\Phi \in \Gamma(\operatorname{end}(\mathcal{E}))$. Then, for even n, we have*

$$\operatorname*{LIM}_{t\to 0}\operatorname{str}\bigl(\Phi e^{-tD_{\mathbb{A}}^2}\bigr) = (2\pi\mathbf{i})^{-n/2}\int_M \operatorname{str}_{\mathcal{E}/S}\bigl(\sigma(\Phi\tilde{k}_{n/2})_{[n]}\bigr),$$

whereas $\mathrm{LIM}_{t\to 0}\operatorname{str}\bigl(\Phi e^{-tD_{\mathbb{A}}^2}\bigr) = 0$ *if n is odd. Here* LIM *denotes the renormalized limit* [1, Section 9.6] *which in this case is just the constant term in the asymptotic expansion for $t \to 0$.*

Proof. For odd n this follows immediately from (61). So assume n is even. Recall from [1, Proposition 2.32] that

$$\operatorname{str}\bigl(\Phi e^{-tD_{\mathbb{A}}^2}\bigr) = \int_M \operatorname{str}_{\mathcal{E}}(\Phi k_t)\cdot\Omega_g. \tag{84}$$

Combining this with (82) we obtain

$$\operatorname{str}\bigl(\Phi e^{-tD_{\mathbb{A}}^2}\bigr) = (\mathbf{i}/2)^{-n/2}\int_M \operatorname{str}_{\mathcal{E}/S}\bigl(\sigma(\Phi k_t)_{[n]}\bigr)$$

We thus get an asymptotic expansion, see (61),

$$\operatorname{str}\bigl(\Phi e^{-tD_{\mathbb{A}}^2}\bigr) \sim (2\pi\mathbf{i}t)^{-n/2}\sum_{i\geq 0} t^i\int_M \operatorname{str}_{\mathcal{E}/S}\bigl(\sigma(\Phi\tilde{k}_i)_{[n]}\bigr) \qquad \text{as } t\to 0,$$

from which the desired formula follows at once. □

Corollary 7.4. *Let $D_{\mathbb{A}}$ be a Dirac operator as in Theorem 7.1. Moreover, let $U \in \Gamma(\mathrm{end}_{\mathrm{Cl}}(\mathcal{E}))$. Then, for even n, we have*

$$\operatorname*{LIM}_{t\to0}\mathrm{str}(Ue^{-tD^2_{\mathbb{A}}}) = (2\pi\mathbf{i})^{-n/2}\int_M \hat{A}_g\wedge\mathrm{str}_{\mathcal{E}/S}\big(U\exp(-F^{\mathcal{E}/S}_\nabla)\big), \tag{85}$$

whereas $\mathrm{LIM}_{t\to0}\mathrm{str}(Ue^{-tD^2_{\mathbb{A}}}) = 0$ *if n is odd.*

Proof. For odd n this follows immediately from Lemma 7.3. So assume n is even. Since $\sigma(U\tilde{k}_i)_{[n]} = U\sigma(\tilde{k}_i)_{[n]}$ Theorem 7.1 yields

$$\mathrm{str}_{\mathcal{E}/S}\big(\sigma(U\tilde{k}_{n/2})_{[n]}\big) = \Big(\hat{A}_g\wedge\mathrm{str}_{\mathcal{E}/S}\big(U\exp(-F^{\mathcal{E}/S}_\nabla)\big)\Big)_{[n]}.$$

Equation (85) then follows from Lemma 7.3. □

Corollary 7.5. *Let $D_{\mathbb{A}}$ be a Dirac operator as in Theorem 7.1. Moreover, suppose $V\in\Omega^1(M;\mathrm{end}_{\mathrm{Cl}}(\mathcal{E}))$, let $c(V)\in\Gamma(\mathrm{end}(\mathcal{E}))$ denote Clifford multiplication with V, and consider $\nabla V\in\Omega^2(M;\mathrm{end}_{\mathrm{Cl}}(\mathcal{E}))$. Then, for even n, we have*

$$\begin{aligned}&\operatorname*{LIM}_{t\to0}\mathrm{str}\Big(c(V)e^{-tD^2_{\mathbb{A}}}\Big) =\\ &-(2\pi\mathbf{i})^{-n/2}\int_M\hat{A}_g\wedge\mathrm{str}_{\mathcal{E}/S}\bigg(\bigg(\frac{e^{\mathrm{ad}(-F^{\mathcal{E}/S}_\nabla)}-1}{\mathrm{ad}(-F^{\mathcal{E}/S}_\nabla)}A\bigg)\wedge\exp\big(-F^{\mathcal{E}/S}_\nabla\big)\wedge\nabla V\bigg),\end{aligned}\tag{86}$$

whereas $\mathrm{LIM}_{t\to0}\mathrm{str}\Big(c(V)e^{-tD^2_{\mathbb{A}}}\Big) = 0$ *if n is odd.*

Proof. If n is odd the statement follows immediately from Lemma 7.3. So assume n is even. Since $\sigma\big(c(V)\tilde{k}_i\big)_{[n]} = V\wedge\sigma(\tilde{k}_i)_{[n-1]}$ Theorem 7.1 yields

$$\begin{aligned}&\mathrm{str}_{\mathcal{E}/S}\Big(\sigma\big(c(V)\tilde{k}_{n/2}\big)_{[n]}\Big)\\ &\qquad= -\,\mathrm{str}_{\mathcal{E}/S}\bigg(V\wedge\nabla\bigg(\hat{A}_g\wedge\bigg(\frac{e^{\mathrm{ad}(-F^{\mathcal{E}/S}_\nabla)}-1}{\mathrm{ad}(-F^{\mathcal{E}/S}_\nabla)}A\bigg)\wedge\exp\big(-F^{\mathcal{E}/S}_\nabla\big)\bigg)\bigg)_{[n]}.\\ &\qquad= -\,\mathrm{str}_{\mathcal{E}/S}\bigg(\nabla\bigg(\hat{A}_g\wedge\bigg(\frac{e^{\mathrm{ad}(-F^{\mathcal{E}/S}_\nabla)}-1}{\mathrm{ad}(-F^{\mathcal{E}/S}_\nabla)}A\bigg)\wedge\exp\big(-F^{\mathcal{E}/S}_\nabla\big)\bigg)\wedge V\bigg)_{[n]}.\end{aligned}$$

Applying Lemma 7.3 and using Stokes' theorem we obtain (86). □

8. Application to Laplacians

Below we will see that the Laplacians $\Delta_{E,g,b}$ introduced in Section 4 are the squares of Dirac operators of the kind considered in Section 7. Applying Corollaries 7.4 and 7.5 will lead to a proofs of Propositions 6.1 and 6.2, respectively.

The exterior algebra as Clifford module

Let (M, g) be a Riemannian manifold of dimension n. In order to understand the Clifford module structure of $\Lambda := \Lambda^* T^* M$ we first note that Λ is a Clifford module for $\hat{\mathrm{Cl}} := \mathrm{Cl}(T^*M, -g)$ too. Let us write $\hat{c}$ for the Clifford multiplication of $\hat{\mathrm{Cl}}$ on Λ. Explicitly, for $a \in T^*_x M \subseteq \hat{\mathrm{Cl}}$ and $\alpha \in \Lambda^* T^*_x M$ we have $\hat{c}(a)\alpha = a \wedge \alpha + i_{\sharp a}\alpha$, where $\sharp a := g^{-1} a \in T_x M$ and $i_{\sharp a}$ denotes contraction with $\sharp a$. It follows from this formula that every $\hat{c}(a)$ commutes with the Clifford action of Cl. We thus obtain an isomorphism of $\mathbb{Z}_2$-graded filtered algebras

$$\hat{c} : \hat{\mathrm{Cl}} \to \mathrm{end}_{\mathrm{Cl}}(\Lambda).$$

Let us write

$$\hat{\sigma} : \hat{\mathrm{Cl}} \to \Lambda, \qquad \hat{\sigma}(a) := \hat{c}(a) \cdot 1$$

for the symbol map of $\hat{\mathrm{Cl}}$.

As in (59) define $R^{\hat{\mathrm{Cl}}} \in \Omega^2(M; \hat{\mathrm{Cl}})$ by

$$R^{\hat{\mathrm{Cl}}}(X, Y) := -\tfrac{1}{4} \sum_{i,j} g\big(R(X,Y)e_i, e_j\big)\hat{c}^i \hat{c}^j$$

where X and Y are two vector fields, $\{e_i\}$ is a local orthonormal frame, $\{e^i\}$ denotes its dual local coframe, and $\hat{c}^i := \hat{c}(e^i)$. For the twisting curvature $F^{\Lambda/S}_{\nabla^g} \in \Omega^2(M; \mathrm{end}_{\mathrm{Cl}}(\Lambda))$ we then have, see [1, Page 145],

$$F^{\Lambda/S}_{\nabla^g} = (1 \otimes \hat{c})(R^{\hat{\mathrm{Cl}}}) \in \Omega^2(M; \mathrm{end}_{\mathrm{Cl}}(\Lambda)), \tag{87}$$

where $(1 \otimes \hat{c}) : \Omega(M; \hat{\mathrm{Cl}}) \to \Omega(M; \mathrm{end}_{\mathrm{Cl}}(\Lambda))$. Indeed, the curvature of Λ, $R^\Lambda \in \Omega^2(M; \mathrm{end}(\Lambda))$, can be written as

$$R^\Lambda(X, Y) = \sum_{i,j} g\big(R(X,Y)e_i, e_j\big) \tfrac{1}{2}(\varepsilon^j \iota^i - \varepsilon^i \iota^j) \in \Gamma(\mathrm{end}(\Lambda))$$

where $\varepsilon^j \in \Gamma(\mathrm{end}(\Lambda))$ denotes exterior multiplication with e^j, and $\iota^i \in \Gamma(\mathrm{end}(\Lambda))$ denotes contraction with e_i. Using $\varepsilon^i = \frac{1}{2}(c^i + \hat{c}^i)$ and $\iota^i = -\frac{1}{2}(c^i - \hat{c}^i)$ one easily deduces

$$\tfrac{1}{2}(\varepsilon^j \iota^i - \varepsilon^i \iota^j) = \tfrac{1}{4}\big(\tfrac{1}{2}(c^i c^j - c^j c^i)\big) - \tfrac{1}{4}\big(\tfrac{1}{2}(\hat{c}^i \hat{c}^j - \hat{c}^j \hat{c}^i)\big)$$

from which we read off (87), see (58). Also note that we have

$$(1 \otimes \hat{\sigma})(R^{\hat{\mathrm{Cl}}}) = -\tfrac{1}{2} R \in \Omega^2(M; \Lambda^2 T^* M), \tag{88}$$

where $(1 \otimes \hat{\sigma}) : \Omega(M; \hat{\mathrm{Cl}}) \to \Omega(M; \Lambda)$.

If n is even then the relative super trace

$$\mathrm{str}_{\Lambda/S} : \mathrm{end}_{\mathrm{Cl}}(\Lambda) \to \mathcal{O}^{\mathbb{C}}_M$$

is given by

$$\mathrm{str}_{\Lambda/S}(\hat{c}(a)) = (\mathbf{i}/2)^{-n/2} T(\hat{\sigma}(a)) \qquad a \in \hat{\mathrm{Cl}}, \tag{89}$$

where $T:\Lambda\to\mathcal{O}_M^{\mathbb{C}}$ denotes the Berezin integration associated with g. Indeed, since $[\hat{\mathrm{Cl}},\hat{\mathrm{Cl}}]=\hat{\mathrm{Cl}}_{n-1}$, see (83), both sides of (89) vanish for $a\in\hat{\mathrm{Cl}}_{n-1}$. Checking (89) on $\hat{\mathrm{Cl}}/\hat{\mathrm{Cl}}_{n-1}$ is straight forward. We will also make use of the formula

$$\mathrm{str}_{\Lambda/S}\big(\exp((1\otimes\hat{c})a)\big)=(\mathbf{i}/2)^{-n/2}T\big(\exp_\Lambda((1\otimes\hat{\sigma})a)\big)\qquad a\in\Omega^2(M;\hat{\mathrm{Cl}}_2),\quad (90)$$

where $1\otimes\hat{c}:\Omega(M;\hat{\mathrm{Cl}})\to\Omega(M;\mathrm{end}_{\mathrm{Cl}}(\Lambda))$, $1\otimes\hat{\sigma}:\Omega(M;\hat{\mathrm{Cl}})\to\Omega(M;\Lambda)$ and $T:\Omega(M;\Lambda)\to\Omega(M;\mathcal{O}_M^{\mathbb{C}})$ denotes Berezin integration. To check this equation note that the assumption on the form degree and the filtration degree of a implies:

$$\mathrm{str}_{\Lambda/S}\big(\exp((1\otimes\hat{c})a)\big)=\mathrm{str}_{\Lambda/S}\big(\tfrac{1}{n!}((1\otimes\hat{c})a)^{n/2}\big)$$
$$T\big(\exp_\Lambda((1\otimes\hat{\sigma})a)\big)=T\big(\tfrac{1}{n!}((1\otimes\hat{\sigma})a)^{n/2}\big)$$

Using the fact that $1\otimes\hat{c}$ is an algebra isomorphism and (89) we obtain

$$\begin{aligned}\mathrm{str}_{\Lambda/S}\big(((1\otimes\hat{c})a)^{n/2}\big)&=\mathrm{str}_{\Lambda/S}\big((1\otimes\hat{c})(a^{n/2})\big)\\&=(\mathbf{i}/2)^{-n/2}T\big((1\otimes\hat{\sigma})(a^{n/2})\big)=(\mathbf{i}/2)^{-n/2}T\big(((1\otimes\hat{\sigma})a)^{n/2}\big)\end{aligned}$$

where we made use of the fact that $1\otimes\hat{\sigma}$ induces an isomorphism on the level of associated graded algebras, for the last equality. Combined with the previous two equations this proves (90).

Lemma 8.1. *Let (M,g) be a Riemannian manifold of even dimension n. Then*[3]

$$\mathrm{e}(g)=(2\pi\mathbf{i})^{-n/2}\,\mathrm{str}_{\Lambda/S}\big(\exp\big(-F^{\Lambda/S}_{\nabla^g}\big)\big).$$

Proof. Consider the negative of the Riemannian curvature $-R\in\Omega^2(M;\Lambda^2T^*M)$ and its exponential $\exp_\Lambda(-R)\in\Omega(M;\Lambda)$. Recall that

$$\mathrm{e}(g):=(2\pi)^{-n/2}T(\exp_\Lambda(-R))\in\Omega^n(M;\mathcal{O}_M^{\mathbb{C}}).$$

Using (88), (90) and (87) we conclude:

$$\begin{aligned}\mathrm{e}(g)&=(2\pi)^{-n/2}T\Big(\exp_\Lambda\big((1\otimes\hat{\sigma})(2R^{\hat{\mathrm{Cl}}})\big)\Big)\\&=(-\pi)^{-n/2}T\Big(\exp_\Lambda\big((1\otimes\hat{\sigma})(-R^{\hat{\mathrm{Cl}}})\big)\Big)\\&=(2\pi\mathbf{i})^{-n/2}\,\mathrm{str}_{\Lambda/S}\Big(\exp\big(-(1\otimes\hat{c})(R^{\hat{\mathrm{Cl}}})\big)\Big)\\&=(2\pi\mathbf{i})^{-n/2}\,\mathrm{str}_{\Lambda/S}\big(\exp\big(-F^{\Lambda/S}_{\nabla^g}\big)\big)\end{aligned}$$

□

Lemma 8.2. *Let (M,g) be a Riemannian manifold of even dimension n. Suppose $\tilde{\xi}\in\Gamma(T^*M\otimes T^*M)$ is symmetric, use $1\otimes\hat{c}:T^*M\otimes T^*M\to T^*M\otimes\mathrm{end}_{\mathrm{Cl}}(\Lambda)$ to define $V:=\frac{1}{2}(1\otimes\hat{c})(\tilde{\xi})\in\Omega^1(M;\mathrm{end}_{\mathrm{Cl}}(\Lambda))$, and consider $\nabla^gV\in\Omega^2(M;\mathrm{end}_{\mathrm{Cl}}(\Lambda))$. Then, for every closed one form $\omega\in\Omega^1(M;\mathbb{C})$, we have*

$$\omega\wedge(\partial_2\,\mathrm{cs})(g,\tilde{\xi})=\tfrac{1}{2}(2\pi\mathbf{i})^{-n/2}\,\mathrm{str}_{\Lambda/S}\Big(\hat{c}(\omega)\wedge\exp\big(-F^{\Lambda/S}_{\nabla^g}\big)\wedge\nabla^gV\Big)$$

[3]Since the degree 0 part of $\hat{A}_g$ is 1, this formula is easily seen to be equivalent to $\mathrm{e}(g)=(2\pi\mathbf{i})^{-n/2}\hat{A}_g\wedge\mathrm{str}_{\Lambda/S}\big(\exp\big(-F^{\Lambda/S}_{\nabla^g}\big)\big)$ which can be found in [1, Proposition 4.6].

in $\Omega^n(M;\mathcal{O}_M^{\mathbb{C}})/d\Omega^{n-1}(M;\mathcal{O}_M^{\mathbb{C}})$.

Proof. Set $\tilde M := M\times\mathbb{R}$ and consider the two natural projections $p:\tilde M\to M$ and $t:\tilde M\to\mathbb{R}$. Consider the bundle $\tilde TM := p^*TM$ over $\tilde M$, and equip it with the fiber metric $\tilde g := p^*(g+t\tilde\xi)$. For sufficiently small t, this will indeed be non-degenerate. For $t\in\mathbb{R}$ let $\operatorname{inc}_t : M\to\tilde M$ denote the inclusion $x\mapsto(x,t)$. Define a connection $\tilde\nabla$ on $\tilde TM$ so that $\operatorname{inc}_t^*\tilde\nabla=\nabla^{g+t\tilde\xi}$ for sufficiently small t, where $\nabla^{g+t\tilde\xi}$ denotes the Levi–Civita connection of $g+t\tilde\xi$, and so that $\tilde\nabla_{\partial t}=\partial_t+\frac12\tilde g^{-1}(p^*\tilde\xi)$. It is not hard to check that $\tilde g$ is parallel with respect to $\tilde\nabla$, i.e. $\tilde\nabla\tilde g=0$. Let $\mathrm{e}(\tilde TM,\tilde g,\tilde\nabla)\in\Omega^n(\tilde M;\mathcal{O}_{\tilde TM})$ denote the Euler form of this Euclidean bundle. Recall that

$$\operatorname{cs}(g,g+\tau\tilde\xi)=\int_0^\tau \operatorname{inc}_t^* i_{\partial_t}\,\mathrm{e}(\tilde TM,\tilde g,\tilde\nabla)dt$$

and thus

$$\begin{aligned}(\partial_2\operatorname{cs})(g,\tilde\xi)&=\operatorname{inc}_0^* i_{\partial_t}\,\mathrm{e}(\tilde TM,\tilde g,\tilde\nabla)\\&=(2\pi)^{-n/2}\cdot\operatorname{inc}_0^* i_{\partial_t}T\big(\exp_\Lambda(-R^{\tilde\nabla})\big)\\&=(-2\pi)^{-n/2}\cdot T\big(\operatorname{inc}_0^* i_{\partial_t}\exp_\Lambda(R^{\tilde\nabla})\big)\\&=(-2\pi)^{-n/2}\cdot T\big(\operatorname{inc}_0^*\big(\exp_\Lambda(R^{\tilde\nabla})\wedge i_{\partial_t}R^{\tilde\nabla}\big)\big)\\&=(-2\pi)^{-n/2}\cdot T\big(\exp_\Lambda(\operatorname{inc}_0^* R^{\tilde\nabla})\wedge\operatorname{inc}_0^* i_{\partial_t}R^{\tilde\nabla}\big)\end{aligned}$$

where $R^{\tilde\nabla}\in\Omega^2(\tilde M;\Lambda^2\tilde TM)$ denotes the curvature of $\tilde\nabla$. Let

$$S:\Lambda\otimes\Lambda\to\Lambda\otimes\Lambda,\qquad S(\alpha\otimes\beta):=(-1)^{|\alpha||\beta|}\beta\otimes\alpha$$

denote the isomorphism of graded algebras obtained by interchanging variables. Consider $\tilde\xi\in\Omega^1(M;T^*M)$, $\nabla^g\tilde\xi\in\Omega^2(M;T^*M)$ and $S(\nabla^g\tilde\xi)\in\Omega^1(M;\Lambda^2T^*M)$. With this notation we have

$$\begin{aligned}\operatorname{inc}_0^* R^{\tilde\nabla}&=R\in\Omega^2(M;\Lambda^2T^*M)\\\operatorname{inc}_0^* i_{\partial_t}R^{\tilde\nabla}&=S(\tfrac12\nabla^g\tilde\xi)\in\Omega^1(M;\Lambda^2T^*M)\end{aligned}$$

where R denotes the Riemannian curvature of g. We obtain

$$(\partial_2\operatorname{cs})(g,\tilde\xi)=(-2\pi)^{-n/2}T\big(\exp_\Lambda(R)\wedge S(\tfrac12\nabla^g\tilde\xi)\big)$$

and wedging with ω we get

$$\omega\wedge(\partial_2\operatorname{cs})(g,\tilde\xi)=(-2\pi)^{-n/2}T\big(\exp_\Lambda(R)\wedge\omega\wedge S(\tfrac12\nabla^g\tilde\xi)\big).$$

Next, note that for $a\in\Lambda^nT^*M\otimes\Lambda^nT^*M$ we have $T(S(a))=T(a)$, for n is supposed to be even. Together with the symmetries of the Riemann curvature,

$S(R)=R$, we obtain

$$\begin{aligned}\omega\wedge(\partial_2\,\mathrm{cs})(g,\tilde\xi) &= (-2\pi)^{-n/2}T\Big(S\big(\exp_\Lambda(R)\wedge\omega\wedge S(\tfrac12\nabla^g\tilde\xi)\big)\Big)\\ &= (-2\pi)^{-n/2}T\Big(S(\exp_\Lambda(R))\wedge S(\omega)\wedge\tfrac12\nabla^g\tilde\xi\Big)\\ &= (-2\pi)^{-n/2}T\Big(\exp_\Lambda(R)\wedge(1\otimes\omega)\wedge\tfrac12\nabla^g\tilde\xi\Big)\\ &= (-2\pi)^{-n/2}\frac{\partial}{\partial s}\Big|_{s=0}T\Big(\exp_\Lambda\Big(R+s(1\otimes\omega)\wedge\tfrac12\nabla^g\tilde\xi\Big)\Big)\end{aligned}$$

In view of (88) we have:

$$R+s(1\otimes\omega)\wedge\tfrac12(\nabla^g\tilde\xi)=(1\otimes\hat\sigma)\Big(-2R^{\hat{\mathrm{Cl}}}+s(1\otimes\omega)\wedge\tfrac12\nabla^g\tilde\xi\Big)$$

Moreover, using (87) and $(1\otimes\hat c)(\frac12\nabla^g\tilde\xi)=\nabla^gV$ we also have:

$$(1\otimes\hat c)\Big(-2R^{\hat{\mathrm{Cl}}}+s(1\otimes\omega)\wedge\tfrac12\nabla^g\tilde\xi\Big)=-2F^{\Lambda/S}_{\nabla^g}+s\hat c(\omega)\wedge\nabla^gV$$

Using these two equations and applying (90) we obtain

$$\begin{aligned}\omega\wedge(\partial_2\,\mathrm{cs})(g,\tilde\xi) &= (4\pi\mathbf{i})^{-n/2}\frac{\partial}{\partial s}\Big|_{s=0}\mathrm{str}_{\Lambda/S}\Big(\exp\Big(-2F^{\Lambda/S}_{\nabla^g}+s\hat c(\omega)\wedge\nabla^gV\Big)\Big)\\ &= (4\pi\mathbf{i})^{-n/2}\,\mathrm{str}_{\Lambda/S}\Big(\exp\big(-2F^{\Lambda/S}_{\nabla^g}\big)\wedge\hat c(\omega)\wedge\nabla^gV\Big)\\ &= \tfrac12(2\pi\mathbf{i})^{-n/2}\,\mathrm{str}_{\Lambda/S}\Big(\exp\big(-F^{\Lambda/S}_{\nabla^g}\big)\wedge\hat c(\omega)\wedge\nabla^gV\Big)\\ &= \tfrac12(2\pi\mathbf{i})^{-n/2}\,\mathrm{str}_{\Lambda/S}\Big(\hat c(\omega)\wedge\exp\big(-F^{\Lambda/S}_{\nabla^g}\big)\wedge\nabla^gV\Big)\end{aligned}$$

□

The Laplacians as squares of Dirac operators

Let E be a flat complex vector bundle equipped with a fiber wise non-degenerate symmetric bilinear form b. Let ∇^E denote the flat connection on E. Consider $b^{-1}\nabla^Eb\in\Omega^1(M;\mathrm{end}(E))$ and introduce the connection, cf. [2, Section 4],

$$\nabla^{E,b}:=\nabla^E+\tfrac12 b^{-1}\nabla^Eb$$

on E. Consider the Clifford bundle $\mathcal{E}:=\Lambda\otimes E$ with Clifford connection

$$\nabla^{E,g,b}:=\nabla^g\otimes 1_E+1_\Lambda\otimes\nabla^{E,b}.$$

Since $(\nabla^{E,g,b})^2=(\nabla^g)^2+(\nabla^{E,b})^2$ the twisting curvature is

$$F^{\mathcal{E}/S}_{\nabla^{E,g,b}}=F^{\Lambda/S}_{\nabla^g}+(\nabla^{E,b})^2. \tag{91}$$

Since the two summands commute we obtain

$$\exp\big(-F^{\mathcal{E}/S}_{\nabla^{E,g,b}}\big)=\exp\big(-F^{\Lambda/S}_{\nabla^g}\big)\wedge\exp\big(-(\nabla^{E,b})^2\big). \tag{92}$$

An easy computation shows that the Dirac operator associated to the Clifford connection $\nabla^{E,g,b}$ is

$$D_{\nabla^{E,g,b}}=d_E+d^\sharp_{E,g,b}+\hat c\big(\tfrac12 b^{-1}\nabla^Eb\big). \tag{93}$$

Setting
$$A_{E,g,b} := -\hat{c}\big(\tfrac{1}{2}b^{-1}\nabla^E b\big) \in \Omega^0(M; \mathrm{end}^-_{\mathrm{Cl}}(\mathcal{E})) \tag{94}$$
we obtain a Clifford super connection
$$\mathbb{A}_{E,g,b} := \nabla^{E,g,b} + A_{E,g,b}. \tag{95}$$
For the associated Dirac operator $D_{\mathbb{A}_{E,g,b}} = d_E + d^\sharp_{E,g,b}$ we find
$$(D_{\mathbb{A}_{E,g,b}})^2 = (d_E + d^\sharp_{E,g,b})^2 = \Delta_{E,g,b}. \tag{96}$$
So we see that the Laplacians introduced in Section 4 are indeed squares of Dirac operators of the type considered in Theorem 7.1.

Proof of Proposition 6.1

For odd n the statement follows immediately from Lemma 7.3. So let us assume that n is even. We will apply Corollary 7.4 to the Clifford super connection (95) and $U := \phi$. From (92) and Lemma 8.1 we get:
$$\begin{aligned}\mathrm{str}_{\mathcal{E}/S}\Big(\phi\exp\big(-F^{\mathcal{E}/S}_{\nabla^{E,g,b}}\big)\Big) &= \mathrm{str}_{\mathcal{E}/S}\Big(\phi\exp\big(-F^{\Lambda/S}_{\nabla^g}\big)\wedge\exp\big(-(\nabla^{E,b})^2\big)\Big)\\ &= \mathrm{str}_{\Lambda/S}\Big(\exp\big(-F^{\Lambda/S}_{\nabla^g}\big)\Big)\wedge\mathrm{tr}_E\Big(\phi\exp\big(-(\nabla^{E,b})^2\big)\Big) = (2\pi\mathbf{i})^{n/2}\,\mathrm{e}(g)\,\mathrm{tr}(\phi)\end{aligned}$$
Here we also used the fact that the form $\mathrm{str}_{\Lambda/S}\big(\exp\big(-F^{\Lambda/S}_{\nabla^g}\big)\big) = (2\pi\mathbf{i})^{n/2}\,\mathrm{e}(g)$ has degree n, and thus the only contributing part of $\mathrm{tr}_E\big(\phi\exp\big(-(\nabla^{E,b})^2\big)\big)$ is the one of form degree 0, which is just $\mathrm{tr}(\phi)$. Using again the fact that $\mathrm{e}(g)$ has maximal form degree, we conclude
$$(2\pi\mathbf{i})^{-n/2}\hat{A}_g\wedge\mathrm{str}_{\mathcal{E}/S}\Big(\phi\exp\big(-F^{\mathcal{E}/S}_{\nabla^{E,g,b}}\big)\Big) = \mathrm{tr}(\phi)\,\mathrm{e}(g),$$
since the degree 0 part of $\hat{A}_g$ is just 1. Proposition 6.1 now follows from Corollary 7.4 and (96).

Proof of Proposition 6.2

For odd n the statement follows immediately from Lemma 7.3. So let us assume n is even. Consider $\tilde{\xi} := g\xi \in \Gamma(T^*M\otimes T^*M)$, and use the bundle map $1\otimes\hat{c} : T^*M\otimes T^*M \to T^*M\otimes\mathrm{end}^-_{\mathrm{Cl}}(\Lambda)$ to define
$$V := \tfrac{1}{2}(1\otimes\hat{c})(\tilde{\xi}) \in \Omega^1(M; \mathrm{end}^-_{\mathrm{Cl}}(\Lambda)).$$
We claim
$$c(V) = \Lambda^*\xi - \tfrac{1}{2}\,\mathrm{tr}(\xi). \tag{97}$$
To check this let $\{e_i\}$ be a local orthonormal frame and let $\{e^i\}$ be its dual local coframe. Then
$$\Lambda^*\xi = \sum_{i,j} g(\xi e_i, e_j)\tfrac{1}{2}\big(\varepsilon^i\iota^j + \varepsilon^j\iota^i\big),$$

where $\varepsilon^i \in \Gamma(\mathrm{end}(\Lambda))$ denotes exterior multiplication with e^i, and $\iota^i \in \Gamma(\mathrm{end}(\Lambda))$ denotes contraction with e_i. Writing $c^i := c(e^i)$, $\hat{c}^i := \hat{c}(e^i)$ and using $\varepsilon^i = \frac{1}{2}(c^i+\hat{c}^i)$ as well as $\iota^i = -\frac{1}{2}(c^i - \hat{c}^i)$ one easily checks

$$\tfrac{1}{2}\big(\varepsilon^i\iota^j + \varepsilon^j\iota^i\big) = \tfrac{1}{4}\big(c^i\hat{c}^j + c^j\hat{c}^i\big) + \tfrac{1}{2}\delta^{ij}.$$

We conclude

$$\Lambda^*\xi = \sum_{i,j} g(\xi e_i, e_j)\Big(\tfrac{1}{4}\big(c^i\hat{c}^j + c^j\hat{c}^i\big) + \tfrac{1}{2}\delta^{ij}\Big) = \sum_{i,j} g(\xi e_i, e_j)\tfrac{1}{2}c^i\hat{c}^j + \tfrac{1}{2}\,\mathrm{tr}(\xi).$$

On the other hand we clearly have $c(V) = \sum_{i,j} g(\xi e_i, e_j)\frac{1}{2}c^i\hat{c}^j$ and thus (97) is established. We will apply Corollary 7.5 to the Clifford super connection (95) and this V.

Next we claim that for all integers $k \geq 1$ and $l \geq 0$ we have

$$\mathrm{str}_{\mathcal{E}/S}\Big(\big(\big(\mathrm{ad}(-F^{\mathcal{E}/S}_{\nabla^{E,g,b}})\big)^k A_{E,g,b}\big) \wedge \big(-F^{\mathcal{E}/S}_{\nabla^{E,g,b}}\big)^l \wedge \nabla V\Big) = 0. \tag{98}$$

To see this let us write $\mathrm{end}_{\mathrm{Cl}}(\mathcal{E})_i$ for the subspace of $\mathrm{end}_{\mathrm{Cl}}(\mathcal{E})$ which via the isomorphism $\hat{c}\otimes 1 : \hat{\mathrm{Cl}} \otimes \mathrm{end}(E) \to \mathrm{end}_{\mathrm{Cl}}(\Lambda)\otimes\mathrm{end}(E) = \mathrm{end}_{\mathrm{Cl}}(\mathcal{E})$ corresponds to the filtration subspace $\hat{\mathrm{Cl}}_i \otimes \mathrm{end}(E)$. Then $-F^{\mathcal{E}/S}_{\nabla^{E,g,b}} \in \Omega^2(M;\mathrm{end}_{\mathrm{Cl}}(\mathcal{E})_2)$, $\nabla V \in \Omega^2(M;\mathrm{end}_{\mathrm{Cl}}(\mathcal{E})_1)$ and $A_{E,g,b} \in \Omega^0(M;\mathrm{end}_{\mathrm{Cl}}(\mathcal{E})_1)$. Looking at the form degree, we see that (98) holds whenever $2k+2l+2 > n$. Moreover, since $k \geq 1$ we have $\big(\mathrm{ad}(-F^{\mathcal{E}/S}_{\nabla^{E,g,b}})\big)^k A_{E,g,b} \in \Omega^{2k}(M;\mathrm{end}_{\mathrm{Cl}}(\mathcal{E})_{2k})$, for $[\hat{\mathrm{Cl}}_2, \hat{\mathrm{Cl}}_1] \subseteq \hat{\mathrm{Cl}}_2$. Thus, considering the filtration degree, we see that (98) holds whenever $2k+2l+1 < n$, for $\mathrm{str}_{\mathcal{E}/S}$ vanishes on $\Omega(M;\mathrm{end}_{\mathrm{Cl}}(\mathcal{E})_{n-1})$. This establishes (98). We conclude

$$\begin{aligned}\mathrm{str}_{\mathcal{E}/S}\Bigg(\Bigg(\frac{e^{\mathrm{ad}(-F^{\mathcal{E}/S}_{\nabla^{E,g,b}})}-1}{\mathrm{ad}(-F^{\mathcal{E}/S}_{\nabla^{E,g,b}})}A_{E,g,b}\Bigg) \wedge \exp\big(-F^{\mathcal{E}/S}_{\nabla^{E,g,b}}\big) \wedge \nabla V\Bigg)& \\ = \mathrm{str}_{\mathcal{E}/S}\Big(A_{E,g,b} \wedge \exp\big(-F^{\mathcal{E}/S}_{\nabla^{E,g,b}}\big)\wedge \nabla^g V\Big)&\end{aligned} \tag{99}$$

Here we wrote $\nabla V = \nabla^g V$ to emphasize that this form does not depend on the flat connection on E, but only on the Levi–Civita connection. Using (92) and $(\nabla^{E,b})^2 \in \Omega^2(M;\mathrm{end}_{\mathrm{Cl}}(\mathcal{E})_0)$ and considering form and filtration degree we easily obtain:

$$\begin{aligned}&\mathrm{str}_{\mathcal{E}/S}\Big(A_{E,g,b}\wedge\exp\big(-F^{\mathcal{E}/S}_{\nabla^{E,g,b}}\big)\wedge\nabla^g V\Big)\\ &\qquad = \mathrm{str}_{\mathcal{E}/S}\Big(A_{E,g,b}\wedge\exp\big(-F^{\Lambda/S}_{\nabla^g}\big)\wedge\nabla^g V\Big)\\ &\qquad = \mathrm{str}_{\Lambda/S}\Big(\mathrm{tr}_E(A_{E,g,b})\wedge\exp\big(-F^{\Lambda/S}_{\nabla^g}\big)\wedge\nabla^g V\Big)\end{aligned} \tag{100}$$

Using (94) and applying Lemma 8.2 to the closed one form $\operatorname{tr}(b^{-1}\nabla^E b)$ we find

$$\begin{aligned}\operatorname{str}_{\Lambda/S}&\Big(\operatorname{tr}_E(A_{E,g,b})\wedge\exp\big(-F^{\Lambda/S}_{\nabla^g}\big)\wedge\nabla^g V\Big)\\&=-\tfrac{1}{2}\operatorname{str}_{\Lambda/S}\Big(\hat{c}\big(\operatorname{tr}(b^{-1}\nabla^E b)\big)\wedge\exp\big(-F^{\Lambda/S}_{\nabla^g}\big)\wedge\nabla^g V\Big)\\&=-(2\pi\mathbf{i})^{n/2}\operatorname{tr}(b^{-1}\nabla^E b)\wedge(\partial_2\operatorname{cs})(g,\tilde{\xi})\end{aligned}\tag{101}$$

Combining (99), (100) and (101) we conclude:

$$\begin{aligned}-(2\pi\mathbf{i})^{-n/2}\hat{A}_g\wedge\operatorname{str}_{\mathcal{E}/S}\Bigg(\Bigg(\frac{e^{\operatorname{ad}(-F^{\mathcal{E}/S}_{\nabla^{E,g,b}})}-1}{\operatorname{ad}(-F^{\mathcal{E}/S}_{\nabla^{E,g,b}})}A_{E,g,b}\Bigg)\wedge\exp\big(-F^{\mathcal{E}/S}_{\nabla^{E,g,b}}\big)\wedge\nabla V\Bigg)&\\=\operatorname{tr}(b^{-1}\nabla^E b)\wedge(\partial_2\operatorname{cs})(g,g\xi)&\end{aligned}$$

Now apply Corollary 7.5 and use (97) as well as (96) to complete the proof of Proposition 6.2.

9. The case of non-vanishing Euler–Poincaré characteristics

It is not necessary to restrict to manifolds with vanishing Euler characteristics. In the general situation [11, 12] Euler structures, coEuler structures, the combinatorial torsion and the analytic torsion depend on the choice of a base point. Given a path connecting two such base points everything associated with the first base point identifies in an equivariant way with the everything associated to the other base point. However, these identifications do depend on the homotopy class of such a path. Below we sketch a natural way to conveniently deal with this situation.

In general the the set of Euler structures $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ depends on a base point $x_0\in M$. One defines the set of Euler structures based at x_0 as equivalence classes $[X,c]$ where X is a vector field with non-degenerate zeros and $c\in C_1^{\mathrm{sing}}(M;\mathbb{Z})$ is such that $\partial c=\mathrm{e}(X)-\chi(M)x_0$. Two such pairs (X_1,c_1) and (X_2,c_2) are equivalent iff $c_2-c_1=\operatorname{cs}(X_1,X_2)$ mod boundaries. Again this is an affine version of $H_1(M;\mathbb{Z})$, the action is defined as in Section 2. Given a path σ from x_0 to x_1, the assignment $[X,c]\mapsto[X,c-\chi(M)\sigma]$ defines an $H_1(M;\mathbb{Z})$-equivariant isomorphism from $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ to $\mathfrak{Eul}_{x_1}(M;\mathbb{Z})$. Since this isomorphism depends on the homotopy class of σ only, we can consider the set of Euler structures as a flat principal bundle $\mathfrak{Eul}(M;\mathbb{Z})$ over M with structure group $H_1(M;\mathbb{Z})$. Its fiber over x_0 is just $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$, and its holonomy is given by the composition

$$\pi_1(M)\to H_1(M;\mathbb{Z})\xrightarrow{-\chi(M)}H_1(M;\mathbb{Z}).$$

Similarly, the set of Euler structures with complex coefficients can be considered as a flat principal bundle $\mathfrak{Eul}(M;\mathbb{C})$ over M with structure group $H_1(M;\mathbb{C})$ and holonomy given by the composition

$$\pi_1(M)\to H_1(M;\mathbb{Z})\xrightarrow{-\chi(M)}H_1(M;\mathbb{Z})\to H_1(M;\mathbb{C}).$$

There is an obvious parallel homomorphism of flat principal bundles over M

$$\iota : \mathfrak{Eul}(M;\mathbb{Z}) \to \mathfrak{Eul}(M;\mathbb{C}) \tag{102}$$

which is equivariant over the homomorphism of structure groups $H_1(M;\mathbb{Z}) \to H_1(M;\mathbb{C})$.

The set of coEuler structures $\mathfrak{Eul}^*_{x_0}(M;\mathbb{C})$ depends on the choice of a base point $x_0 \in M$. It can be defined as the set of equivalence classes $[g,\alpha]$, where g is a Riemannian metric and $\alpha \in \Omega^{n-1}(M \setminus \{x_0\};\mathcal{O}^{\mathbb{C}}_M)$ is such that $\mathrm{e}(g) = d\alpha$ on $M \setminus \{x_0\}$. Two such pairs $[g_1,\alpha_1]$ and $[g_2,\alpha_2]$ are equivalent iff $\alpha_2 - \alpha_1 = \mathrm{cs}(g_1,g_2)$ mod coboundaries, see [11, Section 3.2]. Every homotopy class of paths connecting x_0 and x_1 provides an identification between $\mathfrak{Eul}^*_{x_0}(M;\mathbb{C})$ and $\mathfrak{Eul}^*_{x_1}(M;\mathbb{C})$. Again, one can consider the set of coEuler structures as a flat principal bundle $\mathfrak{Eul}^*(M;\mathbb{C})$ over M with structure group $H^{n-1}(M;\mathcal{O}^{\mathbb{C}}_M)$. Its fiber over x_0 is $\mathfrak{Eul}^*_{x_0}(M;\mathbb{C})$, and its holonomy is given by the composition

$$\pi_1(M) \to H_1(M;\mathbb{Z}) \xrightarrow{-\chi(M)} H_1(M;\mathbb{Z}) \to H_1(M;\mathbb{C}) \to H^{n-1}(M;\mathcal{O}^{\mathbb{C}}_M)$$

where the last arrow indicates Poincaré duality.

The affine version of Poincaré duality introduced in Section 2 can be consider as a parallel isomorphism of flat principal bundles over M

$$P : \mathfrak{Eul}(M;\mathbb{C}) \to \mathfrak{Eul}^*(M;\mathbb{C}) \tag{103}$$

which is equivariant over the homomorphism of structure groups $H_1(M;\mathbb{C}) \to H^{n-1}(M;\mathcal{O}^{\mathbb{C}}_M)$ provided by Poincaré duality. We have $P([X,c]) = [g,\alpha]$ iff

$$\int_{M \setminus (\mathcal{X} \cup \{x_0\})} \omega \wedge (X^*\Psi(g) - \alpha) = \int_c \omega$$

for all closed one forms ω which vanish in a neighborhood of $\mathcal{X} \cup \{x_0\}$.

If E is a flat complex vector bundle over M we consider the flat line bundle

$$\mathrm{Det}(M;E) := \det H^*(M;E) \otimes (\det E)^{-\chi(M)}.$$

Let $\mathrm{Det}^\times(M;E)$ denote its frame bundle, a flat principal bundle over M with structure group $\mathbb{C}^\times$ and holonomy given by

$$\pi_1(M) \to H_1(M;\mathbb{Z}) \xrightarrow{(\theta_E)^{\chi(M)}} \mathbb{C}^\times.$$

We will also consider the flat principal bundle $\mathrm{Det}^\times(M;E)^{-2}$ over M with structure group $\mathbb{C}^\times$ and holonomy given by the composition

$$\pi_1(M) \to H_1(M;\mathbb{Z}) \xrightarrow{(\theta_E)^{-2\chi(M)}} \mathbb{C}^\times.$$

Note that elements in $\mathrm{Det}^\times(M;E)^{-2}$ can be considered as non-degenerate bilinear forms on the corresponding fiber of $\mathrm{Det}(M;E)$.

The combinatorial torsion defines a parallel homomorphism of flat principal bundles

$$\tau^{\mathrm{comb}}_E : \mathfrak{Eul}(M;\mathbb{Z}) \to \mathrm{Det}^\times(M;E)^{-2} \tag{104}$$

which is equivariant over the homomorphism of structure groups

$$(\theta_E)^2 : H_1(M;\mathbb{Z}) \to \mathbb{C}^\times.$$

This formulation encodes in a rather natural way the combinatorial torsion's dependence on the Euler structure and its base point. Concerning the definition of (104), recall that the corresponding construction in Section 3 assigns to an Euler structure $\mathfrak{e}_{x_0} \in \mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ and a bilinear form b_{x_0} on E_{x_0} a bilinear form on $\det H^*(M;E)$. Tensorizing this with the bilinear form on $(\det E_{x_0})^{-\chi(M)}$ induced by b_{x_0}, we obtain an element of $\mathrm{Det}^\times_{x_0}(M;E)^{-2}$ which does not depend on the choice of b_{x_0}. By definition this is the combinatorial torsion $\tau^{\mathrm{comb}}_E(\mathfrak{e}_{x_0})$ in (104).

If b is a fiber wise non-degenerate symmetric bilinear form on E, its analytic torsion provides a parallel homomorphism of flat principal bundles

$$\tau^{\mathrm{an}}_{E,[b]} : \mathfrak{Eul}^*(M;\mathbb{C}) \to \mathrm{Det}^\times(M;E)^{-2} \tag{105}$$

which is equivariant over the homomorphism of structure groups

$$H^{n-1}(M;\mathcal{O}^{\mathbb{C}}_M) \to \mathbb{C}^\times, \quad \beta \mapsto \Big(e^{\langle [\omega_{E,b}]\cup\beta,[M]\rangle}\Big)^2.$$

The definition of (105) is essentially the same as in Section 4. To be more precise, we represent the coEuler structure $\mathfrak{e}^*_{x_0} \in \mathfrak{Eul}^*_{x_0}(M;\mathbb{C})$ as $\mathfrak{e}^*_{x_0} = [g,\alpha]$, where $\alpha \in \Omega^{n-1}(M\setminus\{x_0\};\mathcal{O}^{\mathbb{C}}_M)$ is such that $\mathrm{e}(g) = d\alpha$. We write $b_{(\det E_{x_0})^{-\chi(M)}}$ for the induced bilinear form on $(\det E_{x_0})^{-\chi(M)}$, and set

$$\tau^{\mathrm{an}}_{E,g,b,\alpha} := \tau^{\mathrm{an}}_{E,g,b}(0)\cdot\prod_q\big(\det{}'(\Delta_{E,g,b,q})\big)^{(-1)^q q}\cdot\exp\Big(-2\int_M \omega_{E,b}\wedge\alpha\Big)\otimes b_{(\det E_{x_0})^{-\chi(M)}}.$$

If $\chi(M) \neq 0$, then α will be singular at x_0 and the integral $\int_M \omega_{E,b}\wedge\alpha$ has to be regularized, see [11, 12]. Due to this regularization the additional term $\chi(M)\,\mathrm{tr}(b_u^{-1}\dot b_u)(x_0)$ will appear on the right hand side of (56) and cancel the variation of $b_{(\det E_{x_0})^{-\chi(M)}}$. Other than that the proof of Theorem 4.2 remains the same. Thus $\tau_{E,g,b,\alpha}$ depends on E, $\mathfrak{e}^*_{x_0}$ and $[b]$ only. By definition this is the analytic torsion $\tau^{\mathrm{an}}_{E,[b]}(\mathfrak{e}^*_{x_0})$ in (105).

In this language the extension of Conjecture 5.1 to non-vanishing Euler–Poincaré characteristics asserts that for all b we have

$$\tau^{\mathrm{an}}_{E,[b]}\circ P\circ\iota = \tau^{\mathrm{comb}}_E$$

as an equality of homomorphism of principal bundles over M, see (102), (103), (104) and (105).

As in Section 5 one defines the relative torsion as the quotient of analytic and combinatorial torsion. This is a non-vanishing complex number independent of the Euler structure and its base point. Its properties in Proposition 5.7 remain true as stated. With little more effort one shows that the relative torsion in general is given by the formula in Proposition 5.11. Proving the generalization of Conjecture 5.1 thus amounts to show that the right hand side of the equation in Proposition 5.11

equals 1, even if $\chi(M) \neq 0$. In view of the anomaly formula it suffices to check this for a single Riemannian metric and any representative of the homotopy class $[b]$.

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