COMPLEX VALUED RAY–SINGER TORSION II

DAN BURGHELEA AND STEFAN HALLER

Abstract. In this paper we extend Witten–Helffer–Sjöstrand theory from selfadjoint Laplacians based on fiber wise Hermitian structures, to non-selfadjoint Laplacians based on fiber wise non-degenerate symmetric bilinear forms. As an application we verify, up to sign, the conjecture about the comparison of the Milnor–Turaev torsion with the complex valued analytic torsion, for odd dimensional manifolds. This is done along the lines of Burghelea, Friedlander and Kappeler's proof of the Cheeger–Müller theorem.

Contents

1. Introduction 1
2. Witten–Helffer–Sjöstrand theory 5
3. Asymptotic of the large torsion and the proof of Theorem 1.4 24
4. Appendix; Remarks on the proof of Conjecture 1.3 33
References 34

1. Introduction

Let (M, g) be a closed connected smooth Riemannian manifold of dimension n, and suppose E is a flat complex vector bundle over M, equipped with a (not necessarily parallel) fiber wise non-degenerate symmetric bilinear form b. The flat connection of E will be denoted by ∇^E. Let $\Omega(M; E)$ denote the deRham complex of E-valued differential forms on M, and write d_E for the deRham differential.

The Riemannian metric g and the bilinear form b provide a fiber wise non-degenerate symmetric bilinear form on the complex vector bundle $\Lambda^*T^*M \otimes E$ which will be denoted by b_g. If vol_g denotes the volume density associated with g, then

$$\beta(v, w) := \int_M b_g(v, w) \,\mathrm{vol}_g \qquad v, w \in \Omega(M; E),$$

1991 *Mathematics Subject Classification.* 57R20, 58J52.

Part of this work was done while the second author enjoyed the worm hospitality of the Ohio State University.

defines a symmetric bilinear form on $\Omega(M;E)$. Let $d^\sharp_{E,g,b}$ denote the formal transposed of d_E with respect to β,

$$\beta(d_E v, w) = \beta(v, d^\sharp_{E,g,b} w), \qquad v, w \in \Omega(M;E).$$

The Laplace–Beltrami operator

$$\Delta_{E,g,b} := (d_E + d^\sharp_{E,g,b})^2 = d_E d^\sharp_{E,g,b} + d^\sharp_{E,g,b} d_E \tag{1}$$

is not necessarily selfadjoint.

In [11] the following complex valued analogue of the square of the Ray–Singer torsion [20] was introduced and studied:

$$\prod_q \bigl(\det{}'(\Delta_{E,g,b,q})\bigr)^{(-1)^q q} \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\}. \tag{2}$$

Here $\det'(\Delta_{E,g,b,q}) \in \mathbb{C}^\times$ denotes the zeta regularized product of all non-vanishing eigen values of the Laplacian acting on $\Omega^q(M;E)$. In the definition of the determinant one can use any non-zero Agmon angle, the resulting $\det'(\Delta_{E,g,b,q})$ will be independent of this choice.

Let X be a Morse–Smale vector field[1] on M, and write $\mathcal{X}$ for the set of critical points, i.e. zeros of X. Denote by $C(X;E)$ the associated Morse complex. The fiber wise symmetric bilinear form b induces a non-degenerate bilinear form $b_\mathcal{X}$ on $C(X;E)$. Recall that the integration homomorphism

$$\mathrm{Int}_{E,X} : \Omega(M;E) \to C(X;E) \tag{3}$$

induces an isomorphism in cohomology, see [21]. Let $\Omega_{g,b}(M;E)(0)$ denote the (generalized) zero eigen space of $\Delta_{E,g,b}$. Due to ellipticity of $\Delta_{E,g,b}$, the space $\Omega_{g,b}(M;E)(0)$ is finite dimensional and consists of smooth forms only. Since d_E commutes with $\Delta_{E,g,b}$ the zero eigen space $\Omega_{g,b}(M;E)(0)$ is a subcomplex of $\Omega(M;E)$. As $\Delta_{E,g,b}$ fails to be selfadjoint, the differential on $\Omega_{g,b}(M;E)(0)$ will in general not vanish. However, the inclusion $\Omega_{g,b}(M;E)(0) \to \Omega(M;E)$ induces an isomorphism in cohomology. It follows that the restriction of (3) to $\Omega_{g,b}(M;E)(0)$ induces an isomorphism in cohomology too. Because $\Delta_{E,g,b}$ is symmetric with respect to β, the bilinear form β will restrict to a non-degenerate symmetric bilinear form on $\Omega_{g,b}(M;E)(0)$. We thus have a quasi isomorphism between finite dimensional complexes

$$\mathrm{Int}_{X,E} : \Omega_{g,b}(M;E)(0) \to C_b(X;E)$$

each of which is equipped with a non-degenerate symmetric bilinear form. This permits to define the square of the relative torsion which we will denote by

$$\tau\Bigl(\Omega_{g,b}(M;E)(0) \xrightarrow{\mathrm{Int}_{X,E}} C_b(X;E)\Bigr) \in \mathbb{C}^\times.$$

A more detailed discussion of these facts can be found in [11, Section 4].

[1] That is, with respect to some Riemannian metric, X is the negative gradient of a Morse function, and satisfies the Smale transversality condition, i.e. stable and unstable manifolds intersect transversally.

Recall the Mathai–Quillen form, see [18] or [2, Section III],

$$\Psi_g \in \Omega^{n-1}(TM \setminus M; \mathcal{O}_M),$$

and the Kamber–Tondeur form

$$\omega_{E,b} := -\tfrac{1}{2}\operatorname{tr}_E(b^{-1}\nabla^E b) \in \Omega^1(M;\mathbb{C}). \tag{4}$$

Note that $\omega_{E,b}$ is closed since ∇^E is flat, see [11, Section 2]. The integral

$$\int_{M\setminus\mathcal{X}} \omega_{E,b} \wedge (-X)^*\Psi_g$$

will in general not converge, but can easily be regularized, see [9, Section 2], [10, Section 3] or [2, Section III]. Consider the non-vanishing complex number

$$\mathcal{S}_{E,g,b,X} := \tau\Big(\Omega_{g,b}(M;E)(0) \xrightarrow{\operatorname{Int}_{X,E}} C_b(X;E)\Big) \cdot \prod_q \big(\det{}'(\Delta_{E,g,b,q})\big)^{(-1)^q q} \cdot \exp\Big(-2\int_{M\setminus\mathcal{X}} \omega_{E,g} \wedge (-X)^*\Psi_g\Big).$$

In [11] the following result, analogous to the anomaly formula for the classical Ray–Singer torsion [2, Theorem 0.1], has been established.[2]

Theorem 1.1. *The quantity $\mathcal{S}_{E,g,b,X}$ is independent of the Morse–Smale vector field X, independent of the Riemannian metric g and locally constant in b. It thus depends on the flat bundle E and the homotopy class $[b]$ of the fiber wise non-degenerate bilinear form only, and will be denoted by $\mathcal{S}_{E,[b]}$.*

Remark 1.2. There is a conceptual interpretation of $\mathcal{S}_{E,[b]}$ as the quotient of two invariants, see [11, Section 5].

In analogy with a result of Cheeger [12, 13], Müller [19] and Bismut–Zhang [2, Theorem 0.2], the following conjecture was raised in [11, Conjecture 5.1].

Conjecture 1.3. *We have $\mathcal{S}_{E,[b]} = 1$ for every flat complex vector bundle E and every fiber wise non-degenerate symmetric bilinear form b on E.*

This conjecture has been verified in several non-trivial situations, see [4] and [11, Section 5]. One purpose of this paper is to establish Conjecture 1.3 for odd dimensional manifolds, up to sign. More precisely, we will show

Theorem 1.4. *Suppose M is odd dimensional. Then $\mathcal{S}_{E,[b]} = \pm 1$ for every flat complex vector bundle E and every fiber wise non-degenerate symmetric bilinear form b on E.*[3]

[2]Strictly speaking, this was done for vanishing Euler–Poincaré characteristics only. However, with few additional elementary arguments Theorem 1.1 below can be proved exactly as in [11, Section 6], the crucial analytic results [11, Proposition 6.1 and 6.2] have been established without any restriction on the Euler–Poincaré characteristics.

[3]In the appendix we show how one can remove the sign ambiguity by extending Witten–Helffer–Sjöstrand theory to generalized Morse functions, a project partially realized in [16]. One also note that an extension of the theorem above to compact manifolds with boundary implies the result for closed even dimensional manifolds

In fact we actually show that the strategy of proving the Cheeger–Müller theorem presented in [5] works here too, almost identically. However, the spectral properties of the Witten deformation of the Laplace operators associated with a Riemannian metric and a non-degenerate symmetric bilinear form can not benefit from the methods described in more details in [8], based on selfadjointness. Fortunately, the key geometric consequences continue to hold. The other purpose of this paper is to extend Witten–Helffer–Sjöstrand theory to this non-selfadjoint situation, which may be of independent interest.

This paper is organized as follows. In section 2 we will extend Witten–Helffer–Sjöstrand theory [2, 3, 5, 7, 8, 14, 15] to this non-selfadjoint situation. More precisely, we choose a Morse function f on M and fix a Riemannian metric g which has a standard form near the critical points $\mathcal{X}$ of f. We assume that the gradient vector field $X := -\operatorname{grad}_g(f)$ satisfies the Smale transversality condition. Finally, we assume that the bilinear form b is parallel in a neighbourhood of $\mathcal{X}$, with respect to ∇^E. In view of Theorem 1.1 these assumptions do not cause a loss of generality for the purpose of computing $\mathcal{S}_{E,[b]}$. We then consider the family of Witten deformed flat connections on E

$$\nabla^{E_u} := \nabla^E + udf, \qquad u \geq 0. \tag{5}$$

Let us write E_u for the complex vector bundle E equipped with the flat connection $\nabla^E + udf$. Since $e^{uf} : (E_u, b) \to (E, e^{-2uf}b)$ is an isomorphism of flat vector bundles with bilinear forms, it follows from Theorem 1.1 that $\mathcal{S}_{E_u,[b]}$ is constant in u.

Let us introduce the *large analytic torsion*

$$\tau_{\mathrm{la},u} := \prod_q \big(\det{}^{\mathrm{la}}(\Delta_{E_u,g,b,q})\big)^{(-1)^q q} \tag{6}$$

where $\det^{\mathrm{la}}(\Delta_{E,g,b,q})$ denotes the zeta regularized product of eigen values whose real part is larger than 1. In view of Proposition 2.18 below, $\tau_{\mathrm{la},u}$ is analytic in u for sufficiently large u. Moreover, let us write $\Omega_{\mathrm{sm}}(M;E_u)$ for the sum of eigen spaces corresponding to eigen values with real part at most 1. We refer to $\Omega_{\mathrm{sm}}(M;E_u)$ as the *small complex,* see Proposition 2.18 below. This is a finite dimensional subcomplex of $\Omega(M;E_u)$, β restricts to a non-degenerate symmetric bilinear form $\beta_{\mathrm{sm},u}$ on $\Omega_{\mathrm{sm}}(M;E_u)$, and the restriction of (3) to $\Omega_{\mathrm{sm}}(M;E_u)$ provides a quasi isomorphism

$$\mathrm{Int}_{\mathrm{sm},u} \colon \Omega_{\mathrm{sm}}(M;E_u) \to C(X;E_u). \tag{7}$$

Let us write $\tau(\mathrm{Int}_{\mathrm{sm},u}) \in \mathbb{C}^\times$ for the relative torsion of (7). It is not hard to show, see [11, Proposition 5.10], that

$$\mathcal{S}_{E,[b]} = \mathcal{S}_{E_u,[b]} = \tau(\mathrm{Int}_{\mathrm{sm},u}) \cdot \tau_{\mathrm{la},u} \cdot \exp\Big(-2\int_{M\setminus\mathcal{X}} \omega_{E_u,b} \wedge (-X)^*\Psi_g\Big). \tag{8}$$

The Witten–Helffer–Sjöstrand estimates show that for sufficiently large u the integration (7) is an isomorphism of complexes, and they provide an asymptotic comparison of the bilinear form $\beta_{\mathrm{sm},u}$ on $\Omega_{\mathrm{sm}}(M;E_u)$ and the bilinear form $b_{\mathcal{X}}$ on $C(X;E_u)$. The precise statement is contained in Theorem 2.1 below. This result permits to compute the asymptotic expansion of $\tau(\mathrm{Int}_{\mathrm{sm},u})$ as $u\to\infty$, see Corollary 2.2. In view of (8) this yields a formula for $\mathcal{S}_{E,[b]}$ in terms of the free term of the asymptotic expansion of $\tau_{\mathrm{la},u}$ as $u\to\infty$, see Corollary 2.3.

Unfortunately we are unable at this time to calculate directly this constant term and check whether $\mathcal{S}_{E,[b]}$ is one or not. However, in Section 3 we show that for two systems (M,E,g,b,f) and $(\tilde M,\tilde E,\tilde g,\tilde b,\tilde f)$ with M and $\tilde M$ of the same dimension, E and $\tilde E$ of the same rank, f and $\tilde f$ with the same number of critical points in each index

$$\log\tau_{\mathrm{la},u}-\log\tilde\tau_{\mathrm{la},u}$$

has an asymptotic expansion whose free term is computable as integral of local quantities, see Theorem 3.6. This is done as in [5]; precisely, combining the fact that the Witten deformation is elliptic with parameter away from the critical points with a Mayer–Vietoris formula for the zeta regularized determinants of elliptic operators, yields a result as formulated in Theorem 3.6. Note that we have an unambiguously defined logarithm, given by the formula:

$$\log\tau_{\mathrm{la},u}:=\sum_q(-1)^q q\frac{\partial}{\partial s}\Big|_{s=0}\sum_{\lambda\in\mathrm{Spec}\,\Delta_{u,q},\mathrm{Re}\,\lambda>1}\lambda^{-s}$$

Playing with its symmetry, as in [5], we derive that, for odd dimensional manifolds,

$$\mathcal{S}^2_{E,[b]}=\mathcal{S}^2_{\tilde E,[\tilde b]}.$$

We will then use this to give a proof of Theorem 1.4.

2. Witten–Helffer–Sjöstrand theory

Let f be a Morse function on a closed connected smooth manifold M of dimension n. Fix a Riemannian metric g on M so that the vector field $X:=-\mathrm{grad}_g(f)$ satisfies the Smale transversality condition. Let $\mathcal{X}\subseteq M$ denote the set of critical points of f, and denote by $\mathrm{ind}(x)\in\{0,1,\dots,n\}$ the Morse index of a critical point $x\in\mathcal{X}$. For every critical point $x\in\mathcal{X}$ we fix an open neighbourhood U_x of x and a diffeomorphisms (Morse chart) $\varphi_x=(\varphi_x^1,\dots,\varphi_x^n)\colon U_x\to\mathbb{R}^n$ so that $\varphi_x(x)=0$ and

$$f=f(x)-\frac12\sum_{i\le\mathrm{ind}(x)}(\varphi_x^i)^2+\frac12\sum_{i>\mathrm{ind}(x)}(\varphi_x^i)^2 \tag{9}$$

in a neighbourhood of x. We will assume that every $x\in\mathcal{X}$ admits a neighbourhood on which the Riemannian metric takes the form

$$g=\sum_{i=1}^n d\varphi_x^i\otimes d\varphi_x^i. \tag{10}$$

Finally, we will assume that, in a neighbourhood of $\mathcal{X}$, we have

$$\nabla^E b = 0. \tag{11}$$

Recall the family of integration homomorphisms (7) associated with the Witten deformed flat bundles E_u, see (5). For $u > 0$ let us introduce the scaling isomorphism

$$\eta_u \colon C(X; E_u) \to C(X; E_u) \tag{12}$$

defined by

$$\eta_u(w) := \Big(\frac{\pi}{u}\Big)^{n/4-q/2} w, \qquad w \in C^q(X; E_u).$$

The aim of this section is to provide a proof of the following

Theorem 2.1. *For sufficiently large u, the restriction of the integration map*

$$\operatorname{Int}_{u,\mathrm{sm}} \colon \Omega_{\mathrm{sm}}(M; E_u) \to C(X; E_u) \tag{13}$$

is an isomorphism of complexes. Moreover, there exists a constant $\varepsilon > 0$ so that

$$(\eta_u \operatorname{Int}_{u,\mathrm{sm}})_* \beta_{u,\mathrm{sm}} = b_{\mathcal{X}} + \mathrm{O}(e^{-\varepsilon u}) \qquad \text{as } u \to \infty.$$

Theorem 2.1 generalizes the classical Witten–Helffer–Sjöstrand theorem for selfadjoint Laplacians, see [14], [15], [2], [3] and [8, Theorem 5.5], to this non-selfadjoint situation. The proof of this result will be similar to the one in [8]. A few additional arguments, however, are necessary since the Laplacians $\Delta_u := \Delta_{E_u,g,b}$ are not necessarily selfadjoint. A key step is to establish a widening gap in the spectrum of Δ_u, as $u \to \infty$. A finite number of eigen values will exponentially fast approach 0, while the real part of the remaining will grow linearly with u. For the precise statement see Proposition 2.18 below. This justifies the term *small complex.*

In order to spell out two corollaries let us introduce the notation

$$\chi := \sum_q (-1)^q \dim C^q(X; E) = \chi(M) \operatorname{rank}(E)$$

$$\chi' := \sum_q (-1)^q q \dim C^q(X; E) = \sum_q (-1)^q q |\mathcal{X}_q| \operatorname{rank}(E)$$

where $|\mathcal{X}_q|$ denotes the number of critical points of index q.

Corollary 2.2. *There exists a constant $\varepsilon > 0$ so that, as $u \to \infty$,*

$$\tau(\operatorname{Int}_{u,\mathrm{sm}}) = \Big(\frac{\pi}{u}\Big)^{\frac{n}{2}\chi - \chi'} \big(1 + \mathrm{O}(e^{-\varepsilon u})\big).$$

Proof. In view of Theorem 2.1 the integration (13) is an isomorphism of complexes, assuming u is sufficiently large. Hence, we have

$$\tau(\operatorname{Int}_{u,\mathrm{sm}}) = \operatorname{sdet}\Big((b_{\mathcal{X}})^{-1}\big((\operatorname{Int}_{u,\mathrm{sm}})_* \beta_{u,\mathrm{sm}}\big)\Big). \tag{14}$$

Here the right hand side denotes the super determinant[4] of the composition

$$C(X;E_u)\xrightarrow{(\mathrm{Int}_{u,\mathrm{sm}})_*\beta_{u,\mathrm{sm}}}C(X;E_u)'\xrightarrow{(b_{\mathcal X})^{-1}}C(X;E_u)$$

and the non-degenerate bilinear forms are considered as isomorphisms between the vector space $C(X;E_u)$ and its dual, $C(X;E_u)'$. Note that as isomorphisms $C(X;E_u)\to C(X;E_u)'$ we have

$$(\mathrm{Int}_{u,\mathrm{sm}})_*\beta_{u,\mathrm{sm}}=\big((\eta_u\,\mathrm{Int}_{u,\mathrm{sm}})_*\beta_{u,\mathrm{sm}}\big)\circ\eta_u^2$$

From (14) we thus conclude

$$\tau(\mathrm{Int}_{u,\mathrm{sm}})=\mathrm{sdet}\Big((b_{\mathcal X})^{-1}\big((\eta_u\,\mathrm{Int}_{u,\mathrm{sm}})_*\beta_{u,\mathrm{sm}}\big)\Big)\cdot(\mathrm{sdet}\,\eta_u)^2. \tag{15}$$

From Theorem 2.1 we obtain a constant $\varepsilon>0$ so that, as $u\to\infty$,

$$\mathrm{sdet}\Big((b_{\mathcal X})^{-1}\big((\eta_u\,\mathrm{Int}_{u,\mathrm{sm}})_*\beta_{u,\mathrm{sm}}\big)\Big)=1+\mathrm{O}(e^{-\varepsilon u}).$$

One readily checks:

$$(\mathrm{sdet}\,\eta_u)^2=\Big(\frac{\pi}{u}\Big)^{\frac n2\chi-\chi'}$$

Combining the latter two with (15) completes the proof of the corollary. □

Corollary 2.3. *There exist (unique) constants $a_0\in\mathbb C^\times$, $a_1,a_2\in\mathbb C$ with the following property. There exists a constant $\varepsilon>0$ so that, as $u\to\infty$,*

$$\tau_{\mathrm{la},u}=a_0\cdot e^{a_1u}\cdot u^{a_2}\cdot\big(1+\mathrm O(e^{-\varepsilon u})\big).$$

These constants are given by:

$$a_0=\mathcal S_{E,[b]}\cdot\pi^{-(\frac n2\chi-\chi')}\cdot\exp\Big(2\int_{M\setminus\mathcal X}\omega_{E,b}\wedge(-X)^*\Psi_g\Big)$$
$$a_1=2\,\mathrm{rank}(E)\int_{M\setminus\mathcal X}df\wedge(-X)^*\Psi_g$$
$$a_2=\tfrac n2\chi-\chi'$$

Proof. Combining Corollary 2.2 with (8) we obtain a constant $\varepsilon>0$ so that

$$\tau_{\mathrm{la},u}=\mathcal S_{E,[b]}\cdot\Big(\frac{u}{\pi}\Big)^{\frac n2\chi-\chi'}\cdot\exp\Big(2\int_{M\setminus\mathcal X}\omega_{E_u,b}\wedge(-X)^*\Psi_g\Big)\cdot\big(1+\mathrm O(e^{-\varepsilon u})\big) \tag{16}$$

as $u\to\infty$. From $\nabla^{E_u}b=\nabla^Eb-2udf\otimes b$, see (5), we obtain

$$b^{-1}\nabla^{E_u}b=b^{-1}\nabla^Eb-2udf\otimes 1_E \tag{17}$$

[4]If $\varphi_k:V^k\to V^k$ is a linear endomorphism of a graded vector space which preserves the grading, then its super (or graded) determinant is given by $\mathrm{sdet}(\varphi)=\prod_k(\det\varphi_k)^{(-1)^k}$.

Therefore, see (4), $\omega_{E_u,b} = \omega_{E,b} + u\,\mathrm{rank}(E)df$ and thus

$$\int_{M\setminus\mathcal{X}} \omega_{E_u,b} \wedge (-X)^*\Psi_g = \int_{M\setminus\mathcal{X}} \omega_{E,b} \wedge (-X)^*\Psi_g + u\,\mathrm{rank}(E)\int_{M\setminus\mathcal{X}} df \wedge (-X)^*\Psi_g.$$

Combining this with (16) the corollary follows. □

A formula for the Witten perturbed Laplacian. Recall that for every critical point $x \in \mathcal{X}$ we have fixed a Morse chart

$$\varphi_x = (\varphi_x^1, \dots, \varphi_x^n)\colon\ U_x \to \mathbb{R}^n.$$

Choose $\rho > 0$ so that with

$$B_x := \varphi_x^{-1}\big(\{z \in \mathbb{R}^n \mid |z| < \rho\}\big), \qquad x \in \mathcal{X},$$

equations (9), (10) and (11) hold on B_x, for every critical point $x \in \mathcal{X}$. We assume ρ was chosen sufficiently small so that the closures of B_x are mutually disjoint. It will be convenient to further assume, by choosing ρ sufficiently small, that for all $x, y \in \mathcal{X}$ with $\mathrm{ind}(x) = \mathrm{ind}(y)$, we have

$$W_x^- \cap B_y = \begin{cases} \emptyset & \text{if } x \neq y \\ \varphi_x^{-1}(\mathbb{R}^{\mathrm{ind}(x)}) \cap B_x & \text{if } x = y \end{cases} \tag{18}$$

This is possible in view of the Smale transversality property of X. Here W_x^- denotes the unstable manifold of x. Set

$$B := \bigcup_{x \in \mathcal{X}} B_x. \tag{19}$$

For each $x \in \mathcal{X}$ we introduce the smooth radial function

$$r_x^2\colon\ B_x \to \mathbb{R}, \qquad r_x^2 := \sum_{i=1}^n (\varphi_x^i)^2. \tag{20}$$

Over B_x, $x \in \mathcal{X}$, we decompose the cotangent bundle as

$$T^*M|_{B_x} = V_x^- \oplus V_x^+,$$

where the subbundle $V_x^- \subseteq T^*M|_{B_x}$ is spanned by $d\varphi_x^i$, $1 \leq i \leq \mathrm{ind}(x)$, and the subbundle $V_x^+ \subseteq T^*M|_{B_x}$ is spanned by $d\varphi_x^i$, $\mathrm{ind}(x) < i \leq n$. Let us write $\Lambda := \Lambda^* T^*M$. We obtain an induced decomposition

$$\Lambda|_{B_x} = \Lambda_x^- \otimes \Lambda_x^+, \qquad \text{where} \qquad \Lambda_x^\pm := \Lambda^* V_x^\pm.$$

Let us write

$$N_x^\pm \in \Gamma(\mathrm{end}(\Lambda_x^\pm))$$

for the grading operator acting by multiplication with q on $\Lambda^q V_x^\pm$. We will denote the operators $N_x^- \otimes \mathrm{id}_{\Lambda_x^+} \otimes \mathrm{id}_E$ and $\mathrm{id}_{\Lambda_x^-} \otimes N_x^+ \otimes \mathrm{id}_E$ acting on $\Lambda \otimes E|_{B_x}$ by N_x^- and N_x^+ too.

Lemma 2.4. *There exists a zero order operator $L \in \Gamma(\mathrm{end}(\Lambda \otimes E))$ so that for all $u \geq 0$ we have*

$$\Delta_u = \Delta_0 + uL + u^2|df|^2.$$

For every critical point $x \in \mathcal{X}$ we have

$$L|_{B_x} = 2\big(N_x^+ + \mathrm{ind}(x) - N_x^-\big) - n,$$

and, for $u \geq 0$,

$$\Delta_u = \Delta_0 - un + u^2 r_x^2 + 2u\big(N_x^+ + \mathrm{ind}(x) - N_x^-\big) \qquad \text{over } B_x.$$

Proof. As in the selfadjoint situation this can be verified in coordinates, see [15]. Alternatively, one can give a conceptual proof based on the observation, see [2, Section IV], that the Laplacians Δ_u are the squares of Dirac operators associated to Clifford super connections on $\Lambda \otimes E$. □

Remark 2.5. Let $x \in \mathcal{X}$ be a critical point. Use the flat connection ∇^E and the (flat) Levi–Civita connection to identify $\Omega(B_x; E) = C^\infty(B_x, \Lambda \otimes E_x)$. Then, via this identification, we have

$$\Delta_0 v = -\sum_i \frac{\partial^2 v}{(\partial \varphi_x^i)^2}, \qquad v \in C^\infty(B_x, \Lambda \otimes E_x).$$

Compatible Hermitian structure. We will now introduce a Hermitian structure on the vector bundle E. It is possible to choose such a Hermitian structure to be compatible with the symmetric bilinear form. To this end we start with

Lemma 2.6. *There exists a fiber wise complex anti-linear involution $v \mapsto \bar{v}$ on E such that, for e_1 and e_2 in the same fiber of E, $e_1 \neq 0$, we have*

$$b(\bar{e}_1, \bar{e}_2) = \overline{b(e_1, e_2)} \qquad \text{and} \qquad b(e_1, \bar{e}_1) > 0.$$

Moreover, this involution can be chosen to be parallel over B, see (19). *That is, for $e \in E|_B$ we have*

$$\nabla^E \bar{e} - \overline{\nabla^E e} = 0.$$

Proof. The fiber wise non-degenerate symmetric bilinear form b provides a reduction of the structure group to $O_k(\mathbb{C})$, where $k = \mathrm{rank}(E)$. The natural inclusion $O_k(\mathbb{R}) \to O_k(\mathbb{C})$ is a homotopy equivalence, hence the structure group can be further reduced to $O_k(\mathbb{R})$. Note that the subgroup $O_k(\mathbb{R}) \subseteq O_k(\mathbb{C})$ consists of those matrices whose action on $\mathbb{C}^k$ commutes with the standard complex conjugation on $\mathbb{C}^k$. The existence of the desired complex anti-linear involution on E follows immediately. □

We fix a complex conjugation as in Lemma 2.6. Then

$$\langle e_1, e_2 \rangle := b(e_1, \bar{e}_2) \tag{21}$$

defines a fiber wise (positive definite) Hermitian structure on E. We will write $|e|$ for the associated fiber wise norm. Note that

$$b(\bar{e}_1, \bar{e}_2) = \overline{b(e_1, e_2)}, \quad \langle \bar{e}_1, \bar{e}_2 \rangle = \overline{\langle e_1, e_2 \rangle} \quad \text{and} \quad |\bar{e}| = |e|. \tag{22}$$

Moreover, this Hermitian structure is parallel over B, with respect to ∇^E.

Remark 2.7. Note that $F := \{e \in E \mid \bar{e} = e\}$ is a real subbundle of E, and that there is a canonical isomorphism $F \otimes \mathbb{C} = E$. The restrictions of b and $\langle\cdot,\cdot\rangle$ to F coincide, and define a (positive definite) Euclidean structure on F. We can understand both, b and $\langle\cdot,\cdot\rangle$, as complexifications of this Euclidean inner product, once complexifying to a bilinear form and once complexifying to a sesquilinear form.

The fiber wise Hermitian structure on E induces a Hermitian inner product on the Morse complex $C(X; E_u)$ in an obvious way. For $a_1, a_2 \in C(X; E_u)$ we will denote this by $\langle a_1, a_2\rangle_{\mathcal{X}}$. Similarly we will write $|a|_{\mathcal{X}}$ for the associated norm, $a \in C(X; E_u)$. Moreover, the fiber wise complex conjugation on E induces a complex conjugation on $C(X; E_u)$. For $a, a_1, a_2 \in C(X; E_u)$ we have, see (21) and (22),

$$\langle a_1, a_2\rangle_{\mathcal{X}} = b_{\mathcal{X}}(a_1, \bar{a}_2) \tag{23}$$

as well as

$$b_{\mathcal{X}}(\bar{a}_1, \bar{a}_2) = \overline{b_{\mathcal{X}}(a_1, a_2)}, \quad \langle \bar{a}_1, \bar{a}_2\rangle_{\mathcal{X}} = \overline{\langle a_1, a_2\rangle_{\mathcal{X}}} \quad \text{and} \quad |\bar{a}|_{\mathcal{X}} = |a|_{\mathcal{X}}. \tag{24}$$

Using the Riemannian metric g, we obtain an induced fiber wise Hermitian inner product on $\Lambda^*T^*M \otimes E$ which will be denoted by $\langle v, w\rangle_g$, $v, w \in \Omega(M; E_u)$. Then

$$\langle\!\langle v, w\rangle\!\rangle := \int_M \langle v, w\rangle_g \operatorname{vol}_g, \qquad v, w \in \Omega(M; E_u)$$

is a Hermitian inner product on $\Omega(M; E_u)$. We will write $\|v\|$ for the associated L_2–norm, $v \in \Omega(M; E_u)$. The complex conjugation induces a complex conjugation on $\Omega(M; E_u)$. For $v, w \in \Omega(M; E_u)$ we clearly have, see (21) and (22),

$$\langle\!\langle v, w\rangle\!\rangle = \beta(v, \bar{w}) \tag{25}$$

as well as

$$\beta(\bar{v}, \bar{w}) = \overline{\beta(v, w)}, \quad \langle\!\langle \bar{v}, \bar{w}\rangle\!\rangle = \overline{\langle\!\langle v, w\rangle\!\rangle} \quad \text{and} \quad \|\bar{v}\| = \|v\|. \tag{26}$$

From $\beta(\Delta_u v, w) = \beta(v, \Delta_u w)$ we thus also obtain

$$\langle\!\langle \Delta_u v, w\rangle\!\rangle = \langle\!\langle \Delta_u \bar{w}, \bar{v}\rangle\!\rangle = \langle\!\langle v, \overline{\Delta_u \bar{w}}\rangle\!\rangle, \qquad v, w \in \Omega(M; E_u). \tag{27}$$

Remark 2.8. It follows from Lemma 2.4 that, over B, the Laplacian Δ_u commutes with the complex conjugation, that is

$$\Delta_u \bar{w} = \overline{\Delta_u w}, \qquad w \in \Omega(M; E_u),\ \operatorname{supp} w \subseteq B.$$

From (27) we thus get

$$\langle\!\langle \Delta_u v, w\rangle\!\rangle = \langle\!\langle v, \Delta_u w\rangle\!\rangle, \qquad v, w \in \Omega(M; E_u),\ \operatorname{supp} w \subseteq B.$$

It also follows from Lemma 2.4 that over B, the Laplacian Δ_u coincides with the selfadjoint Witten Laplacian associated to the compatible Hermitian structure, see [2] or [8].

Construction of approximate small eigen forms. In this section we will construct as in [14], cf. also [8], finite dimensional subspaces $V_u \subseteq \Omega(M;E_u)$ which approximate $\Omega_{\mathrm{sm}}(M;E_u)$ as $u \to \infty$. The estimates in the section show that Δ_u has small norm on V_u and is large on the orthogonal complement of V_u.

For a critical point $x \in \mathcal{X}$ and $e \in E_x$ we let $\tilde{e} \in \Gamma(E|_{B_x})$ denote the unique parallel section, i.e. $\nabla^E \tilde{e} = 0$, satisfying $\tilde{e}(x) = e$. Moreover, we introduce the differential form

$$\Omega_x^- := d\varphi_x^1 \wedge \cdots \wedge d\varphi_x^{\mathrm{ind}(x)} \in \Omega^{\mathrm{ind}(x)}(B_x;\mathbb{R}).$$

Choose a smooth function $\sigma\colon \mathbb{R} \to [0,1]$ such that $\sigma(t) = 1$ for all $t \leq \rho/3$ and $\sigma(t) = 0$ for all $t \geq 2\rho/3$. For $u \geq 0$, consider the smooth form, see (20),

$$\phi_{u,e} := (\sigma \circ r_x) e^{-ur_x^2/2} \Omega_x^- \otimes \tilde{e} \in \Omega(B_x;E_u). \tag{28}$$

Since $\phi_{u,e}$ has compact support contained in B_x, we can consider it as a globally defined form, $\phi_{u,e} \in \Omega(M;E_u)$.

Definition 2.9. For $u \geq 0$ we let $V_u \subseteq \Omega(M;E_u)$ denote the finite dimensional subspace spanned by the forms $\phi_{u,e}$ where x runs through $\mathcal{X}$ and e runs through (a basis) of E_x.

Observation 2.10. *The complex conjugation preserves V_u. The β- orthogonal complement of V_u coincides with its Hermitian orthogonal complement. It will be denoted by $V_u^\perp \subseteq \Omega(M;E_u)$.*

Proof. The first assertion is immediate from the definition of V_u and the fact that the complex conjugation is parallel over B, see Lemma 2.6. The second claim then follows from (25). □

For $k \in \mathbb{N}$ let $\|w\|_{C^k}$ denote the (a fixed) C^k–norm of $w \in \Omega(M;E_u)$. The following estimates follow easily from the structure of Δ_u in the neighborhood of critical points, cf. Lemma 2.4 and Remark 2.5.

Lemma 2.11. *There exist constants $u_{1,k} \geq 1$ and $\varepsilon_{1,k} > 0$ so that for all $k \in \mathbb{N}$, $u \geq u_{1,k}$ and $v \in V_u$ we have*

$$\|\Delta_u v\|_{C^k} \leq e^{-\varepsilon_{1,k} u} \|v\|.$$

Proof. Let $x \in \mathcal{X}$ and $e \in E_x$. An elementary computation, see Remark 2.5, shows

$$(\Delta_0 - un + u^2 r_x^2) e^{-ur_x^2/2} \Omega_x^- \otimes \tilde{e} = 0.$$

We conclude, see (28), that

$$\operatorname{supp}\bigl((\Delta_0 - un + u^2 r_x^2)\phi_{u,e}\bigr) \subseteq R_x,$$

where $R_x := \varphi_x^{-1}\bigl(\{z \in \mathbb{R}^n \mid \rho/3 \leq |z| \leq 2\rho/3\}\bigr)$. Note that there exist constants $C_k \geq 0$ and $\varepsilon' > 0$ so that, for all $u \geq 0$,

$$\bigl\|(\Delta_0 - un + u^2 r_x^2)\phi_{u,e}\bigr\|_{C^k} \leq C_k e^{-\varepsilon' u} |e|.$$

Together with $(N_x^+ + \operatorname{ind}(x) - N_x^-)\Omega_x^- = 0$ and the formula in Lemma 2.4 we see that, for all $u \geq 0$,

$$\big\|\Delta_u \phi_{u,e}\big\|_{C^k} \leq C_k e^{-\varepsilon' u}|e|. \tag{29}$$

Clearly,

$$\|\phi_{u,e}\|^2 = \int_{B_x} (\sigma \circ r_x)^2 e^{-ur_x^2}|e|^2 \operatorname{vol}_g = \int_{\mathbb{R}^n} \sigma(|z|)^2 e^{-u|z|^2} dz |e|^2,$$

and so there exist constants $C \geq 0$ and $\varepsilon'' > 0$ so that for all $u \geq 1$

$$\Big|\|\phi_{u,e}\| - (\pi/u)^{n/4}|e|\Big| \leq Ce^{-\varepsilon'' u}|e|.$$

Combining this with (29) we find constants $u_k \geq 1$ and $\varepsilon_{1,k} > 0$ so that for $u \geq u_k$, $x \in \mathcal{X}$ and $e \in E_x$ we have

$$\big\|\Delta_u \phi_{u,e}\big\|_{C^k} \leq e^{-\varepsilon_{1,k} u}\|\phi_{u,e}\|.$$

The lemma now follows from the fact that for $x_1, x_2 \in \mathcal{X}$, $x_1 \neq x_2$, and $e_1 \in E_{x_1}$, $e_2 \in E_{x_2}$ the forms ϕ_{u,e_1} and ϕ_{u,e_2} have disjoint support. □

Lemma 2.12. *There exist constants $u_2 \geq 1$ and $\varepsilon_2 > 0$ so that for $u \geq u_2$, $v \in V_u$ and $w \in \Omega(M; E_u)$ we have*

$$|\langle\!\langle \Delta_u v, w\rangle\!\rangle| \leq e^{-\varepsilon_2 u}\|v\|\|w\| \qquad \textit{and} \qquad |\langle\!\langle v, \Delta_u w\rangle\!\rangle| \leq e^{-\varepsilon_2 u}\|v\|\|w\|.$$

Proof. The first statement follows immediately from the Cauchy–Schwarz inequality and Lemma 2.11 for $k = 0$. To see the second inequality, recall from Observation 2.10 that V_u is invariant under the complex conjugation. Hence (27), the first statement and (26) imply

$$|\langle\!\langle v, \Delta_u w\rangle\!\rangle| = |\langle\!\langle \Delta_u \bar v, \bar w\rangle\!\rangle| \leq e^{-\varepsilon_2 u}\|\bar v\|\|\bar w\| = e^{-\varepsilon_2 u}\|v\|\|w\|. \qquad □$$

Introduce the smooth cut-off function, see (20) and (28),

$$\chi\colon M \to [0,1], \quad \chi(y) := \sum_{x \in \mathcal{X}} \sigma \circ r_x.$$

Note that $\chi = 1$ in a neighbourhood of $\mathcal{X}$ and $\operatorname{supp}\chi \subseteq B$, see (19).

Lemma 2.13. *There exist constants $u_3' \geq 1$ and $\varepsilon_3' > 0$ so that for any $u \geq u_3'$ and $v' \in V_u^\perp$ we have*

$$\operatorname{Re}\langle\!\langle \Delta_u(\chi v'), \chi v'\rangle\!\rangle \geq \varepsilon_3' u \|\chi v'\|^2. \tag{30}$$

Proof. It suffices to establish the result in $\mathbb{R}^n$ and for the standard Witten Laplacian only, see Remark 2.8. This is done in [8, Appendix A.2, equation (5.7)]. □

Lemma 2.14. *There exist constants $u_3'' \geq 1$ and $\varepsilon_3'' > 0$ so that for all $w \in \Omega(M; E_u)$ with $\operatorname{supp} w \subseteq \overline{\{y \in M \mid \chi(y) \neq 1\}}$ we have*

$$\operatorname{Re}\langle\!\langle \Delta_u w, w\rangle\!\rangle \geq \varepsilon_3'' u^2 \|w\|^2.$$

Proof. Since $\Delta_0 + \Delta_0^*$ is a selfadjoint operator whose principal symbol is positive definite it follows, see [22, Corollary 9.3], that $\Delta_0 + \Delta_0^*$ is bounded from below, i.e. there exists a constant $C \geq 0$ so that for all $w \in \Omega(M; E_u)$

$$2\,\mathrm{Re}\langle\langle \Delta_0 w, w\rangle\rangle = \langle\langle (\Delta_0 + \Delta_0^*)w, w\rangle\rangle \geq -C\|w\|^2. \tag{31}$$

Using the fact that on $\overline{\{y \in M \mid \chi(y) \neq 1\}}$ the function $|df|^2$ is strictly positive, we conclude from Lemma 2.4 that there exists a constant $\varepsilon > 0$ so that for sufficiently large u and all $w \in \Omega(M; E_u)$ satisfying $\mathrm{supp}\, w \subseteq \overline{\{y \in M \mid \chi(y) \neq 1\}}$ we have

$$\mathrm{Re}\langle\langle \Delta_u - \Delta_0)w, w\rangle\rangle \geq \varepsilon u^2 \|w\|^2. \tag{32}$$

Combining (31) and (32) the lemma follows easily. □

Lemma 2.15. *There exist constants $u_3''' \geq 1$ and $C_3''' \geq 0$ so that for any $u \geq u_3'''$ and $w \in \Omega(M; E_u)$ we have*

$$\mathrm{Re}\langle\langle \Delta_u(\chi w), (1-\chi)w\rangle\rangle \geq -C_3'''\|w\|^2. \tag{33}$$

Proof. It suffices to show this inequality for $w' \in \Omega(M; E_u)$ with $\mathrm{supp}\, w' \subseteq B$. Indeed, choose a smooth cut-off function $\eta\colon M \to [0,1]$ with $\mathrm{supp}\,\eta \subseteq B$ and $\mathrm{supp}\,\chi \subseteq \{y \in M \mid \eta(y) = 1\}$. Consider $w' := \eta w$. Since $\eta\chi = \chi$ we have $\Delta_u(\chi w') = \Delta_u(\chi w)$, and since $\eta = 1$ on $\mathrm{supp}\,\Delta_u(\chi w)$ we obtain

$$\langle\langle \Delta_u(\chi w), (1-\chi)w\rangle\rangle = \langle\langle \Delta_u(\chi w'), (1-\chi)w'\rangle\rangle.$$

Moreover, note that $\|w'\| \leq \|w\|$. Hence, if the desired inequality holds for all $w' \in \Omega(M; E_u)$ with $\mathrm{supp}\, w' \subseteq B$, it will remain true for arbitrary $w \in \Omega(M; E_u)$ with the same constant $C_3''' \geq 0$. In view of Remark 2.8 the Laplacian Δ_u coincides with the standard selfadjoint Witten Laplacian over B. For the latter operator this estimate can be found in [8]. □

The above lemmas imply

Proposition 2.16. *There exist constants $u_3 \geq 1$ and $\varepsilon_3 > 0$ so that for all $u \geq u_3$ and $v' \in V_u^\perp$ we have*

$$\mathrm{Re}\langle\langle \Delta_u v', v'\rangle\rangle \geq \varepsilon_3 u \|v'\|^2.$$

Proof. Suppose $v' \in V_u^\perp$ and write $v' = v_1' + v_2'$ with $v_1' := \chi v'$ and $v_2' := (1-\chi)v'$. In view of Remark 2.8 and since $\mathrm{supp}\, v_1' \subseteq B$ we have

$$\langle\langle \Delta_u v_2', v_1'\rangle\rangle = \langle\langle v_2', \Delta_u v_1'\rangle\rangle = \overline{\langle\langle \Delta_u v_1', v_2'\rangle\rangle}.$$

Therefore $\mathrm{Re}\langle\langle \Delta_u v_2', v_1'\rangle\rangle = \mathrm{Re}\langle\langle \Delta_u v_1', v_2'\rangle\rangle$ and thus

$$\mathrm{Re}\langle\langle \Delta_u v', v'\rangle\rangle = \mathrm{Re}\langle\langle \Delta_u v_1', v_1'\rangle\rangle + 2\,\mathrm{Re}\langle\langle \Delta_u v_1', v_2'\rangle\rangle + \mathrm{Re}\langle\langle \Delta_u v_2', v_2'\rangle\rangle. \tag{34}$$

By Lemma 2.13 we have

$$\mathrm{Re}\langle\langle \Delta_u v_1', v_1'\rangle\rangle \geq \varepsilon_3' u \|v_1'\|^2, \qquad u \geq u_3'. \tag{35}$$

Since $\mathrm{supp}\, v_2' \subseteq \overline{\{y \in M \mid \chi \neq 1\}}$, Lemma 2.14 implies

$$\mathrm{Re}\langle\langle \Delta_u v_2', v_2'\rangle\rangle \geq \varepsilon_3'' u^2 \|v_2'\|^2, \qquad u \geq u_3''. \tag{36}$$

From Lemma 2.15 we get

$$\operatorname{Re}\langle\!\langle \Delta_u v_1', v_2' \rangle\!\rangle \geq -C_3'''\|v'\|^2, \qquad u \geq u_3'''. \tag{37}$$

Combining (34), (35), (36) and (37) we obtain

$$\operatorname{Re}\langle\!\langle \Delta_u v', v' \rangle\!\rangle \geq \varepsilon_3' u\|v_1'\|^2 + \varepsilon_3'' u^2\|v_2'\|^2 - C_3'''\|v'\|^2 \tag{38}$$

for all $u \geq \max\{u_3', u_3'', u_3'''\}$. Note that adding $\|v'\|^2 = \|v_1'\|^2 + \|v_2'\|^2 + 2\operatorname{Re}\langle\!\langle v_1', v_2' \rangle\!\rangle$ and $0 \leq \|v_1' - v_2'\|^2 = \|v_1'\|^2 + \|v_2'\|^2 - 2\operatorname{Re}\langle\!\langle v_1', v_2' \rangle\!\rangle$ yields

$$\|v'\|^2 \leq 2\big(\|v_1'\|^2 + \|v_2'\|^2\big).$$

Combining this with (38) the statement of the proposition follows immediately. □

The spectral gap. The estimates in the preceding section permit to establish a widening gap in the spectrum of Δ_u, as $u \to \infty$. For the precise statement see Proposition 2.18 below. This result is a generalization of a well known “separation of the spectrum property” in the selfadjoint situation, see for instance [8, Proposition 5.2]. We start with the following resolvent estimate.

Lemma 2.17. *There exist constants $u_4 \geq 1$ and $\varepsilon_4 > 0$ so that for all $u \geq u_4$, $\lambda \in \mathbb{C}$ and $w \in \Omega(M; E_u)$ we have*

$$\min\big\{|\lambda| - e^{-\varepsilon_4 u}, \varepsilon_4 u - \operatorname{Re}\lambda\big\}\|w\| \leq \|(\Delta_u - \lambda)w\|.$$

Proof. For $u \geq 0$ let us write

$$\pi_u\colon \Omega(M; E_u) \to V_u \qquad \text{and} \qquad \pi_u^\perp\colon \Omega(M; E_u) \to V_u^\perp$$

for the orthogonal projections onto V_u and $V_u^\perp$, respectively. Let

$$\Delta_u = \begin{pmatrix} \Delta_{1,u} & \Delta_{2,u} \\ \Delta_{3,u} & \Delta_{4,u} \end{pmatrix}$$

denote the decomposition of Δ_u with respect to $\Omega(M; E_u) = V_u \oplus V_u^\perp$. More precisely, we have:

$$\begin{aligned}
&\Delta_{1,u}\colon V_u \to V_u && \Delta_{1,u} := \pi_u \Delta_u|_{V_u} \\
&\Delta_{2,u}\colon V_u^\perp \to V_u && \Delta_{2,u} := \pi_u \Delta_u|_{V_u^\perp} \\
&\Delta_{3,u}\colon V_u \to V_u^\perp && \Delta_{3,u} := \pi_u^\perp \Delta_u|_{V_u} \\
&\Delta_{4,u}\colon V_u^\perp \to V_u^\perp && \Delta_{4,u} := \pi_u^\perp \Delta_u|_{V_u^\perp}
\end{aligned}$$

Define operators

$$A_u := \begin{pmatrix} 0 & 0 \\ 0 & \Delta_{4,u} \end{pmatrix} \qquad \text{and} \qquad B_u := \begin{pmatrix} \Delta_{1,u} & \Delta_{2,u} \\ \Delta_{3,u} & 0 \end{pmatrix}.$$

Clearly, $\Delta_u = A_u + B_u$. ¿From Lemma 2.12 we can see that for $u \geq u_2$ and $w \in \Omega(M; E_u)$ we have

$$\|B_u w\| \leq 2e^{-\varepsilon_2 u}\|w\|. \tag{39}$$

Indeed, using $\pi_u B_u w = \pi_u \Delta_u w$ and $\pi_u^\perp B_u w = \pi_u^\perp \Delta_u(\pi_u w)$ we obtain:

$$\begin{aligned}\|B_u w\|^2 &= \langle\!\langle \pi_u B_u w, \pi_u B_u w\rangle\!\rangle + \langle\!\langle \pi_u^\perp B_u w, \pi_u^\perp B_u w\rangle\!\rangle \\ &= \langle\!\langle \Delta_u w, \pi_u B_u w\rangle\!\rangle + \langle\!\langle \Delta_u(\pi_u w), \pi_u^\perp B_u w\rangle\!\rangle \\ &\leq e^{-\varepsilon_2 u}\|w\|\|\pi_u B_u w\| + e^{-\varepsilon_2 u}\|\pi_u w\|\|\pi_u^\perp B_u w\| \\ &\leq 2e^{-\varepsilon_2 u}\|w\|\|B_u w\|\end{aligned}$$

From Proposition 2.16 and the Cauchy–Schwarz inequality we get

$$\begin{aligned}&\|(\Delta_{4,u} - \lambda)v'\|\|v'\| \geq |\langle\!\langle(\Delta_{4,u} - \lambda)v', v'\rangle\!\rangle| = |\langle\!\langle(\Delta_u - \lambda)v', v'\rangle\!\rangle| \\ &\quad\geq \operatorname{Re}\langle\!\langle(\Delta_u - \lambda)v', v'\rangle\!\rangle = \operatorname{Re}\langle\!\langle \Delta_u v', v'\rangle\!\rangle - \operatorname{Re}\lambda\|v'\|^2 \geq (\varepsilon_3 u - \operatorname{Re}\lambda)\|v'\|^2\end{aligned}$$

and thus

$$\|(\Delta_{4,u} - \lambda)v'\| \geq (\varepsilon_3 u - \operatorname{Re}\lambda)\|v'\|$$

for $u \geq u_3$, all $\lambda \in \mathbb{C}$ and $v' \in V_u^\perp$. We conclude that

$$\|(A_u - \lambda)w\| \geq \min\{|\lambda|, \varepsilon_3 u - \operatorname{Re}\lambda\}\|w\| \tag{40}$$

for $u \geq u_3$, all $\lambda \in \mathbb{C}$ and $w \in \Omega(M; E_u)$. Combining (39) and (40) we find

$$\|(\Delta_u - \lambda)w\| \geq \|(A_u - \lambda)w\| - \|B_u w\| \geq \big(\min\{|\lambda|, \varepsilon_3 u - \operatorname{Re}\lambda\} - 2e^{-\varepsilon_2 u}\big)\|w\|$$

for sufficiently large u, all $\lambda \in \mathbb{C}$ and $w \in \Omega(M; E_u)$. The statement now follows with an appropriate choice of ε_4 and u_4. □

Proposition 2.18. *Let $\varepsilon_4 > 0$ be the constant from Lemma 2.17. Then, for sufficiently large u, we have*

$$\operatorname{Spec}(\Delta_u) \subseteq \big\{\lambda \in \mathbb{C} \;\big|\; |\lambda| \leq e^{-\varepsilon_4 u}\big\} \cup \big\{\lambda \in \mathbb{C} \;\big|\; \operatorname{Re}\lambda \geq \varepsilon_4 u\big\}.$$

Proof. Suppose $\lambda \in \operatorname{Spec}(\Delta_u)$. Then there exists $w \in \Omega(M; E_u)$ with $(\Delta_u - \lambda)w = 0$ and $\|w\| \neq 0$. Assuming u is sufficiently large, see Lemma 2.17, we conclude $\min\big\{|\lambda| - e^{-\varepsilon_4 u}, \varepsilon_4 u - \operatorname{Re}\lambda\big\} \leq 0$, and the statement follows. □

Approximation of the small complex. We will now show that, as $u \to \infty$, the subspace $V_u \subseteq \Omega(M; E_u)$ approximates $\Omega_{\mathrm{sm}}(M; E_u)$ well with respect to every C^k–norm on $\Omega(M; E_u)$. For the precise statements see Proposition 2.21 and Proposition 2.23 below. Recall that $\Omega_{\mathrm{sm}}(M; E_u)$ denotes the sum of eigen spaces of Δ_u whose corresponding eigen values have real part at most 1.

For $s \in \mathbb{N}$ and $w \in \Omega(M; E_u)$ let $\|w\|_s$ denote the (a fixed) Sobolev s-norm. We continue to write $\|w\| = \|w\|_0$. We will start with the following improvement of the resolvent estimate in Lemma 2.17.

Lemma 2.19. *Let $u_4 \geq 1$ and $\varepsilon_4 > 0$ be the constants from Lemma 2.17. For every $s \in \mathbb{N}$ there exists a constant $C_{4,s} \geq 0$ so that for all $u \geq u_4$, $\lambda \in \mathbb{C}$ and $w \in \Omega(M; E_u)$ we have*

$$\min\big\{|\lambda| - e^{-\varepsilon_4 u}, \varepsilon_4 u - \operatorname{Re}\lambda\big\}\|w\|_s \leq C_{4,s}(u^2 + |\lambda|)^s\|(\Delta_u - \lambda)w\|_s. \tag{41}$$

Moreover, there exists a constant $\tilde{C}_{4,2s} \geq 0$ *so that for all* $u \geq u_4$, $\lambda \in \mathbb{C}$ *and* $\omega \in \Omega(M; E_u)$ *we even have*

$$\min\bigl\{|\lambda| - e^{-\varepsilon_4 u}, \varepsilon_4 u - \operatorname{Re}\lambda\bigr\}\|w\|_{2s} \leq \tilde{C}_{4,2s}(u^2 + |\lambda|)^s \|(\Delta_u - \lambda)w\|_{2s}. \quad (42)$$

Proof. We will construct the constants $C_{4,s} \geq 0$ by induction on s. For $s = 0$ the statement was proved in Lemma 2.17. The induction is based on an argument that is used in the selfadjoint situation too, see [8, proof of Proposition 5.4] or [2, proof of Theorem 8.8].

From the ellipticity of Δ_0 we get constants $C'_s \geq 0$ so that for all $w \in \Omega(M; E_u)$

$$\|w\|_{s+1} \leq \|w\|_{s+2} \leq C'_s\bigl(\|\Delta_0 w\|_s + \|w\|_s\bigr). \quad (43)$$

From Lemma 2.4 we obtain constants $C''_s \geq 0$ such that for all $u \geq u_4$ and $w \in \Omega(M; E_u)$

$$\|(\Delta_u - \Delta_0)w\|_s \leq C''_s u^2 \|w\|_s. \quad (44)$$

Combining (43) and (44) we obtain constants $C'''_s \geq 0$ so that for all $u \geq u_4$, $\lambda \in \mathbb{C}$ and $w \in \Omega(M; E_u)$ we have

$$\|w\|_{s+1} \leq C'''_s\Bigl(\|(\Delta_u - \lambda)w\|_s + (u^2 + |\lambda|)\|w\|_s\Bigr). \quad (45)$$

By induction we may assume that there exists a constant $C_{4,s} \geq 0$ so that (41) holds. Combining (41) with (45) and using $\min\bigl\{|\lambda| - e^{-\varepsilon_4 u}, \varepsilon_4 u - \operatorname{Re}\lambda\bigr\} \leq |\lambda|$ we find a constant $C_{4,s+1} \geq 0$ so that for all $u \geq u_4$, $\lambda \in \mathbb{C}$ and $\omega \in \Omega(M; E_u)$

$$\min\bigl\{|\lambda| - e^{-\varepsilon_4 u}, \varepsilon_4 u - \operatorname{Re}\lambda\bigr\}\|w\|_{s+1} \leq C_{4,s+1}(u^2 + |\lambda|)^{s+1}\|(\Delta_u - \lambda)w\|_s$$

Now use $\|(\Delta_u - \lambda)w\|_s \leq \|(\Delta_u - \lambda)w\|_{s+1}$ to complete the induction. The proof of (42) is similar. □

Let $Q_u \colon \Omega(M; E_u) \to \Omega_{\mathrm{sm}}(M; E_u)$ denote the spectral projection.

Lemma 2.20. *For every* $s \in \mathbb{N}$ *there exist constants* $u_{5,s} \geq 1$ *and* $\varepsilon_{5,s} > 0$ *so that for all* $u \geq u_{5,s}$ *and* $v \in V_u$ *we have*

$$\|Q_u v - v\|_s \leq e^{-\varepsilon_{5,s} u}\|v\|.$$

Proof. By Proposition 2.18, for sufficiently large u, we have $\operatorname{Spec}(\Delta_u) \cap S^1 = \emptyset$ and hence Q_u is given by the Riesz projector

$$Q_u = \frac{1}{2\pi\mathbf{i}}\int_{S^1}(\lambda - \Delta_u)^{-1} d\lambda.$$

Since $(\lambda - \Delta_u)^{-1} - \lambda^{-1} = \lambda^{-1}(\lambda - \Delta_u)^{-1}\Delta_u$ we obtain, for $v \in \Omega(M; E_u)$,

$$Q_u v - v = \frac{1}{2\pi\mathbf{i}}\int_{S^1}\lambda^{-1}(\lambda - \Delta_u)^{-1}\Delta_u v\, d\lambda. \quad (46)$$

From Lemma 2.19 and Lemma 2.11 we easily infer the existence of constants $u_{5,s} \geq 1$ and $\varepsilon_{5,s} > 0$ so that for all $s \in \mathbb{N}$, $u \geq u_{5,s}$, $\lambda \in S^1$ and $v \in V_u$ we have

$$\|\lambda^{-1}(\lambda - \Delta_u)^{-1}\Delta_u v\|_s \leq e^{-\varepsilon_{5,s} u}\|v\|.$$

The lemma now follows easily by combining this estimate with (46). □

Proposition 2.21. *There exist constants $u_{6,k} \geq 1$ and $\varepsilon_{6,k} > 0$ so that for $k \in \mathbb{N}$, $u \geq u_{6,k}$ and $v \in V_u$ we have*

$$\|Q_u v - v\|_{C^k} \leq e^{-\varepsilon_{6,k} u}\|v\|.$$

Proof. This follows from Lemma 2.20 and the Sobolev embedding theorem. □

In the selfadjoint case the following estimate is an immediate consequence of Proposition 2.18. In our situation we will have to use the resolvent estimate from Lemma 2.19.

Lemma 2.22. *For every $s \in \mathbb{N}$ there exist constants $u_{7,s} \geq 1$ and $\varepsilon_{7,s} > 0$ so that for all $u \geq u_{7,s}$ and all $w \in \Omega_{\mathrm{sm}}(M; E_u)$ we have*

$$\|\Delta_u w\|_s \leq e^{-\varepsilon_{7,s} u}\|w\|_s.$$

Proof. Let $\varepsilon_4 > 0$ be the constant from Lemma 2.17, and set $\rho_u := 2e^{-\varepsilon_4 u}$. Assume $u \geq 0$ is sufficiently large so that all eigen values with real part at most 1 are contained in the interior of the circle $\rho_u S^1$ of radius ρ_u, see Proposition 2.18. Then

$$\Delta_u Q_u = Q_u \Delta_u = \frac{1}{2\pi \mathbf{i}} \int_{\rho_u S^1} \lambda(\lambda - \Delta_u)^{-1} d\lambda. \tag{47}$$

For $\lambda \in \rho_u S^1$ and sufficiently large u, Lemma 2.19, see (41), provides the estimate

$$\|\lambda(\lambda - \Delta_u)^{-1} w\|_s \leq 2C_{4,s}(u^2 + 2)^s \|w\|_s, \qquad w \in \Omega(M; E_u).$$

Combining this with (47) we obtain, for $w \in \Omega(M; E_u)$,

$$\|\Delta_u Q_u w\|_s \leq 2C_{4,s}(u^2 + 2)^s \rho_u \|w\|_s.$$

Choosing $\varepsilon_{7,s} > 0$ and $u_{7,s} \geq 1$ appropriately we get

$$\|\Delta_u Q_u w\|_s \leq e^{-\varepsilon_{7,s} u}\|w\|_s$$

for all $u \geq u_{7,s}$ and $w \in \Omega(M; E_u)$. The statement follows immediately. □

Proposition 2.23. *For sufficiently large u the restriction of the spectral projection $Q_u \colon V_u \to \Omega_{\mathrm{sm}}(M; E_u)$ is an isomorphism.*

Proof. Recall that $\Omega_{\mathrm{sm}}(M; E_u)$ is the image of the spectral projection Q_u, and that $\beta(Q_u v, w) = \beta(v, Q_u w)$ for all $v, w \in \Omega(M; E_u)$. Hence,

$$\Omega_{\mathrm{sm}}(M; E_u) \cap (Q_u V_u)^{\perp_\beta} = \Omega_{\mathrm{sm}}(M; E_u) \cap V_u^{\perp}. \tag{48}$$

Here $(Q_u V_u)^{\perp_\beta}$ denotes the β-orthogonal complement of $Q_u V_u$, but see also Observation 2.10. It follows from Proposition 2.16 and Lemma 2.22 that $\Omega_{\mathrm{sm}}(M; E_u) \cap V_u^{\perp} = 0$ for sufficiently large u. Together with (48) this shows that $Q_u V_u = \Omega_{\mathrm{sm}}(M; E_u)$, and therefore $Q_u \colon V_u \to \Omega_{\mathrm{sm}}(M; E_u)$ is onto, for sufficiently large u. The injectivity follows from Lemma 2.20 with $s = 0$. □

The integration on the small complex. For the sake of notational simplicity let us introduce the notation, $u > 0$,

$$I_u := \eta_u \mathrm{Int}_u|_{V_u} \colon\ V_u \to C(X; E_u)$$
$$I_{u,\mathrm{sm}} := \eta_u \, \mathrm{Int}_{u,\mathrm{sm}} \colon\ \Omega_{\mathrm{sm}}(M; E_u) \to C(X; E_u)$$

where $\eta_u \colon\ C(X; E_u) \to C(X; E_u)$ denotes the scaling isomorphism introduced at the beginning of this section, see (12). Moreover, we will write $\beta_u := \beta|_{V_u}$ for the restriction of the bilinear form β to V_u, and we will continue to write $\beta_{u,\mathrm{sm}} = \beta|_{\Omega_{\mathrm{sm}}(M;E_u)}$ for the restriction of the bilinear form β to $\Omega_{\mathrm{sm}}(M; E_u)$. Recall that we write $\langle\cdot,\cdot\rangle_{\mathcal{X}}$ and $|\cdot|_{\mathcal{X}}$ for the Hermitian inner product and its associated norm on $C(X; E_u)$. Finally, we recall that $b_{\mathcal{X}}$ denotes the bilinear form on $C(X; E_u)$, see (23).

Lemma 2.24. *For sufficiently large u the mapping $I_u \colon\ V_u \to C(X; E_u)$ is an isomorphism. Moreover, there exists a constant $\varepsilon_8 > 0$ so that, as $u \to \infty$,*

$$(I_u)_* \langle\!\langle \cdot, \cdot \rangle\!\rangle = \langle \cdot, \cdot \rangle_{\mathcal{X}} + \mathrm{O}(e^{-\varepsilon_8 u}) \tag{49}$$

and

$$(I_u)_* \beta_u = b_{\mathcal{X}} + \mathrm{O}(e^{-\varepsilon_8 u}) \tag{50}$$

Proof. We claim that there exist constants $u_8 \geq 1$ and $\varepsilon_8 > 0$ so that for all $u \geq u_8$ and $v_1, v_2 \in V_u$ we have

$$\Big| \langle I_u v_1, I_u v_2 \rangle_{\mathcal{X}} - \langle\!\langle v_1, v_2 \rangle\!\rangle \Big| \leq e^{-\varepsilon_8 u} |I_u v_1|_{\mathcal{X}} |I_u v_2|_{\mathcal{X}}. \tag{51}$$

If such estimate is established we immediately see that $I_u \colon\ V_u \to C(X; E_u)$ is injective for $u \geq u_8$. Since V_u and $C(X; E_u)$ obviously have the same dimension, the first assertion of the lemma follows. ¿From (51) we obtain, for $u \geq u_8$ and $a_1, a_2 \in C(X; E_u)$,

$$\Big| \langle a_1, a_2 \rangle_{\mathcal{X}} - \langle\!\langle I_u^{-1} a_1, I_u^{-1} a_2 \rangle\!\rangle \Big| \leq e^{-\varepsilon_8 u} |a_1|_{\mathcal{X}} |a_2|_{\mathcal{X}}.$$

from which we infer (49). Combining the last equation with (23), (24), (25) and using the fact that $I_u \colon\ V_u \to C(X; E_u)$ intertwines the complex conjugations we also obtain, for $a_1, a_2 \in C(X; E_u)$,

$$\Big| b_{\mathcal{X}}(a_1, a_2) - \beta_u(I_u^{-1} a_1, I_u^{-1} a_2) \Big| \leq e^{-\varepsilon_8 u} |a_1|_{\mathcal{X}} |a_2|_{\mathcal{X}}$$

and thus (50).

It thus remains to establish the estimate (51). To do so, suppose $x \in \mathcal{X}$ and set $q := \mathrm{ind}(x)$. For $e_1, e_2 \in E_x$ we have, see (28),

$$\langle\!\langle \phi_{u,e_1}, \phi_{u,e_2} \rangle\!\rangle = \int_{B_x} (\sigma \circ r_x)^2 e^{-u r_x^2} \langle e_1, e_2 \rangle \, \mathrm{vol}_g = \int_{\mathbb{R}^n} \sigma(|z|)^2 e^{-u|z|^2} dz \langle e_1, e_2 \rangle$$

and thus there exist constants $C' \geq 0$ and $\varepsilon' > 0$ so that for all $u \geq 1$ and $e_1, e_2 \in E_x$

$$\Big| \langle\!\langle \phi_{u,e_1}, \phi_{u,e_2} \rangle\!\rangle - (\pi/u)^{n/2} \langle e_1, e_2 \rangle \Big| \leq C' e^{-u\varepsilon'} |e_1||e_2| \tag{52}$$

Further, keeping (18), (5) and (9) in mind, we have for $e_1, e_2 \in E_x$,

$$\begin{aligned}\big\langle I_u\phi_{u,e_1}, &I_u\phi_{u,e_2}\big\rangle_{\mathcal{X}}\\ &= \left((\pi/u)^{n/4-q/2}\int_{W_x^-\cap B_x}(\sigma\circ r_x)e^{-ur_x^2/2}e^{u(f-f(x))}\Omega_x^-\right)^2\langle e_1,e_2\rangle\\ &= (\pi/u)^{n/2-q}\left(\int_{\mathbb{R}^q}\sigma(|z|)e^{-u|z|^2}dz\right)^2\langle e_1,e_2\rangle\end{aligned}$$

and hence there exist constants $C'' \geq 0$ and $\varepsilon'' > 0$ so that for all $u \geq 1$ and $e_1, e_2 \in E_x$

$$\left|\big\langle I_u\phi_{u,e_1}, I_u\phi_{u,e_2}\big\rangle_{\mathcal{X}} - (\pi/u)^{n/2}\langle e_1,e_2\rangle\right| \leq C''e^{-u\varepsilon''}|e_1||e_2|. \tag{53}$$

Note that this estimate also implies, for $e \in E_x$ and $u \geq 1$

$$\Big((\pi/u)^{n/2} - C''e^{-u\varepsilon''}\Big)|e|^2 \leq |I_u\phi_{u,e}|^2_{\mathcal{X}}. \tag{54}$$

Combining (52), (53) and (54) we find constants $u_8 \geq 1$ and $\varepsilon_8 > 0$ so that for all $u \geq u_8$, $x \in \mathcal{X}$, and $e_1, e_2 \in E_x$ we have

$$\left|\big\langle I_u\phi_{u,e_1}, I_u\phi_{u,e_2}\big\rangle_{\mathcal{X}} - \langle\!\langle \phi_{u,e_1}, \phi_{u,e_2}\rangle\!\rangle\right| \leq e^{-u\varepsilon_8}|I_u\phi_{u,e_1}|_{\mathcal{X}}|I_u\phi_{u,e_2}|_{\mathcal{X}}.$$

For $x, y \in \mathcal{X}$, $x \neq y$, $e_1 \in E_x$, $e_2 \in E_y$ we clearly have $\langle I_u\phi_{u,e_1}, I_u\phi_{u,e_2}\rangle = 0 = \langle\!\langle \phi_{u,e_1}, \phi_{u,e_2}\rangle\!\rangle$, see (18). The estimate (51) now follows easily, see Definition 2.9, and the proof is complete. □

Lemma 2.25. *There exist constants $u_9 \geq 1$ and $\varepsilon_9 > 0$ so that for all $u \geq u_9$ and $v \in V_u$ we have*

$$\big|I_{u,\mathrm{sm}}Q_uv - I_uv\big|_{\mathcal{X}} \leq e^{-\varepsilon_9 u}\|v\|.$$

Proof. There exists a constant $C \geq 0$ so that for all $u \geq 0$ and $w \in \Omega(M; E_u)$

$$|\operatorname{Int}_u w|_{\mathcal{X}} \leq C\|w\|_{C^0}.$$

To see this we use the compactification result for the unstable manifolds. The uniformity in u is guaranteed by the relation $\operatorname{Int}_u w = e^{-uf}\operatorname{Int}_0(e^{uf}w)$ and the fact that the function e^{uf} restricted to an unstable manifold W_x^- will attain its maximum at the critical point $x \in \mathcal{X}$. The statement then follows from Proposition 2.21 with $k = 0$. □

Lemma 2.26. *There exist constant $u_{10} \geq 1$ and $\varepsilon_{10} > 0$ so that for all $u \geq u_{10}$ and $v_1, v_2 \in V_u$ we have*

$$\big|\beta_{u,\mathrm{sm}}(Q_uv_1, Q_uv_2) - \beta_u(v_1,v_2)\big| \leq e^{-\varepsilon_{10}u}\|v_1\|\|v_2\|.$$

Proof. From (25), (26) and the Cauchy–Schwarz inequality we obtain

$$|\beta(w_1,w_2)| \leq \|w_1\|\|w_2\|$$

for all $w_1, w_2 \in \Omega(M; E_u)$. Hence:

$$\begin{aligned}\big|\beta(Q_u w_1, Q_u w_2) - \beta(w_1, w_2)\big| &\\ \leq \big|\beta(Q_u w_1 - w_1, Q_u w_2)\big| &+ \big|\beta(w_1, Q_u w_2 - w_2)\big| \\ \leq \|Q_u w_1 - w_1\| \|Q_u w_2\| &+ \|w_1\| \|Q_u w_2 - w_2\| \\ \leq \|Q_u w_1 - w_1\| \big(\|Q_u w_2 - w_2\| &+ \|w_2\|\big) + \|w_1\| \|Q_u w_2 - w_2\|\end{aligned}$$

The statement thus follows from Lemma 2.20 with $s = 0$. □

We are now in the position to put the pieces together and provide a

Proof of Theorem 2.1. From (49) we obtain a constant $C' \geq 0$ so that for sufficiently large u and $a \in C(X; E_u)$ we have

$$\|I_u^{-1} a\| \leq C' |a|_{\mathcal{X}}. \tag{55}$$

Combining this with Lemma 2.25 we obtain constant $\varepsilon' > 0$ so that, as $u \to \infty$,

$$I_{u,\mathrm{sm}} Q_u I_u^{-1} = \mathrm{id}_{C(X;E_u)} + \mathrm{O}(e^{-\varepsilon' u}). \tag{56}$$

Particularly, the mapping $I_{u,\mathrm{sm}} Q_u I_u^{-1} \colon C(X; E_u) \to C(X; E_u)$ is an isomorphism, for sufficiently large u. Hence $I_{u,\mathrm{sm}} \colon \Omega_{\mathrm{sm}}(M; E_u) \to C(X; E_u)$ is an isomorphism for sufficiently large u, see Proposition 2.23 and Lemma 2.24. Since the scaling η_u is an isomorphism for all $u > 0$, we see that $\mathrm{Int}_{u,\mathrm{sm}} = \eta_u^{-1} I_{u,\mathrm{sm}} \colon \Omega_{\mathrm{sm}}(M; E_u) \to C(X; E_u)$ is an isomorphism too. This proves the first claim of Theorem 2.1.

Next note that (55) and (56) provide a constant $C'' \geq 0$ so that for sufficiently large u and $a \in C(X; E_u)$ we have

$$\|(I_{u,\mathrm{sm}} Q_u)^{-1} a\| \leq C'' |a|_{\mathcal{X}}. \tag{57}$$

Combining this with Lemma 2.26 we find a constant $\varepsilon'' > 0$ so that

$$(I_{u,\mathrm{sm}} Q_u)_* \big(Q_u^* \beta_{u,\mathrm{sm}} - \beta_u\big) = \mathrm{O}(e^{-\varepsilon'' u}). \tag{58}$$

From (56) and (50) we find a constant $\varepsilon''' > 0$ so that, as $u \to \infty$,

$$(I_{u,\mathrm{sm}} Q_u)_* \beta_u = (I_{u,\mathrm{sm}} Q_u I_u^{-1})_* (I_u)_* \beta_u = b_{\mathcal{X}} + \mathrm{O}(e^{-\varepsilon''' u}). \tag{59}$$

Clearly, (58) and (59) imply the existence of a constant $\varepsilon > 0$ so that, as $u \to \infty$,

$$(I_{u,\mathrm{sm}})_* \beta_{u,\mathrm{sm}} = (I_{u,\mathrm{sm}} Q_u)_* (Q_u^* \beta_{u,\mathrm{sm}} - \beta_u) + (I_{u,\mathrm{sm}} Q_u)_* \beta_u = b_{\mathcal{X}} + \mathrm{O}(e^{-\varepsilon u}).$$

This completes the proof of Theorem 2.1. □

A uniform estimate for the heat trace. The aim of this section is to establish Theorem 2.27 below. This estimate generalizes [2, Theorem 7.7 and Theorem 7.8] to the non-selfadjoint situation. Similar estimates can be found in [5]. We will proceed in the spirit of [2], see also [1, Section 9].

Let $P_u := 1 - Q_u$ denote the spectral projection onto the sum of eigen spaces whose corresponding eigen values have real part larger than 1. Moreover, for a trace class operator A, let $\|A\|_{\mathrm{tr}} := \mathrm{tr}(A^*A)$ denote the trace norm, see [22, Appendix A.3.4].

Theorem 2.27. *There exist constants $\varepsilon > 0$ and $u_0 \geq 1$ with the following property. For every $p > n/2$, $p \in \mathbb{N}$, there exists a constant $C_p \geq 0$ so that for all $t > 0$ and $u \geq u_0$ we have*

$$\|\exp(-t\Delta_u)P_u\|_{\mathrm{tr}} \leq C_p e^{-\varepsilon t u}\Big(\frac{u^{p(p+1)+2}}{t^{p-1}} + \frac{u}{t^p}\Big). \tag{60}$$

The proof of Theorem 2.27 is based on estimates for the resolvent of Δ_u which are uniform in u, see Proposition 2.28 and 2.31 below. Proposition 2.28 is of very general nature, whereas Proposition 2.31 makes use of the Witten–Helffer–Sjöstrand estimates in an essential way. We fix an angle $0 < \theta < \pi/4$. For an operator A and $s_1, s_2 \in \mathbb{N}$ we will write $\|A\|_{s_1,s_2} = \sup_{w\neq 0} \|Aw\|_{s_2}/\|w\|_{s_1}$, where $\|w\|_s$ denotes the Sobolev s-norm of w.

Proposition 2.28. *For every $p \in \mathbb{N}$ there exists constants $\alpha_p \geq 0$ and $C'_p \geq 0$ so that the following holds. If $u \geq 1$, $\lambda \in \mathbb{C}$, $|\arg\lambda| \geq \theta$ and $\alpha_p u^2 \leq |\lambda|$ then $\lambda \notin \mathrm{Spec}(\Delta_u)$ and*

$$\|(\Delta_u - \lambda)^{-p}\|_{0,2p} \leq C'_p.$$

For the proof of Proposition 2.28 we need the following two lemmas.

Lemma 2.29. *For every $s \in \mathbb{N}$ there exists a constant $\tilde{C}_s \geq 0$ with the following property. If $u \geq 1$ and $\lambda \notin \mathrm{Spec}(\Delta_u)$ then*

$$\|(\Delta_u - \lambda)^{-1}\|_{s,s+2} \leq \tilde{C}_s\Big(1 + (|\lambda| + u^2)\|(\Delta_u - \lambda)^{-1}\|_{s,s}\Big).$$

Proof. By ellipticity there exists a constant $\bar{C}_s \geq 0$ so that for every $w \in \Omega(M; E_u)$ we have:

$$\begin{aligned}
\|w\|_{s+2} &\leq \bar{C}_s\big(\|\Delta_0 w\|_s + \|w\|_s\big)\\
&= \bar{C}_s\Big(\|(\Delta_u - \lambda + \lambda - \Delta_u + \Delta_0)(w)\|_s + \|w\|_s\Big)\\
&\leq \bar{C}_s\Big(\|(\Delta_u - \lambda)w\|_s + \big(1 + |\lambda| + \|\Delta_u - \Delta_0\|_{s,s}\big)\|w\|_s\Big)\\
&\leq \bar{C}_s\Big(1 + \big(1 + |\lambda| + \|\Delta_u - \Delta_0\|_{s,s}\big)\|(\Delta_u - \lambda)^{-1}\|_{s,s}\Big)\|(\Delta_u - \lambda)w\|_s
\end{aligned}$$

Since $\|\Delta_u - \Delta_0\|_{s,s} = O(u^2)$ we find a constant $\tilde{C}_s \geq 0$ so that

$$\|w\|_{s+2} \leq \tilde{C}_s\Big(1 + (|\lambda| + u^2)\|(\Delta_u - \lambda)^{-1}\|_{s,s}\Big)\|(\Delta_u - \lambda)w\|_s,$$

and this implies the statement of the lemma. □

Lemma 2.30. *For every $s \in \mathbb{N}$ there exist constants $\tilde{\alpha}_s \geq 0$ and $\tilde{C}'_s \geq 0$ with the following property. For all $u \geq 1$ and $\lambda \in \mathbb{C}$ with $\tilde{\alpha}_s u^2 \leq |\lambda|$ and $|\arg \lambda| \geq \theta$ we have $\lambda \notin \operatorname{Spec}(\Delta_u)$ and*

$$\|(\Delta_u - \lambda)^{-1}\|_{s,s} \leq \frac{\tilde{C}'_s}{|\lambda|}.$$

Proof. In view of [22, Corollary 9.2] there exists constants $R \geq 0$ and $\tilde{C}'_s > 0$ so that the following holds. If $|\lambda| \geq R$ and $|\arg \lambda| \geq \theta$ then $\lambda \notin \operatorname{Spec}(\Delta_0)$ and

$$\|(\Delta_0 - \lambda)^{-1}\|_{s,s} \leq \frac{\tilde{C}'_s}{2|\lambda|}. \tag{61}$$

Choose $\tilde{\alpha}_s \geq R$ such that for all $u \geq 1$ we have

$$\|\Delta_u - \Delta_0\|_{s,s} \leq \frac{\tilde{\alpha}_s}{\tilde{C}'_s} u^2. \tag{62}$$

If $u \geq 1$, $\tilde{\alpha}_s u^2 \leq |\lambda|$, $|\arg \lambda| \geq \theta$ and $\omega \in \Omega(M; E_u)$, then combining (61) and (62) we obtain

$$\begin{aligned}\|(\Delta_u - \lambda)w\|_s &\geq \|(\Delta_0 - \lambda)w\|_s - \|(\Delta_u - \Delta_0)w\|_s \\ &\geq \Big(\frac{2|\lambda|}{\tilde{C}'_s} - \frac{\tilde{\alpha}_s u^2}{\tilde{C}'_s}\Big)\|w\|_s \geq \frac{|\lambda|}{\tilde{C}'_s}\|w\|_s\end{aligned}$$

and the lemma follows. □

Proof of Proposition 2.28. For $s \in \mathbb{N}$ set $\tilde{C}''_s := \tilde{C}_s(1 + (1 + \frac{1}{\tilde{\alpha}_s})\tilde{C}'_s)$ where $\tilde{C}_s$, $\tilde{\alpha}_s$ and $\tilde{C}'_s$ are the constants from Lemma 2.29 and 2.30. Then

$$\|(\Delta_u - \lambda)^{-1}\|_{s,s+2} \leq \tilde{C}''_s$$

for all $u \geq 1$, $\tilde{\alpha}_s u^2 \leq |\lambda|$ and $|\arg \lambda| \geq \theta$. Set $\alpha_p := \max\{\tilde{\alpha}_0, \tilde{\alpha}_2, \tilde{,} \dots, \tilde{\alpha}_{2p-2}\}$. Then

$$\|(\Delta_u - \lambda)^{-p}\|_{0,2p} \leq \prod_{s=0}^{p-1} \|(\Delta_u - \lambda)^{-1}\|_{2s,2s+2} \leq \prod_{s=0}^{p-1} \tilde{C}''_s$$

for all $u \geq 1$, $\alpha_p u^2 \leq |\lambda|$ and $|\arg \lambda| \geq \theta$. The proposition now follows with $C'_p := \prod_{s=0}^{p-1} \tilde{C}''_s$. □

Let us introduce the notation:

$$\operatorname{Spec}_{\mathrm{sm}}(\Delta_u) := \big\{\lambda \in \operatorname{Spec}(\Delta_u) \,\big|\, \operatorname{Re}\lambda \leq 1\big\}$$
$$\operatorname{Spec}_{\mathrm{la}}(\Delta_u) := \big\{\lambda \in \operatorname{Spec}(\Delta_u) \,\big|\, \operatorname{Re}\lambda > 1\big\}$$

Proposition 2.31. *There exist constants $\varepsilon > 0$, $u_0 \geq 1$ and for every $p \in \mathbb{N}$ a constant $C''_p \geq 0$ so that the following holds. If $u \geq u_0$ then (by Proposition 2.18)*

$$\operatorname{Spec}_{\mathrm{sm}}(\Delta_u) \subseteq \big\{\lambda \in \mathbb{C} \,\big|\, \operatorname{Re}\lambda < \varepsilon u\big\}, \tag{63}$$

$$\operatorname{Spec}_{\mathrm{la}}(\Delta_u) \subseteq \big\{\lambda \in \mathbb{C} \,\big|\, \operatorname{Re}\lambda > 2\varepsilon u\big\}, \tag{64}$$

and for $\lambda \in \mathbb{C}$ with $\varepsilon u \leq \operatorname{Re}\lambda \leq 2\varepsilon u$, we have

$$\|(\Delta_u - \lambda)^{-p}\|_{0,2p} \leq C''_p(|\lambda| + u^2)^{p(p+1)/2}.$$

Proof of Proposition 2.31. Set $\varepsilon := \varepsilon_4/4$ where $\varepsilon_4 > 0$ denotes the constant in Proposition 2.18, and choose $u_0 \geq 1$ sufficiently large so that for $u \geq u_0$ (63) and (64) hold. In view of Lemma 2.19, see (42), we can increse u_0 so that for $u \geq u_0$, $\varepsilon u \leq \operatorname{Re}\lambda \leq 2\varepsilon u$ we have

$$\|(\Delta_u - \lambda)^{-1}\|_{2s,2s} \leq (u^2 + |\lambda|)^s, \qquad s = 0, 1, \dots, p-1.$$

Then, using Lemma 2.29,

$$\|(\Delta_u - \lambda)^{-1}\|_{2s,2s+2} \leq \tilde{C}_{2s}\big(1 + (|\lambda| + u^2)^{s+1}\big) \leq 2\tilde{C}_{2s}(|\lambda| + u^2)^{s+1}$$

and we obtain

$$\|(\Delta_u - \lambda)^{-p}\|_{0,2p} \leq \prod_{s=0}^{p-1} \|(\Delta_u - \lambda)^{-1}\|_{2s,2s+2} \leq (|\lambda| + u^2)^{p(p+1)/2} \prod_{s=0}^{p-1} 2\tilde{C}_{2s}.$$

The proposition thus follows with $C''_p := \prod_{s=0}^{p-1} 2\tilde{C}_{2s}$. □

We will now use Propositions 2.28 and 2.31 to provide a

Proof of Theorem 2.27. We are going to use the constants $\alpha_p \geq 0$, $C'_p \geq 0$, $\varepsilon > 0$, $u_0 \geq 1$ and $C''_p \geq 0$ from Propositions 2.28 and 2.31. Increasing α_p, we may assume that $\alpha_p \geq \varepsilon$. For $u \geq u_0$ we consider the contour[5] Γ_u parametrized by

$$\lambda_u\colon \mathbb{R} \to \mathbb{C}, \qquad \lambda_u(x) := \varepsilon u + \varepsilon u|x| - \alpha_p u^2 x\mathbf{i}.$$

Note that for $u \geq u_0$, $t > 0$ and $x \in \mathbb{R}$ we have

$$|e^{-t\lambda_u(x)}| = e^{-\varepsilon t u} e^{-\varepsilon t u|x|} \qquad \text{and} \qquad |\lambda'_u(x)| \leq \sqrt{2}\alpha_p u^2. \tag{65}$$

Observe that for $|x| \leq 1$ we have $\varepsilon u \leq \operatorname{Re}\lambda_u(x) \leq 2\varepsilon u$ and $|\lambda_u(x)| \leq 3\alpha_p u^2$, hence Proposition 2.31 tells that

$$\|(\lambda_u(x) - \Delta_u)^{-p}\|_{0,2p} \leq C''_p(3\alpha_p + 1)u^{p(p+1)} \tag{66}$$

for all $|x| \leq 1$ and $u \geq u_0$. Moreover, if $|x| \geq 1$ then $\alpha_p u^2 \leq |\lambda_u(x)|$ and $|\arg\lambda_u(x)| \geq \pi/4 \geq \theta$, hence Proposition 2.28 yields

$$\|(\lambda_u(x) - \Delta_u)^{-p}\|_{0,2p} \leq C'_p \tag{67}$$

for all $|x| \geq 1$ and $u \geq u_0$. Further we have:

$$\begin{aligned}
\exp(-t\Delta_u)P_u &= \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_u} e^{-t\lambda}(\lambda - \Delta_u)^{-1}\, d\lambda \\
&= \frac{(p-1)!}{2\pi\mathbf{i}} \frac{(-1)^{p-1}}{t^{p-1}} \int_{\Gamma_u} e^{-t\lambda}(\lambda - \Delta_u)^{-p}\, d\lambda \\
&= \frac{(p-1)!}{2\pi\mathbf{i}} \frac{(-1)^{p-1}}{t^{p-1}} \int_{-\infty}^{\infty} e^{-t\lambda_u(x)}\big(\lambda_u(x) - \Delta_u\big)^{-p}\lambda'_u(x)\, dx
\end{aligned}$$

[5]The angular contour bisected by the positive real axis.

Using (65) we thus get

$$\begin{aligned}\big\|\exp(-t\Delta_u)P_u\big\|_{0,2p} \\ \leq \frac{\sqrt{2}\alpha_p(p-1)!}{2\pi}\frac{u^2}{t^{p-1}}e^{-\varepsilon tu}\int_{-\infty}^{\infty} e^{-\varepsilon tu|x|}\|(\lambda_u(x)-\Delta_u)^{-p}\|_{0,2p}\,dx \end{aligned}\tag{68}$$

From (66) we obtain

$$\int_{-1}^{1} e^{-\varepsilon tu|x|}\|(\lambda_u(x)-\Delta_u)^{-p}\|_{0,2p}\,dx \leq 2C''_p(3\alpha_p+1)u^{p(p+1)}.\tag{69}$$

From (67) we obtain

$$\int_{|x|\geq 1} e^{-\varepsilon tu|x|}\|(\lambda_u(x)-\Delta_u)^{-p}\|_{0,2p}\,dx \leq \frac{2C'_p}{\varepsilon}\frac{1}{tu}.\tag{70}$$

Combining (68), (69) and (70) we find a constant $\bar{C}_p \geq 0$ so that for all $u \geq u_0$ and $t > 0$ we have

$$\|\exp(-t\Delta_u)P_u\|_{0,2p} \leq \bar{C}_p e^{-\varepsilon tu}\Big(\frac{u^{p(p+1)+2}}{t^{p-1}}+\frac{u}{t^p}\Big).\tag{71}$$

Choose $\lambda_0 \notin \operatorname{Spec}(\Delta_0)$, i.e. $\Delta_0-\lambda_0$ is invertible. Then, see [22, Proposition A.3.7],

$$\|\exp(-t\Delta_u)P_u\|_{\mathrm{tr}} \leq \|(\Delta_0-\lambda_0)^{-p}\|_{\mathrm{tr}}\|(\Delta_0-\lambda_0)^p\|_{2p,0}\|\exp(-t\Delta_u)P_u\|_{0,2p}.$$

To complete the proof of Theorem 2.27 combine this with (71) and note that $(\Delta_0-\lambda_0)^{-p}$ is trace class since we assumed $p > n/2$. □

3. Asymptotic of the large torsion and the proof of Theorem 1.4

Given the special role of the variable u in this section we will replace the notation $\Delta_{q,u}$, $\tau_{\mathrm{la},u}$ etc. by $\Delta_q(u)$, $\tau_{\mathrm{la}}(u)$ etc.

We will consider functions $t(u)$, $u \in (0,\infty)$ of the form

$$\sum_{j=0}^{n} A_j u^j + \sum_{j=0}^{n} B_j u^j \log u + o(1), \qquad \text{as } u \to \infty,$$

and refer to A_0 as the free term of $t(u)$ and denote it by $\mathrm{FT}(t(u))$. Note that $\log_\pi \tau_{\mathrm{la}}(u)$ is by Corollary 2.3 such a function. The key result of this section is Theorem 3.6 below whose proof, although the same as of Theorem B in [5], is supported by estimates derived in Section 2. This theorem calculates the free term

$$\mathrm{FT}\big(\log_\pi \tau_{\mathrm{la}}(u) - \log_\pi \tilde{\tau}_{\mathrm{la}}(u)\big)$$

provided M and $\tilde{M}$ have the same dimension, E and $\tilde{E}$ the same rank, f and $\tilde{f}$ the same number of critical points in each index.

Here $\tau_{\mathrm{la}}(u)$ and $\tilde{\tau}_{\mathrm{la}}(u)$ denote the large torsions associated with two systems (M,E,g,b,f) and $(\tilde{M},\tilde{E},\tilde{g},\tilde{b},\tilde{f})$ as in Section 2 which satisfy (9), (10) and (11).

Asymptotic expansion of $\log\det$ **for elliptic with parameter.** Suppose $E \to M$ is a rank k complex vector bundle over (M, g) a smooth Riemannian manifold of dimension n, $\mathbb{D}$ a second order elliptic operator of Laplace–Beltrami type (cf. [5]) (i.e. the principal symbol $\sigma(\mathbb{D})(\xi) = -\|\xi\|^2\,\mathrm{id}$), $L\colon E \to E$ a bundle map and $F\colon M \to \mathbb{R}$ a smooth function. The operator $\mathbb{D}$ has always π as a principal angle.[6] We denote also by L resp. F the zero order (differential) operators defined by the bundle map L, resp. the multiplication by F.

For any $u \in [0, \infty)$, let $\mathbb{D}(u) := \mathbb{D}^{L,F}(u) = \mathbb{D} + uL + u^2F$. If F is strictly positive then the family $\mathbb{D}(u)$ is elliptic with parameter in the sense of [22] or [6].

We apply the considerations below to $\mathbb{D} = \Delta + \varepsilon$ where $\Delta := \Delta_{E,g,b,q}$ is the q–Laplacian associated with the flat connection ∇^E, the non-degenerate symmetric bilinear form b, and the Riemannian metric g as defined in Section 1, and ε a positive real number. The smooth function F will be $|df|^2$ where $f\colon M \to \mathbb{R}$ is a smooth function on M and the endomorphism L given by Lemma 2.4. In this case the family $\mathbb{D}(u)$ is elliptic with parameter away from the set of critical points of f.

With respect to a coordinate chart $U \subseteq M$ with coordinates $x_1, \dots, x_n$ and a trivialization, $E|_U = U \times \mathbb{C}^k$ the symbol of the operator $\mathbb{D}$ is given by

$$\sigma(\mathbb{D})(x, \xi) = -\|\xi\|^2 + \sum_{i=1}^{n} \alpha_i(x)\xi^i + \beta(x),$$

where $\|\xi\|^2 = \sum g^{ij}\xi^i\xi^j$, g^{ij} the Riemannian metric, $\alpha_i(x), \beta_i(x)$ are smooth $\mathrm{end}(\mathbb{C}^n)$–valued functions. The operator $\mathbb{D}(u)|_U$ is given by

$$\mathbb{D}(u)|_U = a_2(x, D, u) + a_1(x, D, u) + a_0(x, u),$$

$x = (x_1, \dots, x_n)$, $D = (D_{x_1}, \dots, D_{x_n})$, where $a_{2-j}(x, \xi, u)$, $\xi \in \mathbb{R}^n$, are $\mathrm{end}(\mathbb{C}^k)$–valued smooth functions with the homogeneity property

$$a_{2-j}(x, \tau\xi, \tau u) = \tau^{2-j} a_{2-j}(x, \xi, u),$$

$\tau \in \mathbb{R}$. Precisely a_2, a_1, a_0 are given by

$$\begin{aligned} a_2(x, \xi, u) &= -\,\|\xi\|^2 + u^2F \\ a_1(x, \xi, u) &= uL(x) + \sum_{i}^{n} \alpha_i(x)\xi^i \\ a_0(x, \xi, u) &= \varepsilon + \beta(x). \end{aligned} \tag{72}$$

Suppose that F is strictly positive, hence $\mathbb{D}(u)$ is elliptic with parameter. As in [6] we define inductively the functions $r_{-2-j}(x, \xi, u, \mu)$, $\mu \in \mathbb{C}$, with

[6]In fact any angle $\theta \neq 0$ is a principal angle, i.e. for any x and $\xi \neq 0$ the spectrum of the finite dimensional linear map $\sigma(\mathbb{D})(\xi, x) : E_x \to E_x$ is disjoint from the ray of angle θ.

values in $\mathrm{end}(\mathbb{C}^k)$ by:

$$r_{-2}(x,\xi,u,\mu) := (\mu - a_2(x,\xi,u))^{-1}$$

$$r_{-2-j}(x,\xi,u,\mu) := r_{-2}(x,\xi,u,\mu) \cdot \left(\sum_{k=0}^{j-1} \sum_{|\alpha|+l+k=j} \tfrac{1}{\alpha!} \partial_\xi^\alpha a_{2-l}(x,\xi,u,\mu) D_x^\alpha r_{-2-k}(x,\xi,u,\mu) \right)$$

with α a multi index $\alpha = (i_1, \dots, i_n)$. Clearly r_{-2-j} is homogeneous of degree $(-2-j)$ in $(\xi, u, \mu^{1/2})$.

We also define the smooth complex valued function

$$a_{L,F}(x) := -\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} d\xi \int_0^\infty d\mu \, \mathrm{tr}\big(r_{-2-n}(x,\xi,1,-\mu)\big) \tag{73}$$

and as in [5], page 352, equation (3.11) a simple calculation shows

$$a_{L,F}(x) + a_{-L,F}(x) = 0. \tag{74}$$

Suppose M is closed and $(E, \mathbb{D}, F, L)$ as above. We can consider the complex valued function $\log \det_\pi \mathbb{D}(u)$. The following result was established in the appendix of [6], see also [17].

Theorem 3.1 ([6]). *The functions $a_{L,F}(x)$ define a density on M^n. If $\mathbb{D}(u)$ is invertible for u large enough then $\log \det_\pi \mathbb{D}(u)$ has an asymptotic expansion of the form*

$$\sum_{j=0}^{n} A_j u^j + \sum_{j=0}^{n} B_j u^j \log u + \sum_{i=1}^{\infty} C_i u^{-i}, \qquad \text{as } u \to \infty, \tag{75}$$

with $A_0 = \int_M a_{L,H}$.[7] *In particular $\log \det_\pi \mathbb{D}(u)$ is of the form*

$$\sum_{j=0}^{n} A_j u^j + \sum_{j=0}^{n} B_j u^j \log u + o(1) \qquad \text{as } u \to \infty \tag{76}$$

(The result proved in [6] is formulated under more general hypotheses and with stronger conclusions). The result can be extended to compact manifolds with boundaries and Dirichlet boundary condition and leads to the following relative version of Theorem 3.1 stated under more restrictive hypotheses (satisfied in our situation).

Theorem 3.2 ([5]). *Suppose $(E_i, \mathbb{D}_i, L_i, F_i)$, $i = 1, 2$, are two systems as above. Suppose that there exist compact sets $K_i \subseteq M_i$ and open neighborhoods U_i of of K_i so that:*

1) $F_i|_{M \setminus K_i}$ are strictly positive, and

2) there exists the diffeomorphisms $\varphi\colon U_1 \to U_2$, $\tilde{\varphi}\colon E_1|_{U_1} \to E_2|_{U_2}$ bundle isomorphism above φ which intertwines

$$(E_1|_{U_1}, \mathbb{D}_1, L_1|_{U_1}, F_1|_{U_1}) \quad \text{with} \quad (E_2|_{U_2}, \mathbb{D}_2, L_2|_{U_2}, F_2|_{U_2}).$$

[7] Actually all terms A_j and B_j are integrals on M of densities explicitly computable in each chart in terms of the symbol of $\mathbb{D}(u)$.

Let V_i compact domains with smooth boundaries, $K_i \subseteq V_i \setminus \partial V_i \subseteq V_i \subseteq U_i$ with $\varphi(V_1) = V_2$. If $\mathbb{D}_i(u)$ are invertible for u large enough then

$$\log \det_\pi \mathbb{D}_1(u) - \log \det_\pi \mathbb{D}_2(u)$$

has an asymptotic expansion of the form (75) *with*

$$A_0 = \int_{M_1 \setminus V_1} a_{L_1,F_1} - \int_{M_2 \setminus V_2} a_{L_2,F_2}$$

A relative result for the asymptotic expansion of the large torsion. Let (M, E, g, b, f) be consisting of a closed smooth manifold M, complex flat bundle E, Riemannian metric g, non-degenerate symmetric bilinear form b and Morse function f. We will refer to such collection (M, E, g, b, f) as a "system" provided (9), (10), (11) hold, and to a collection of neighborhoods $B_{q;j}$ of the critical points $x_{q;j} \in \mathcal{X}_q^f$ as ρ-admissible neighborhoods if they have disjoint closures and are the source of Morse charts $\varphi_{x_{q;j}}$ of radius ρ so that on them (9), (10) and (11) hold. Clearly such neighborhoods exists for ρ small enough.

We define $B'_{q;j} := \varphi_{x_{q;j}}(D_{\rho/2})$, where D_r denotes the disc of radius r in $\mathbb{R}^n$ centered at 0.

Given a collection of ρ–admissible neighborhoods $B_{q;j}$ introduce the manifolds

$$M_I := M \setminus \cup_{q,j} B'_{q;j}; \quad M_{II} := \cup_{q,j} \overline{B'_{q;j}},$$

where $B'_{q;j}$ is defined as in the above definition. Both manifolds M_I and M_{II} have the same boundary, given by a disjoint union of spheres of dimension $n-1$.

Fix $\varepsilon > 0$ and consider the operator $\Delta_q(u) + \varepsilon$. If u is large enough, in view of Proposition 2.18 $\Delta_q(u)+\varepsilon$ has π as an Agmon angle and is invertible. Its symbol with respect to arbitrary coordinates (φ, ψ) of $(M, E \to M)$ is of the form

$$a_2(x, \xi) + u^2 \|\nabla f\|^2 + a_1(x, \xi) + uL(x) + \varepsilon$$

where $a_i : B_{3\alpha} \times \mathbb{R}^d \to \operatorname{end}(\Lambda^q(\mathbb{R}^d) \otimes \mathbb{C}^N)$, $i = 1, 2$, are homogeneous of degree i in ξ, where $\|\nabla f\|^2 : B_{3\alpha} \to \mathbb{R}$ is given by

$$\|\nabla f\|^2 = \sum_{1 \le i,j \le d} g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}$$

and $L\colon\ B_{2\alpha} \to \operatorname{end}(\Lambda^q(\mathbb{R}^d) \otimes \mathbb{C}^N)$.

Therefore, away from the critical points of f, this operator is elliptic with parameter and away from the critical points we can consider the densities $a_{L,H}$ associated with $(\Delta_q(u)+\varepsilon)$; they will be denoted here by $a^q(f, \varepsilon, x) := a^q(b, g, f, \varepsilon, x)$. As in [5] we can establish the following intermediary results:

Proposition 3.3. *Assume that the systems (M, E, b, g, f) and $(\tilde{M}, \tilde{E}, \tilde{b}, \tilde{g}, \tilde{f})$ satisfy* $\dim M = \dim \tilde{M} = n$, $\sharp\mathcal{X}_q(f) = \sharp\mathcal{X}_q(\tilde{f})$, $0 \le q \le n$, *and* $\operatorname{rank} E = \operatorname{rank} \tilde{E}$. *Choose $\rho > 0$ so that we have ρ–admissible neighborhoods of critical* *points for both systems. Denote by* $\Delta_q(u)$ *resp.* $\tilde{\Delta}_q(u)$ *the Witten Laplacians associated with the two systems. Then, for any* $\varepsilon > 0$,

$$\log\det_\pi(\Delta_q(u)+\varepsilon) - \log\det_\pi(\tilde{\Delta}_q(u)+\varepsilon)$$

has an asymptotic expansion of the form (75) *for* $u \to \infty$ *whose free term equals*

$$\overline{a}^q(f,\tilde{f},\varepsilon) = \int_{M_I} a^q(f,\varepsilon,x) - \int_{\tilde{M}_I} a^q(\tilde{f},\varepsilon,\tilde{x}).$$

For any $\varepsilon > 0$ introduce

$$A(f,u,\varepsilon) := \frac{1}{2}\sum_{q=0}^{d}(-1)^{q+1} q \log\det(\Delta_q(u)+\varepsilon), \tag{77}$$

which can be written as $A(f,u,\varepsilon) = A_{\mathrm{sm}}(f,u,\varepsilon) + A_{\mathrm{la}}(f,u,\varepsilon)$ with

$$A_{\mathrm{sm/la}}(f,u,\varepsilon) := \frac{1}{2}\sum_{q=0}^{d}(-1)^{q+1} q \log\det_\pi\bigl(\Delta_{\mathrm{sm/la},q}(u)+\varepsilon\bigr)$$

and $\Delta_{\mathrm{sm/la},q}(u) := \Delta_q(u)|_{\Omega^q_{\mathrm{sm/la}}(M;E_u)}$. Clearly $A_{\mathrm{la}}(f,u,\varepsilon=0) = \log\tau_{f,\mathrm{la}}(u)$.

Proposition 3.4. *With the assumptions in Proposition 3.3, for any* $\varepsilon > 0$ *the quantities* $A(f,u,\varepsilon) - A(\tilde{f},u,\varepsilon)$ *and* $A_{\mathrm{la}}(f,u,\varepsilon) - A_{\mathrm{la}}(\tilde{f},u,\varepsilon)$ *have asymptotic expansions of the form* (75) *for* $u \to \infty$ *which are identical. In particular*

$$\mathrm{FT}\bigl(A(f,u,\varepsilon) - A(\tilde{f},u,\varepsilon)\bigr) = \mathrm{FT}\bigl(A_{\mathrm{la}}(f,u,\varepsilon) - A_{\mathrm{la}}(\tilde{f},u,\varepsilon)\bigr).$$

Proposition 3.5. *With the assumptions in Proposition 3.4:*

(i) The limit

$$\lim_{\varepsilon\to 0}\mathrm{FT}\bigl(A_{\mathrm{la}}(f,u,\varepsilon) - A_{\mathrm{la}}(\tilde{f},u,\varepsilon)\bigr) \tag{78}$$

exists and is equal to $\mathrm{FT}\bigl(\log\tau_{f,\mathrm{la}}(u) - \log\tau_{\tilde{f},\mathrm{la}}(u)\bigr)$.

(ii) This limit is given by

$$a(f,\tilde{f}) = \int_{M_I}\sum_q(-1)^q q a^q(f,\varepsilon=0,x) - \int_{\tilde{M}_I}\sum_q(-1)^q q a^q(\tilde{f},\varepsilon=0,\tilde{x}),$$

(ρ *the same for both manifolds and small enough) with* $a^q(f,\varepsilon,x)$ *and* $a^q(\tilde{f},\varepsilon,\tilde{x})$ *the densities considered above.*

Combining the above propositions and (74) we have

Theorem 3.6. *Assume that the systems* (M^n,E,b,g,f) *and* $(\tilde{M}^n,\tilde{E},\tilde{b},\tilde{g},\tilde{f})$ *satisfy* $\sharp\mathcal{X}_q(f) = \sharp\mathcal{X}_q(\tilde{f})$, $0 \le q \le n$, *and* $\operatorname{rank}E = \operatorname{rank}\tilde{E}$. *Then*

$$\mathrm{FT}\bigl(\log\tau_{\mathrm{la}}(u) - \log\tilde{\tau}_{\mathrm{la}}(u)\bigr)$$

has an asymptotic expansion of the form (75) *whose free term is*

$$a(f,\tilde{f}) = \int_{M_I}\sum_q(-1)^q q a_0^q(f,\varepsilon=0,x) - \int_{\tilde{M}_I}\sum_q(-1)^q q a_0(\tilde{f},\varepsilon=0,\tilde{x})$$

with $a^q(f,\varepsilon,x)$ *and* $a^q(\tilde f,\varepsilon,\tilde x)$ *the densities considered in Proposition 3.3 above.*

(iii) When $\dim M = n$ *is odd we have*

$$a(f,\tilde f)+a(n-f,n-\tilde f)=0.$$

Proposition 3.3 is similar to Proposition 3.1 in [5] (but for non-selfadjoint Witten Laplacians) so the proof is the same. It is actually a straightforward consequence of the Theorem 3.2 above.

The proof of Proposition 3.4 goes as follows. As the eigenvalues of the operator $\Delta_{\mathrm{sm},q}(u)$ tend exponentially fast to 0 as $u\to\infty$ in view of Proposition 2.18

$$\log_\pi \det(\Delta_{q,\mathrm{sm}}(u)+\varepsilon)=m_q\log\varepsilon+O\left(\frac{1}{\varepsilon}\alpha e^{-\beta u}\right)$$

for some positive constants α,β, and therefore, $A_{\mathrm{sm}}(f,u,\varepsilon)-A_{\mathrm{sm}}(\tilde f,u,\varepsilon)$ is exponentially small as $u\to\infty$. Therefore for any $\varepsilon>0$,

$$A(f,u,\varepsilon)-A(\tilde f,u,\varepsilon)$$

and

$$A_{\mathrm{la}}(f,u,\varepsilon)-A_{\mathrm{la}}(\tilde f,u,\varepsilon)$$

have asymptotic expansions of the form (75) for $u\to\infty$ which are identical. q.e.d.

Proposition 3.5 is more elaborated and the remaining of the subsection elaborate on this proof.

Proof of Proposition 3.5. To check (i) we verify that the function $H(u,\varepsilon)$, defined for $\varepsilon>0$ and u sufficiently large by

$$H(u,\varepsilon):=A_{\mathrm{la}}(f,u,\varepsilon)-A_{\mathrm{la}}(\tilde f,u,\varepsilon)+\log\tau_{\mathrm{la}}(u)-\log\tilde\tau_{\mathrm{la}}(u)$$

is of the form

$$H(u,\varepsilon)=\sum_{k=1}^{T}\varepsilon^k f_k(u)+g(u,\varepsilon), \tag{79}$$

where $g(u,\varepsilon)=O(u^{-1+\delta})$ uniformly in ε. The statement of Proposition 3.5 can be deduced from this formula as follows: Recall that for $\varepsilon>0$, $H(u,\varepsilon)$ has an asymptotic expansion for $u\to\infty$ because $A_{\mathrm{la}}(f,u,\varepsilon)-A_{\mathrm{la}}(\tilde f,u,\varepsilon)$ and $\log\tau_{\mathrm{la}}(u)$ and $\log\tilde\tau_{\mathrm{la}}(u)$ have, the first by Theorem 3.6 the last two by Corollary 2.3. As $g(u,\varepsilon)=O(u^{-1+\delta})$ uniformly in ε we conclude that for any $\varepsilon>0$, $\sum_{k=1}^T\varepsilon^k f_k(u)$ has an asymptotic expansion for $u\to\infty$. By taking T different values $0<\varepsilon_1<\cdots<\varepsilon_T$ for ε and using that the Vandermonde determinant is nonzero

$$\det\begin{pmatrix}\varepsilon_1 & \dots & \varepsilon_1^T\\ \vdots & & \vdots\\ \varepsilon_d & \dots & \varepsilon_d^T\end{pmatrix}\neq 0$$

we conclude that for any $1 \leq k \leq T$, $f_k(u)$ has an asymptotic expansion for $u \to \infty$ and that for any $\varepsilon > 0$

$$\mathrm{FT}(H(u,\varepsilon)) = \sum_{k=1}^{d} \varepsilon^k \, \mathrm{FT}(f_k(u)).$$

Hence $\lim_{\varepsilon\to 0} \mathrm{FT}(H(u,\varepsilon))$ exists and $\lim_{\varepsilon\to 0} \mathrm{FT}(H(u,\varepsilon)) = 0$. It remains to prove (79).

Recall that if

$$\theta_{q,\mathrm{la}}(u,\mu) := \mathrm{tr}(e^{-\mu\Delta_q(u)} P_u)$$

and

$$\zeta_{q,\mathrm{la}}(u,\varepsilon,s) := \frac{1}{\Gamma(s)} \int_0^\infty \mu^{s-1} \theta_{q,\mathrm{la}}(u,\mu) e^{-\varepsilon\mu} d\mu \tag{80}$$

then

$$\log \det\nolimits_\pi(\Delta_{q,\mathrm{la}}(u) + \varepsilon) = \frac{d}{ds}\Big|_{s=0} \zeta_{q,\mathrm{la}}(u,\varepsilon,s). \tag{81}$$

We have the estimate

Lemma 3.7. *There exists a constant* $C_1, C_2, \beta, > 0$, $0 < \delta < 1$, *and integer* $T \geq n$ *so that*

(i) for u *large enough and* $0 \leq \mu \leq u^{-1+\delta}$

$$\theta_q(u,\mu) \leq C_1 \mu^{-T} \tag{82}$$

(ii) for u *large enough, and* $\mu \geq u^{-1+\delta}$

$$\theta_q(u,\mu) \leq C_2 e^{-\beta u \mu}. \tag{83}$$

Proof. Note that Theorem 2.27 implies that there exists an integer $N \geq n$ and the constants $C', \beta > 0$ so that for u large enough

$$|\theta(u,\mu)| \leq C' e^{-\gamma u \mu} u^N / \mu^n$$

with $\gamma > 0$. Choose $0 < \delta < 1$, (for example $\delta = 1/2$).

Suppose $\mu \leq u^{-1+\delta}$. This implies $\mu u^{1-\delta} \leq 1$; and supposing u large enough we have $\mu \leq 1/2$.

As $\mu \leq u^{\delta-1}$, hence $u^N \leq \mu^{-N/(1-\delta)}$, we get

$$|\theta(u,\mu)| \leq C' e^{-\gamma\mu u} / u^{n+N/(1-\delta)} \leq C' e / u^T,$$

where $T > n + N/(1-\delta)$. This establishes (i).

Suppose $\mu \geq u^{-1+\delta}$. This implies $\mu^{-n} \leq u^{n(1-\delta)}$ and therefore

$$e^{-\gamma/2 \ \mu u} u^N / \mu^n \leq e^{-\gamma/2 \ u^\delta} u^{N+n(1-\delta)}.$$

Clearly for u large enough we have

$$e^{-\gamma/2 \ u^\delta} u^{N+n-\delta} \leq 1$$

since $\delta > 0$. Take $C_2 = C'$, $\beta = \gamma/2$ and (ii) is verified. $\square$

To establish (79) we decompose the function $\zeta_{q,\mathrm{la}}(u,\varepsilon,s)$ into two parts

$$\zeta^{I}_{q,\mathrm{la}}(u,\varepsilon,s)=\frac{1}{\Gamma(s)}\int_{u^{-1+\delta}}^{\infty}\mu^{s-1}\theta_q(u,\mu)e^{-\varepsilon\mu}d\mu \tag{84}$$

and

$$\zeta^{II}_{q,\mathrm{la}}(u,\varepsilon,s)=\frac{1}{\Gamma(s)}\int_{0}^{u^{-1+\delta}}\mu^{s-1}\theta_q(u,\mu)e^{-\varepsilon\mu}d\mu. \tag{85}$$

First let us consider

$$\zeta^{I}_{q,\mathrm{la}}(u,\varepsilon,s)-\zeta^{I}_{q,\mathrm{la}}(u,\varepsilon=0,s)=\frac{1}{\Gamma(s)}\int_{u^{-1+\delta}}^{\infty}\mu^{s}\theta_q(u,\mu)\frac{e^{-\varepsilon\mu}-1}{\mu}d\mu$$

Note that

$$\zeta^{I}_{q,\mathrm{la}}(u,\varepsilon,s)-\zeta^{I}_{q,\mathrm{la}}(u,\varepsilon=0,s)$$

is by Lemma 3.7(ii) an entire function of s and so is $1/\Gamma(s)$.

Clearly, $\frac{1}{\Gamma(s)}\big|_{s=0}=0$, $\frac{d}{ds}\big|_{s=0}\frac{1}{\Gamma(s)}=1$ and $1-e^{-\varepsilon\mu}\leq\varepsilon\mu$. Therefore by Lemma 3.7 we have

$$\begin{aligned}&\left|\frac{d}{ds}\Big|_{s=0}\big(\zeta^{I}_{q,\mathrm{la}}(u,\varepsilon,s)-\zeta^{I}_{q,\mathrm{la}}(u,\varepsilon=0,s)\big)\right|\\ &\qquad=\Big|\int_{u^{-1+\delta}}^{\infty}\theta_q(u,\mu)\frac{e^{-\varepsilon\mu}-1}{\mu}d\mu\Big|\leq\varepsilon C_2\int_{u^{-1+\delta}}^{\infty}e^{-\beta u\mu}d\mu=\frac{\varepsilon C_2}{\beta u}e^{-\beta u^{\delta}}.\end{aligned}$$

Concerning the term

$$\frac{d}{ds}\Big|_{s=0}\big(\zeta^{II}_{q,\mathrm{la}}(u,\varepsilon,s)-\zeta^{II}_{q,\mathrm{la}}(u,\varepsilon=0,s)\big),$$

expand $(e^{-\varepsilon\mu}-1)/\mu$

$$(e^{-\varepsilon\mu}-1)/\mu=\sum_{k=1}^{T}\frac{(-1)^k}{k!}\varepsilon^k\mu^{k-1}+\varepsilon^{T+1}\mu^{d}e(\varepsilon,\mu)$$

where the error term is given by

$$e(\varepsilon,\mu)=\left(\sum_{k=T+1}^{\infty}\frac{(-1)^k}{k!}\varepsilon^k\mu^{k-1}\right)/\varepsilon^{T+1}\mu^{T}.$$

Note that according to Lemma 3.7(i) we have $\mu^T\theta_q(u,\mu)\leq C_1$.

Therefore

$$\int_0^{u^{-1+\delta}}\mu^s\theta_q(u,\mu)\varepsilon^{d+1}\mu^Te(\varepsilon,\mu)d\mu$$

is a meromorphic function of s, with $s=0$ a regular point and, for sufficiently large u we have

$$\left|\frac{d}{ds}\Big|_{s=0}\left(\frac{1}{\Gamma(s)}\int_0^{u^{-1+\delta}}\mu^s\theta_q(u,\mu)\varepsilon^{T+1}\mu^Te(\varepsilon,\mu)d\mu\right)\right|\leq\varepsilon^{T+1}C_1u^{-1+\delta}$$

with C_1 is independent of u and ε, $0\leq\varepsilon\leq1$.

Finally, recall that $\theta_q(u,\mu)$ admits an expansion for $\mu\to 0+$ of the form

$$\theta_q(u,\mu)=\sum_{j=0}^{T}C_j(u)\mu^{(j-d)/2}+\theta'_q(u,\mu)$$

where $\theta'_q(u,\mu)$ is continuous in $\mu\geq 0$. Therefore, for $1\leq k\leq T$,

$$\frac{1}{\Gamma(s)}\int_0^{u^{-1+\delta}}\mu^s\theta_q(u,\mu)\frac{(-1)^k}{k!}\varepsilon^k\mu^{k-1}d\mu$$

is analytic with respect to s at $s=0$ and

$$\sum_{k=1}^{T}\frac{d}{ds}\Big|_{s=0}\left(\frac{1}{\Gamma(s)}\int_0^{u^{-1+\delta}}\mu^s\theta_q(u,\mu)\frac{(-1)^k}{k!}\varepsilon^k\mu^{k-1}d\mu\right)$$

is of the form $\sum_{k=1}^{T}\varepsilon^k f_k(u)$. This establishes (79). Part (ii) follows from Proposition 3.3 and part (iii) from (74). □

Proof of Theorem 1.4. In this section we want to check that $S^2_{E,[b]}=1$. Consider a system (M,E,g,b,f). Clearly $(M,E,g,b,-f)$ is also a system and denote by $\tau^+_{\mathrm{la}}(u)$ resp $\tau^-_{\mathrm{la}}(u)$ the large torsion for the first resp. of the second. Similarly we write $\tau(\mathrm{Int}^+_{\mathrm{sm},u})$ resp. $\tau(\mathrm{Int}^-_{\mathrm{sm},u})$ for the relative torsion of (13) in Section 1 when applied to the first resp. second system.

Suppose now we have two systems (M,E,g,b,f) and $(\tilde M,\tilde E,\tilde g,\tilde b,\tilde f)$, and suppose that f and $\tilde f$ have the same number of critical points in each index. Since $\dim M=n$ is odd we have $(-X)^*\Psi_g=-X^*\Psi_g$, and therefore Corollary 2.3 yields

$$S^2_{E,[b]}/S^2_{\tilde E,[\tilde b]}=\frac{\tau^+_{\mathrm{la}}(u)\cdot\tau^-_{\mathrm{la}}(u)}{\tilde\tau^+_{\mathrm{la}}(u)\cdot\tilde\tau^-_{\mathrm{la}}(u)}\cdot e^{\beta u}\cdot\big(1+O(e^{-\varepsilon u})\big) \tag{86}$$

with a real number β. We know that the left side of (86) is constant and

$$\log_\pi\frac{\tau^+_{\mathrm{la}}(u)\cdot\tau^-_{\mathrm{la}}(u)}{\tilde\tau^+_{\mathrm{la}}(u)\cdot\tilde\tau^-_{\mathrm{la}}(u)}$$

has an asymptotic expansion whose free part is by Theorem 3.6 equal to 0. This implies that

$$\mathcal{S}^2_{E,[b]}/S^2_{\tilde E,[\tilde b]}=1.$$

We choose $\tilde M$ to be the sphere, $\tilde E$ the trivial flat bundle of the same rank as E and $\tilde b$ the canonical symmetric bilinear form. We note that since $\dim M=n$ is odd, one can always provide a Morse function $\tilde f$ on $\tilde M:=S^{\dim M}$ with the same number of critical points as f in each index. Since $\mathcal{S}^2_{\tilde E,[\tilde b]}=1$ we conclude that $\mathcal{S}^2_{E,[b]}=1$. q.e.d.

4. Appendix; Remarks on the proof of Conjecture 1.3

It is likely that in Theorem 1.4 one can replace ± 1 by 1. For this purpose notice that for each Morse function f one can produce a square root $S'_{E,[b],f}$ of $S_{E[b]}$ and by the same arguments as in the proof of Theorem 1.4 one can show that

$$S'_{E,[b],f} \cdot S'_{E,[b],-f} = 1.$$

Clearly, if we show that $S'_{E,[b],f}$ is independent of f, then $S_{E,[b]} = 1$.

To define a square root of $S_{E,[b]}$ we use the formulas (7) and (8). We need two additional data: a Morse–Smale vector field X and an orientation of the total space of the mapping cone complex associated with the quasi isomorphism $\mathrm{Int}_{\mathrm{sm},u} \colon \Omega_{\mathrm{sm}}(M; E_u) \to C(X; E_u)$. The orientation provides a square root of $\tau(\mathrm{Int}_{\mathrm{sm},u})$. Both $\tau_{\mathrm{la},u}$ and $\exp\bigl(-2\int_{M\setminus\mathcal{X}} \omega_{E_u,b} \wedge (-X)^*\Psi_g\bigr)$ have an unambiguous square root. We can choose $X = -\operatorname{grad}_g f$ in which case $\mathrm{Int}_{\mathrm{sm},u}$ is an isomorphism for u large enough and a canonical orientation is implicit in the construction of the mapping cone. The product of these square roots give our desired square root.

It is possible to show that this construction extends to generalized Morse functions. Recall that by a result of H. Chaltin [16] any two Morse functions f_1 and f_2 can be joined by a homotopy f_t with f_t Morse function for all $t \in [1,2]$ but $t_1, t_2, \dots, t_k$, and generalized Morse function for $t = t_1, t_2, \dots, t_k$. One can even arrange that each generalized Morse function has only one birth/death or death/birth critical point. Then the independence of $S'(E,[b],f)$ of the Morse function f follows for from the continuity in t of $S'_{E,[b],f_t}$ for a homotopy of the type provided by Chatlin result.

Recall that a generalized Morse function is a smooth function whose critical points are either non-degenerate or are birth-death/death-birth critical points, cf. [16]. For each q, let m_q be the number of non-degenerate critical points of index q and m'_q the number of degenerate (birth/death) critical points of index q. It was established in [16] that for u large the spectrum of $\Delta_{E_u,g,b,q}$ decomposes in three disjoint parts $\mathrm{Spec}_{\mathrm{sm},u}$, $\mathrm{Spec}_{\mathrm{mla,u}}$, $\mathrm{Spec}_{\mathrm{vla,u}}$, called small spectrum, moderately large spectrum and and very large spectrum. The small spectrum $\mathrm{Spec}_{\mathrm{sm},u}$ consists of $(\operatorname{rank} E)\cdot m_q$ complex numbers converging exponentially fast to 0, $\mathrm{Spec}_{\mathrm{sm},u}$ consists of $2(\operatorname{rank} E)\cdot m'_q$ complex numbers whose both real part and absolute value are converging to ∞ but not faster than $Cu^{2/3}$ and $\mathrm{Spec}_{\mathrm{vla,u}}$, the rest of the spectrum, consists of complex numbers whose real part is converging to ∞ faster than $C'u$, with C, C' some constants. Actually this was established for a flat vector bundle equipped with a Hermitian structure but we expect the statements hold true for a non-degenerate symmetric bilinear form also.

To define the square root of $S_{E,[b]}$ for such generalized Morse function one can use instead of $\Omega_{\mathrm{sm}}(M; E_u)$ and $\Omega_{\mathrm{la}}(M; E_u)$ either $\Omega_{\mathrm{sm}}(M; E_u)$ and $\Omega_{\mathrm{mla}}(M; E_u) \oplus \Omega_{\mathrm{vla}}(M; E_u)$ or $\Omega_{\mathrm{sm}}(M; E_u) \oplus \Omega_{\mathrm{mla}}(M; E_u)$ and $\Omega_{\mathrm{vla}}(M; E_u)$. One can choose conveniently a Morse–Smale vector field, for example a vector field which away from the degenerate critical points is gradient like for the generalized Morse function; in this case canonical orientations exist for the mapping cone of the corresponding integration morphisms in both cases. The square roots obtained by either choice are the same.

While the continuity at t' when $f_{t'}$ is a Morse function is straightforward at t' when $f_{t'}$ is generalized Morse function is more subtle. It is easier to check this continuity when there is only one birth/death or dearth/birth critical point and this can be done separately from left and from right using the two possible definitions of the square root.

An extension of the result $S_{E,[b]} = 1$ to smooth odd dimensional manifolds with boundary combined with a product formula for $S_{E,b}$, (like the product formula for for analytic or combinatorial torsion,) will imply the result for even dimensional manifolds as well. The details of the above remarks will be presented in a forthcoming paper.

References

[1] J.M. Bismut and G. Lebeau, *Complex immersions and Quillen metrics*, Inst. Hautes Édtudes Sci. Publ. Math. **74**, 1991.

[2] J.M. Bismut and W. Zhang, *An extension of a theorem by Cheeger and Müller*, Astérisque **205**, Société Mathématique de France, 1992.

[3] J.M. Bismut and W. Zhang, *Milnor and Ray–Singer metrics on the equivariant determinant of a flat vector bundle,* Geom. Funct. Anal. **4**(1994), 136–212.

[4] M. Braverman and T. Kappeler, *Comparison of the refined analytic and the Burghelea–Haller torsions,* preprint `arXiv:math.DG/0606398`

[5] D. Burghelea, L. Friedlander and T. Kappeler, *Asymptotic expansion of the Witten deformation of the analytic torsion,* J. Funct. Anal. **137**(1996), 320–363.

[6] D. Burghelea, L. Friedlander and T. Kappeler, *Mayer–Vietoris type formula for the determinants of elliptic differential operators,* J. Funct. Anal. **107**(1992), 34–66.

[7] D. Burghelea, L. Friedlander and T. Kappeler, *Relative Torsion*, Commun. Contemp. Math. **3**(2001), 15–85.

[8] D. Burghelea, L. Friedlander, T. Kappeler and P. McDonald, *Analytic and Reidemeister torsion for representations in finite type Hilbert modules,* Geom. Funct. Anal. **6**(1996), 751–859.

[9] D. Burghelea and S. Haller, *Euler structures, variety of representations and the Milnor–Turaev torsion,* Geom. Topol. **10**(2006), 1185–1238.

[10] D. Burghelea and S. Haller, *A Riemannian invariant, Euler structures and some topological applications,* pages 37–60 in the proceedings of the conference "C^*-Algebras and Elliptic Theory (Będlewo, Poland 2004)," Trends in Mathematics, Birkhäuser Verlag, Basel, 2006.

[11] D. Burghelea and S. Haller, *Complex valued Ray–Singer torsion*, preprint `arXiv: math.DG/0604484`

[12] J. Cheeger, *Analytic torsion and Reidemeister torsion,* Proc. Nat. Acad. Sci. U.S.A. **74**(1977), 2651–2654.

[13] J. Cheeger, *Analytic torsion and the heat equation,* Ann. of Math. **109**(1979), 259–322.

[14] B. Helffer and J. Sjöstrand, *Multiple wells in the semiclassical limit. I,* Comm. Partial Differential Equations **9**(1984), 337–408.

[15] B. Helffer and J. Sjöstrand, *Puits multiples en mécanique semi-classique. IV. Étude du complexe de Witten,* Comm. Partial Differential Equations **10**(1985), 245–340.

[16] H. Wai, *Witten–Helffer–Sjöstrand theory for a generalized Morse functions,* preprint arXiv:math.dg-ga/9503006

[17] Y. Lee, *Mayer–Vietoris formula for determinants of elliptic operators of Laplace–Beltrami type (after Burghelea, Friedlander, Kappeler),* Differential Geom. Appl. **7**(1997), 325–340.

[18] V. Mathai and D. Quillen, *Superconnections, Thom classes, and equivariant differential forms*, Topology **25**(1986), 85–110.

[19] W. Müller, *Analytic torsion and R-torsion of Riemannian manifolds,* Adv. in Math. **28**(1978), 233–305.

[20] D.B. Ray and I.M. Singer, *R-torsion and the Laplacian on Riemannian manifolds,* Adv. Math. **7**(1971), 145–210.

[21] M. Schwarz, *Morse homology.* Progress in Mathematics **111**, Birkhäuser Verlag, Basel, 1993.

[22] M.A. Shubin, *Pseudodifferential operators and spectral theory.* Translated from the 1978 Russian original by Stig I. Andersson. Second edition. Springer-Verlag, Berlin, 2001.

Department of Mathematics, The Ohio State University,, 231 West 18th Avenue, Columbus, OH 43210, USA.

E-mail address: burghele@mps.ohio-state.edu

Department of Mathematics, University of Vienna,, Nordbergstrasse 15, A-1090 Vienna, Austria.

E-mail address: stefan.haller@univie.ac.at