DIFFEOMORPHISMS K THEORY AND FREE-LOOP SPACES

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Abstract. This is more or less what I know about automorphisms of compact smooth and topological manifolds and their relationship with Algebraic K- theory reformulated as equivariant cohomology of the free loop spaces.All the results described here were actually obtained before 1985. This material represents my Notes for a three (survey) lectures I have given at Göettingen summer school in geometry and groups , June 2000.

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0. A result in geometric topology.

There are two basic categories for studying manifolds, the "smooth" category, denoted by Diff or shorter D, whose the objects are smooth manifolds and morphisms smooth maps and and the "topological" category,denoted by Top or shorter T, whose the objects are topological manifolds and morphisms continuous maps. We will also denote by H the category whose objects (are Poincaré Duality) spaces and morphisms are continuous maps. It is understood that any smooth manifold can be regarded as a topological manifold and any topological manifold as a Poincare Duality space. All our manifolds are compact (possibly with boundary).

The basic group in the topology of Topological manifolds is

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$$Top_n := \text{Homeo}(\mathbb{R}^n, 0)$$

The analogous group for the topology of Smooth manifolds is

$$\text{Diff}(\mathbb{R}^n, 0)$$

$\text{Diff}(\mathbb{R}^n, 0)$ contains the orthogonal group O_n as a deformation retract. The relationship between O_n and Top_n can be explained in terms of the arrows of the diagram

$$\begin{array}{ccccccc}
\cdots Top_n/O_n & \xrightarrow{i_n} & Top_{n+1}/O_{n+1} & \longrightarrow & \cdots & \longrightarrow & Top/O \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots Top_n & \longrightarrow & Top_{n+1} & \longrightarrow & \cdots & \longrightarrow & Top \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots O_n & \longrightarrow & O_{n+1} & \longrightarrow & \cdots & \longrightarrow & O
\end{array}$$

and the map

$$s_n : Top_n/Top_{n-1} \to \Omega Top_{n+1}/Top_n$$

defined by

$$s_n^T(h) := \begin{pmatrix} R_\theta & 0 \\ o & Id \end{pmatrix} \cdot h \cdot \begin{pmatrix} R_{-\theta} & 0 \\ o & Id \end{pmatrix}$$

with R_θ representing the rotation of angle θ in $\mathbb{R}^2$. Note that s_n^T restricts to

$$s_n^O : O_n/O_{n-1} \to \Omega O_{n+1}/O_n$$

and induces the commutative diagram

$$\begin{array}{ccc}
Top_n/Top_{n-1} & \xrightarrow{s_n^T} & \Omega Top_{n+1}/Top_n \\
{\scriptstyle j_n}\uparrow & & {\scriptstyle \Omega j_{n+1}}\uparrow \\
O_n/O_{n-1} & \xrightarrow{s_n^O} & \Omega O_{n+1}/O_n
\end{array}$$

One denotes by

$$F_n = \Lambda(Top_{n+1}/O_{n+1}, Top_n/O_n) \sim \Lambda(Top_{n+1}/Top_n, O_{n+1}/O_n)$$

the homotopy theoretic fiber of the inclusion $i_n : Top_n/O_n \to Top_{n+1}/O_{n+1} \sim O_{n+1}/O_n \to Top_{n+1}/Top_n$.

Then $s_n^{\cdots}$ induce

$$s_n : F_n \to \Omega F_{n+1}$$

and the commutative diagram

$$\begin{array}{ccc} Top_{n+1}/O_{n+1} & \longrightarrow & \Omega Top_{n+2}/O_{n+2} \\ \uparrow & & \uparrow \\ Top_n/O_n & \longrightarrow & \Omega Top_{n+1}/O_{n+1} \\ \uparrow & & \uparrow \\ F_n & \xrightarrow{s_n} & \Omega F_{n+1} \end{array}$$

It is understood that the relevant topology for Homeo is the C^0 while for Diff is C^∞. Both agree on O_n. Most often we regard the above groups as simplicial groups, cf section 2 below.

Well known facts :

1): $O_{n+1}/O_n = S^n$

2): Bott periodicity:

$$\pi_{i-1}(O) = \left\{ \begin{array}{c} \mathbb{Z} \text{ if } i = 0,4 \text{ (mod 8)} \\ \mathbb{Z}_2 \text{ if } i = 1,2 \text{ (mod 8)} \\ 0 \text{ if } i = 3,5,6,7, \text{ (mod 4)} \end{array} \right\}$$

hence

$$\pi_i(O) \otimes \mathbb{Q} = \left\{ \begin{array}{c} \mathbb{Q} \text{ if } i = 0 \text{ (mod 4)} \\ 0 \text{ if } i = 1,2,3 \text{ (mod 8)} \end{array} \right\}$$

3): For any i, $\pi_i(top/O)$ is finite and identifies to the group of SMOOTH STRUCTURE ON S^i. (when $i \neq 3,4$).

Statements analogue to (1) and (2) are not true about Top_n. However:

$$\pi_i(Top_{n+1}/Top_n) = \pi_i(S^n) \oplus \pi_{i-1}(F_n).$$

which can be derived from the triviality of the bundle

$$\Omega(O_{n+1}/O_n) \to \Omega(Top_{n+1}/Top_n) \to F_n$$

[1] cf [BL1] part II, Prop 5.3. and we have

Theorem. *The canonical map s_n is $(n-9)/3$ connected*

The spaces F_n and the connecting maps s_n define the connected spectrum $\mathbb{F}$, and we have

[1] actually this bundle deloops to $O_{n+1}/O_n \to Top_{n+1}/Top_n \to BF_n$

Theorem.

$$\pi_i(\mathbb{F}) \otimes \mathbb{Q} = \left\{ \begin{array}{l} \mathbb{Q} \ if \ i = 4k, k = 1, 2, \cdots \\ 0 \ if \ i \neq 4k \end{array} \right\}$$

One expects that these theorems be proven (or provable) in the framework of GEOMETRIC TOPOLOGY without the use of differential topology and of hard analysis. So far this was not yet done but most likely it is possible and important.

The first Theorem follows from the work of C.Morlet and Burghelea-Lashof, [BL1] which implies that $BC_D(D^n) \sim \Omega^n(F_n)$ combined with the work of K.Igusa which establishes the stability property for $C_D(D^n)$. This work uses analysis of singularities of smooth maps including the theory of universal unfoldings.

The second Theorem follows from the work of Waldhausen who proved that $\pi_i(F) \otimes Q = Prim(H_*(GLZ; \mathbb{Q}))_{*+1}$ and of Borel who has calculated $H_*(GLZ; \mathbb{Q})$. Again the proof of Borel uses analysis.

Although these two theorems belong entirely to the field of geometric topology (i.e.= topology of topological manifolds) they do permit :

1) to formulate (in algebraic manner) necessary and sufficient conditions for a map between two compact smooth manifolds to be homotopic to a smooth submersion

2) to calculate (at least rationally) the homotopy groups of $B(Diff(M, \partial M))$ which are essentially the same as the homotopy groups of the space of Riemannian structures of M and then have relevance in Riemannian geometry.

Finding the "right" proof of the above theorems, as well as additional implications of the above results in Math Phys.,Geometry, Analysis, will very likely have a considerable impact in topology.

1. Concordances stabilization and the free loop space.

1):The space of concordances was first introduced by Cerf [C] in the smooth category.

2)Hatcher [H], Burghelea-Lashof [B-L], Igusa [I] have noticed (and proven) a remarkable property, of the space of concordances, its homotopical stability (cf below).

3)Morlet, Burghelea-Lashof- Rothenberg [BLR], Goodwillie [G1] have noticed a strong connectivity property of concordances spaces which, together with (2), lead to a homotopy functor with values in the infinite loop spaces (or $\Omega-$ spectra) and then via homotopy groups to graded abelian groups.

4) Waldhausen [W] has identified this functor as the "Whitehead theory", a companion of the "Algebraic K theory of spaces", an other homotopy functor he has introduced.

5)Burghelea [B1],(B+Fedorowicz), [B-F] and Goodwillie [G2] have expressed this K-theory, and therefore the concordance theory in terms of configuration space of the nonparametrized free loop spaces. The homotopy groups of this space are isomorphic to the S^1-equivariant homology groups of the free loop space.

6)The results referred to in 5) represent calculations of homotopy groups tensored by rational numbers $\mathbb{Q}$. Integral results (improvements) have been subsequently obtained by Carlson, Cohen, Goodwillie and Hsiang (cf [CCGH]).

In this lecture I will review these results, (1)-(5), and make few comments about the mathematics behind them.

Definitions and geometric constructions

Let $(M, \partial M)$ be a compact manifold in one of the categories $A =$ D, T.
Denote by $A(M, K)$ the group of automorphisms, (diffeomorphisms or homeomorphisms) which restrict to the identity on $K \subset M$.

Definition 1.
1): $\mathbb{C}_A(M) := A(M \times I, \partial(M \times I)) \ / A_s(M \times I, \partial(M \times I))$
where $A_s(M \times I, \partial(M \times I)) := \{\alpha \in A(M \times I, \partial(M \times I) \ |, p_2\alpha = p_2\}$ *and* $p_2 : M \times I \to I$ *the second factor projection.*
2): $C_A(M) := A(M \times I, \partial(M \times I) \cup (M \times 0))$

Note that the obvious map $C_A(M) \to \mathbb{C}_A(M)$ is a homotopy equivalence and from now on we will not differentiate between $C(M)$ and $\mathbb{C}(M)$.

Definition 1 is good to define the *transfer* and the *involution*, cf C and D below, Definition 2 is good for delooping the space of concordances, cf E below, and for alternative description the *stabilization map.*

A: $N^n \subset M^n$ proper embedding (i.e. $\partial N \pitchfork \partial M$) either one of the definitions 1) or 2) is good to define

$$i_{N,M} : C(N) \to C(M)$$

and then $BC(N) \to BC(M)$. The inclusion $i_{N,M}$ is defined *extending by identity* outside N.

B: The *stabilization map* Σ_M*:*

$$\Sigma_A(M) : C(M) \to C(M \times I)$$

is defined essentially by product with $id : I \to I,\ \ I = [0,1]$ or$[-1,1]$. Equivalently by rotating $M \times [0,1] \times 0$ inside $M \times [0,1] \times [-1.+1]$ about $M \times 0 \times 0$ and taking $\Sigma_A(M)(h)$ to be the rotation of h extended by the identity (outside the domain spanned by $M \times [0,1] \times 0$ inside $M \times [0,1] \times [-1.+1]$).

C: The *transfer:*
For $\xi : E \to B$ a smooth bundle with fibers compact manifolds (possibly with boundary) one can define (cf [BL2])

$$T_\xi : BC(B) \to BC(E)$$

The stabilization map Σ is homotopic to the transfer associated with the trivial bundle with fiber I.

D: The *involution:*

$$\tau : \mathbb{C}(M) \to \mathbb{C}(M)$$

is defined by conjugation with $\tau : M \times I \to M \times I$ given by $\tau(x,t) = (x, 1-t)$

E: $BC(M \times I^k)$ has a natural $k-$ times loop space structure provided by juxtaposition. In fact one can show that[2] $BC(M \times I^k) \sim \Omega^k BC^{bd}(M \times R^k)$

All these maps , $i_{M,N}, \Sigma, T_\xi$ and the deloopings are compatible. More details about the above constructions and about the compatibility of these maps can be found in [BL2].

[2]In case of an open manifolds equipped with a complete metric (N,ρ), one consider bounded automorphisms $\alpha : N \to N$, i.e $d(x, \alpha(x)) < C$ for any x and some C, and denote by $A^{bd}(N)$ the group of such automorphisms with the appropriate topology

The geometric results:

Theorem 1. *(Connectivity Theorem [BLR]) Let $N^n \subset M$ proper embedding so that*
1) $\pi_i(\partial M) = \pi(M)$*, $i = 0, 1$ and $n \geq 5$*
2)$\pi_i(M, N) = 0, \quad i \leq r, \ r \leq n - 4$
3)$\pi_i(M) = 0, \ i \leq k$
Then the map $i_{M,N} : C_A(N) \to C_A(M)$ is
$\inf(2r-3, r+k-2)-$*connected for $A =$ Diff and*
$\inf(2r-3, r+k-2, r+2)-$*connected for $A =$ Top.*

Note that the connectivity in the topological category is worse than in differential category.

A stability range will be a nondecreasing integer valued function $\omega(n)$ for which there exists N, an integer, so that $\omega(n+N) > \omega(n)$ for any n. Theorem 2 below establishes the existence of a stability range $\omega_A(n)$ which makes $\Sigma(M)$ an $\omega_A(n)$ connected map.

Theorem 2. *(Stability for concordances) There exists the stability range functions $\omega_A(n)$ so that $\Sigma(M^n)$ is $\omega_A(n)$ connected and*
1) $\omega_D(n) \geq (n-9)/3 \sim n/3$ *(cf [I])*
2) $2\omega_A(n) + 1 \geq \omega_B(n)$*, A, B being D, T (implicit in [BL2]*
3) the map of pairs $\Sigma_{M,N} : (C(M), C(N)) \to (C(M \times I), C(N \times I))$ is $(n-3)$ connected[3] *when $A =$ Diff. (implicit in [G1]).*

The geometrically constructed maps, i, Σ, T, together with Theorems 1 and 2 above permit to construct two homotopy functors:

$$X \longmapsto Wh_D(X), \text{ and } X \longmapsto Wh_T(X)$$

from the category of compact ANR to the category of ∞-loop spaces (or equivalently $(-1)-$connected $\Omega-$spectra, and a natural transformation

$$w(X) : Wh_D(X) \to Wh_T(X)$$

between these functors (cf [BL2]) so that:

[3]the induced map between the homotopy theoretic fibers is $(n-3)$ connected

Theorem 3.

1) The functors have transfer (and behave as expected with respect to the transfer)

2) The functors have the strong connectivity property[4].

3)Each stable spherical fibration $\xi : X \to BG$ [5] *induces the involution up to homotopy* $\tau_A(\xi) : Wh_A(X) \to Wh_A(X)$ *which is intertwined by* $w(X)$.

4)$Wh_T(pt)$ *is contractible and the homotopy fiber of* $w(X)$ *is* $< X \wedge Wh_D(pt) >$[6]

5) For any compact manifold $(M, \partial M)$ *there exists a natural map*

$$i(M) : BC_A(M) \to \Omega Wh_A(M)$$

which is $\omega_A(n)$ *connected and if* n *is even / odd, intertwines /skew intertwines the involution* τ *on* $BC(M)$ *with the involution on* $Wh_A(M)$, *induced from the tangent spherical fibration of* M *(cf 3 above).*

Actually in [BL2] one construct geometrically the functor $\Omega Wh_A(X)$ In view of E, it is not hard to see that it has a canonical delooping. The functor is defined by assigning to each compact manifold with boundary the space $\mathbb{B}C(M) := \lim_{\Sigma: C(M\times I^k)\to C(M\times I^{k+1})} C(M \times I^k)$, and for an embedding $f : M \to N \times I^k$ with normal disc bundle $D(\nu) \to M$ the composition $T_\nu \cdot i_{D(\nu),M\times I^k}$. One uses the fact that any compact ANR, X, has the homotopy type of a compact manifold (with boundary) M and for any $f : M \to N$ one can produce for k large enough an embedding $\tilde{f} : M \to N \times R^k$ unique up to an isotopy.One take $\Omega Wh_A(X); = \mathbb{B}C(M)$

Waldhausen has provided a purely homotopic construction for $Wh_A(X)$ and for a compact manifold M he has constructed a natural homotopy equivalence from $\mathbb{B}C(M)$ to $\Omega Wh_A(M)$. His construction of Wh_A is based on "Waldhausen algebraic $K-$theory" of a topological space cf [W].

The free loop spaces:

Suppose X an ANR. Denote by:

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[4]this means that $Wh_A(f)$ is $(k-1)-$connected when f is $k-$ connected, and transforms a diagram

$$\begin{array}{ccc} A\cap B & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & A\cup B \end{array}$$

whose horizontal arrows are $n-$connected and vertical arrows $m-$ connected into a commutative diagram whose induced map between the homotopy fibers of the horizontal arrows is $(n+m-2)$ connected

[5]BG denotes the classifying space of such fibrations

[6]$< X \wedge Wh_D(pt) >$ denotes the zero component of the $\Omega-$ spectrum associated with the spectrum $X \wedge BO$

$X^{S^1} := \mathrm{Maps}(S^1, X)$,
$\mu : S^1 \times X^{S^1} \to X^{S^1}$ the obvious action,
$\psi_{-1} : X^{S^1} \to X^{S^1}$ the involution induced by "going reverse",
$\psi_n : X^{S^1} \to X^{S^1}$ the map induced by going around the free loop n-times.
$X^{S^1}//S^1$, the homotopy quotient of μ.[7]
X identifies to the constant free loops = the fixed points set of μ.
Consider the Gysin sequence

$$(*) \qquad \cdots \to H_*(X^{S^1} : \mathbb{Q}) \to H_*^{S^1}((X^{S^1} : \mathbb{Q}) \to H_{*-2}^{S^1}(X^{S^1} : \mathbb{Q}) \\ \to H_{*-1}(X^{S^1} : \mathbb{Q}) \to \cdots$$

The maps ψ_{-1} and ψ_n induce the endomorphisms

$$\tau : H_*(X^{S^1}; \mathbb{Q}) \to H_*(X^{S^1}; \mathbb{Q})$$

$$\tau : H_*^{S^1}(X^{S^1}; \mathbb{Q}) \to H_*^{S^1}(X^{S^1}; \mathbb{Q})$$

and

$$\Psi_n : H_*(X^{S^1}; \mathbb{Q}) \to H_*(X^{S^1}; \mathbb{Q})$$

$$\Psi_n : H_*^{S^1}(X^{S^1} : \mathbb{Q}) \to H_*^{S^1}(X^{S^1} : \mathbb{Q})$$

Theorem4. *[BFG] Suppose X 1-connected.*

1) Ψ_n are isomorphisms and all its eigenvalues are n^r, $r = 0, 1 \cdots n, \cdots$. For each n^r, $n > 1$, the corresponding eigenspace is independent on n (and denoted by $H_*(X^{S^1} : \mathbb{Q})(r)$ resp. $H_*^{S^1}(X^{S^1} : \mathbb{Q})(r)$).

One has the decompositions

$$H_*(X^{S^1}; \mathbb{Q}) = \bigoplus_{r \geq 0} H_*(X^{S^1}; \mathbb{Q})(r)$$

$$H_*^{S^1}(X^{S^1}; \mathbb{Q}) = \bigoplus_{r \geq 0} H_*^{S^1}(X^{S^1}; \mathbb{Q})(r)$$

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[7] $X^{S^1}//S^1 = ES^1 \times_{S^1} X$

where:

$$H_*(X^{S^1};\mathbb{Q})(0) = H_*(X,\mathbb{Q}),\ \ H_*(X^{S^1};\mathbb{Q})(r) = 0, \quad \textit{for } r > *$$

$$H_*^{S^1}(X^{S^1};\mathbb{Q})(0) = H_*^{S^1}(pt;\mathbb{Q}),$$

$$H_*^{S^1}(X^{S^1};\mathbb{Q})(1) = \tilde{H}_{*+1}(X,\mathbb{Q})$$

$$H_*^{S^1}(X^{S^1};\mathbb{Q})(r) = 0, if \ r > *.$$

2) The Gysin sequence $(*)$ *decomposes as a direct sum of the exact sequences*

$$\cdots \to H_*(X^{S^1}:\mathbb{Q})(r) \to H_*^{S^1}((X^{S^1}:\mathbb{Q})(r) \to$$
$$H_{*-2}^{S^1}(X^{S^1}:\mathbb{Q})(r) \to H_{*-1}(X^{S^1}:\mathbb{Q})(r-1) \to \cdots$$

3)The involution τ *is identity on* $H^{\cdots}_{\cdots}(2r)$ *and* $-id$ *on* $H^{\cdots}_{\cdots}(2r+1)$

Denote by

$$\mathcal{H}_*^{S^1}(X^{S^1};Q) := \bigoplus_{r\geq 2} H_*^{S^1}((X^{S^1}:\mathbb{Q})(r).$$

Here is the main result about the calculation of $\pi_i(Wh_A(X)) \otimes \mathbb{Q}$ when X is 1-connected.

Theorem5. *([W], [B], [BF])*

1) $\pi_i(BC_T(D^n)) = 0.$

2): $\pi_i(BC_D(D^n) \otimes \mathbb{Q} = \left\{ \begin{matrix} Q & \textit{if } i = 4k, \\ 0 & \textit{if } i \neq 4k \end{matrix} \right\}$ *if* $i \leq \omega_D(n)$.

3)$\pi_i(BC_D(M^n), (BC_D(D^n)) \otimes \mathbb{Q} = \mathcal{H}_i^{S^1}(X^{S^1};Q)$ *if* X *is 1-connected and* $i \leq \omega_D(n)$.

4)$\pi_i(BC_T(M)) \otimes \mathbb{Q} = \mathcal{H}_i^{S^1}(X^{S^1};Q) \oplus (\tilde{KO}_i(M) \otimes \mathbb{Q})$ *if* X *is 1-connected and* $i \leq \omega_T(n)$
These isomorphisms intertwine / skew intertwine the involutions if $\dim M$ *is even / odd. (The involution on* $\mathbb{Q}$ *in the right side of formula (2) and on* $\tilde{KO}_i(M) \otimes \mathbb{Q}$ *is* $-id$*)*

Let $Y \subset X$, and let

$$\mathcal{C}(X,Y;\mathbb{K}) := \{\alpha : X \setminus Y \to \mathbb{K} | \sharp(\text{ supp}(\alpha) < \infty\}$$

be the set of configurations of points in $X \setminus Y$ with " charges" in $\mathbb{K}$, equipped with the collision topology[8] where $\mathbb{K}$ a commutative unital ring like $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, etc.

By a Theorem of Dold-Thom

$$\pi_i(\mathcal{C}(X, Y; \mathbb{K})) = H_i(X, Y; \mathbb{K}).$$

Theorem 5 suggests the existence of a map $Wh_D(X) \to \mathcal{C}(X^{S^1}//S^1, pt^{S^1}//S^1; \mathbb{R})$ which induces the isomorphisms stated above. It is an interesting problem to construct geometrically such map.This question will be addressed again in the last section.

The proof of Theorem 5 at present is very indirect and can not be done without

1)Waldhausen description of stabilized concordances in terms of Algebraic $K-$theory,

2)the isomorphism between the reduced Waldhausen $K-$theory of a compact ANR X tensored by the rational numbers and S^1-equivariant homology with rational coefficients of the free loop space of X in the case X is 1-connected, established by [B3],

3)the structure of this S^1 equivariant cohomology as established in BFG.

It is however very likely that a direct proof of Theorem 5, in the framework of geometric analysis is possible. Such proof will undoubtedly be very interesting and likely open new connections between topology and analysis (elliptic operators).

2.Automorphisms of manifolds.

Consider $A(M, \partial M)$, $A =$ Diff, Top, H, the space of self automorphisms (diffeomorphisms, homeomorphisms, simple homotopy equivalences) of the compact manifold M which restrict to id on ∂M equipped with the proper topology[9] If $A =$ Diff, Top then $A(M, \partial M)$ is a topological group and if $A =$ H is a topological monoid. We are interested in $A(M, \partial M)$, $A =$ Diff, Top and treat $H(M, \partial)$ as known since this objet belong to homotopy theory[10]

There is a simplicial version of $A(M, \partial M)$ which is very convenient to describe the homotopy type of $A(M, \partial M)$ and its associated homogeneous spaces.

Definition. *A k simplex of $A(M, K)$ is a diffeomorphism, homeomorphism, homotopy equivalence α so that:*

[8]if $\alpha : X \setminus Y \to \mathbb{R}$ is a configuration with support $\{x_1, x_1, \cdots x_r\}$ an ϵ neighborhood of α consists of all configurations whose support lie in the $\epsilon-$ neighborhood of $Y \cup x_1 \cdots \cup x_r$ and for any $0 \leq i \leq r$ satisfies $\alpha(x_i) = \sum_{\{y|d(y,x_i)<\epsilon\}} \beta(y)$

[9]C^∞ if $A =$ Diff and C^0 if $A =$ Top, H.

[10]if X is 1-connected a rational minimal model provides a canonical model (and then a minimal model)for $BH(X)$. Such model has been constructed by Sullivan [S].

(1)

$$\begin{array}{ccc} M \times \Delta(k) & \xrightarrow{alpha} & M \times \Delta(k) \\ \downarrow{\scriptstyle pr_2} & & {\scriptstyle pr_2}\downarrow \\ \Delta(k) & \xrightarrow{id} & \Delta(k) \end{array}$$

(2) $\alpha|K \times \Delta(k) = Id.$

(1) and (2) imply that:

(i): $\alpha(M \times d_I(\Delta(k)) \subset (M \times d_I(\Delta(k)).$

This suggests the definition of the following simplicial group resp.monoid $\tilde{A}(M, \partial M)$.

Definition. *A k simplex of* $\tilde{A}(M, \partial M)$ *is a diffeomorphism, homeomorphism, homotopy equivalence* $\alpha : M \times \Delta(k) \to M \times \Delta(k)$ *so that (2) and (i) above hold.*

Although $\tilde{A}(M, \partial M)$ is not a genuine simplicial group, one can work with it as with a simplicial group, cf[BLR].

It will be convenient sometimes to write $\tilde{A}/A(M, \partial M)$ for $\tilde{A}(M, \partial M)/A(M, \partial M)$

Observations:

(1) the inclusion $H(M, \partial M) \subset \tilde{H}(M, \partial M)$ is a homotopy equivalence,

(2)$\tilde{C}(M, \partial M)$ is contractible

(3) $\tilde{A}(M \times I, \partial(M \times I)) \sim \Omega\tilde{A}(M, \partial M)$. As in section1 D, the map $\tau : M \times I \to M \times I$ induces an involution $\tau : \tilde{A}(M \times I, \partial(M \times I)) \to \tilde{A}(M \times I, \partial(M \times I))$ which, in view of this identification, is homotopic to $-Id$.

The simplicial complexes $\tilde{A}(M, \partial M)$ are particular useful because:

(A) The homotopy type of the quotient simplicial set $\tilde{H}/\tilde{A}$ can (in principle) be described (reduced to homotopy theory) with the help of surgery theory (cf [BLR]). In particular if M is one connected then

$$\pi_i(\tilde{H}(M, \partial M)/\tilde{A}(M, \partial M)) \otimes \mathbb{Q} = KO_{n+1}(M) \otimes \mathbb{Q}$$

where KO_* denotes the Atiyah-Hirtzebruch real (based on real vector bundles) $K-$homology theory, cf[BLR].

(B) The fibration

$$\frac{\tilde{A}(M \times I, \partial(M \times I)}{A(M \times I, \partial(M \times I)} \xrightarrow{i} \frac{\tilde{C}(M)}{C(M)} = BC(M) \xrightarrow{p} \frac{\tilde{A}(M, \partial M)}{A(M, \partial M)}$$

provides the basis of the relationship between $BC(M, \partial M)$ and $A(M, \partial M)$ as described below.

Let X_{odd} be the localization at "2" of X; i.e. $\pi_i(X_{odd}) = \pi_i(X) \otimes \mathbb{Z}_{odd}$ if $\pi_1(X)$ is commutative, where $\mathbb{Z}_{odd} = \{\frac{p}{2^k} \in \mathbb{Q}\}$. Precisely there exists a functor $(\cdot)_{odd}$ (unique up to homotopy) from topological spaces to topological spaces and a natural transformation j, from the functor Id to $(\cdot)_{odd}$ so that the map $j(X) : X \to X_{odd})$ induces (for the homotopy groups of a space with abelian fundamental group) the canonical homomorphism $\pi_i(X) \to \pi_i(X) \otimes \mathbb{Q}$.

Theorem6. *([BL3]) If the* $\dim(M) = n$ *then the fibration*

$$(\frac{\tilde{A}(M \times I, \partial(M \times I)}{A(M \times I, \partial(M \times I)})_{odd} \xrightarrow{i} BC(M)_{odd} \xrightarrow{p} (\frac{\tilde{A}(M, \partial M)}{A(M, \partial M)})_{odd}$$

is trivial in stability range $\omega_A(n)$.

Theorem 6 above will lead to a description of the homotopy groups (actually of homotopy type) of $BA(M, \partial M)$ in terms of the homotopy type of X and surgery theory away of prime 2 and in stability range.

To explain this description we need few definitions and one additional element, the natural map

$$BH(X) \to Wh_A(X)$$

When X is an infinite loop space and $\tau : X \to X$ is an involution up to homotopy then there exists a canonical decomposition $X_{odd} \sim X_{odd,+} \times X_{odd,-}$ characterized by the fact that the projection of X_{odd} onto $X_{odd,\pm}$ induces for homotopy groups the canonical projection of $\pi_*(X) \otimes \mathbb{Z}_{odd}$ onto $(\pi_*(X) \otimes \mathbb{Z}_{odd})\pm$. (If W, τ is a $\mathbb{Z}_{odd}$ module with involution then $W_\pm = \{x \in W \mid \tau(x) = \pm x\}$.)

Theorem 7. *(The structure theorem [BL3])*

1)Let X *be a compact ANR. There exists the natural maps (up to homotopy)* $o_D : BH(X) \to Wh_D(X)$ *and* $o_T : BH(X) \to Wh_T(X)$ *so that:*

i) $w(X) \cdot o_D(X) = o_T(X)$

ii) $\tau_D(\xi) \cdot o_D(X) = o_D(X) \cdot \tau_D(\xi)$

iii) $\tau_T(\xi) \cdot o_T(X) = o_T(X) \cdot \tau_T(\xi)$ *for any* X *compact ANR and any stable spherical fibration* ξ *on* X.

Here $H(X)$ *denotes the space of all homotopy equivalences. The maps* $o_A(X)$ *composed with the canonical projections* $Wh(X) \to Wh_A(X)_odd \to (Wh_A(X))_{odd,\pm}$ *induce the commutative diagrams whose horizontal lines are fibrations*

$$\begin{array}{ccccc} \Omega(Wh_A(X))_{odd,\pm} & \longrightarrow & \mathcal{E}_\pm & \longrightarrow & BH(X) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle id} \\ \Omega(Wh_A(X))_{odd,\pm} & \longrightarrow & \mathcal{E}_\pm & \longrightarrow & BH(X) \end{array}$$

2) Suppose that $(M^n, \partial M^n)$ is a compact manifold in the geometric category A. One has the commutative diagram

$$\begin{array}{ccccc} \Omega(Wh_A(M))_{odd,\epsilon(n)} & \longrightarrow & \mathcal{E}_{\epsilon(n)} & \longrightarrow & B\tilde{H}(M) \\ \uparrow & & \uparrow & & in\uparrow \\ \frac{\tilde{A}(M,\partial M)}{A(M,\partial M)} & \longrightarrow & BA(M,\partial M) & \longrightarrow & B\tilde{A}(M,\partial M) \end{array}$$

where the right vertical arrow is the canonical inclusion of $\tilde{A}(M, \partial M)$ and the left vertical arrow is the same as the right vertical arrow in the diagram

$$\begin{array}{ccccc} \frac{\tilde{A}(M\times I,\partial(M\times I)}{A(M\times I,\partial(M\times I))} & \xrightarrow{i} & BC(M) & \xrightarrow{p} & \frac{\tilde{A}(M,\partial M)}{A(M,\partial M))} \\ \downarrow & & j(\Omega Wh_A(M))i(M)\downarrow & & \downarrow \\ \Omega Wh_A(M)_{odd,-\epsilon(n)} & \longrightarrow & \Omega Wh_A(M)_{odd} & \longrightarrow & \Omega Wh_A(M)_{odd,\epsilon(n)} \end{array}$$

and is $\omega_A(n)$ connected after localization at "2".

As a consequence of the above theorem and of surgery theory one obtains

Theorem 8.

Suppose $(M, \partial M)$ is a compact smooth manifold of dimension n so that M is 1-connected. We have the following short exact sequence of $Q-$vector spaces

$0 \to \mathcal{H}_*^{\epsilon(n)}(M^{S^1}//S^1) \xrightarrow{i} \pi_*(H(M, \partial M)/\mathit{Diff}(M, \partial M)) \otimes \mathbb{Q} \to KO_{N+*}(M) \otimes \mathbb{Q} \to 0$

$0 \to \mathcal{H}_*^{\epsilon(n)}(M^{S^1}//S^1) \oplus (\tilde{KO}_i(M) \otimes \mathbb{Q})^{\epsilon(n)} \to \pi_*(H(M, \partial M)/\mathit{Homeo}(M, \partial M)) \otimes \mathbb{Q}$
$\to KO_{n+*}(M) \otimes \mathbb{Q} \to 0$

Moreover there exists a natural homomorphism

$$\theta : \pi_*(H(M, \partial M)/Diff(M, \partial M)) \otimes \mathbb{Q} \to \tilde{H}_*^{S^1}(M^{S^1}; \mathbb{Q})$$

which provides an inverse to i in the first short exact sequence

Smoothing theory.

Let M be a topological manifold of dimension n. The topological tangent bundle (= microbundle) of M gives a Top_n-principal bundle

(*) $$Top_n \to \mathcal{E}_M \to M$$

and therefore the bundle

$$(**)\qquad Top_n/O_n \to E_M \to M$$

associated with $(*)$ and the obvious action of Top_n on Top_n/O_n.

If M is a manifold with boundary one get the commutative diagram whose horizontal rows are bundles of type $(**)$ and vertical arrows are inclusions.

$$(***)\qquad \begin{array}{ccccc} Top_n/O_n & \longrightarrow & E_M & \xrightarrow{\pi_M} & M \\ \uparrow & & \uparrow & & \uparrow \\ Top_{n-1}/O_{n-1} & \longrightarrow & E_{\partial M} & \xrightarrow{\pi_{\partial M}} & \partial M \end{array}$$

Recall that a section s in the diagram $(***)$ is a section $s : M \to E_M$ of $(**)$ so that $s(\partial M) \subset E_{\partial M}$. Denote by

$$\Gamma(M, \partial M; \mathcal{E}_M, \mathcal{E}_{\partial M})$$

the space of all sections of $(***)$ with the C^0 topology.

Theorem.
There exist smooth structures of M iff the bundle $(***)$ has sections and any such smooth structure defines a section unique up to homotopy.

Suppose $(M, \partial M)$ has a smooth structure and let $s_0 \in \Gamma(M, \partial M; \mathcal{E}_M, \mathcal{E}_{\partial M})$ be the corresponding section. Suppose $K \subset M$ be a codimension zero submanifold.

Denote by $\Gamma_K(M, \partial M; \mathcal{E}_M, \mathcal{E}_{\partial M}, s)$ the subspace of $\Gamma(M, \partial M; \mathcal{E}_M, \mathcal{E}_{\partial M})$ consisting of the sections which agree to s above K.

Theorem 9. *([M],[BL1]) There exists a map*

$$(Homeo(M, K)/Diff(M, K) \to \Gamma_K(M, \partial M; \mathcal{E}_M, \mathcal{E}_{\partial M}, s)$$

which induces an injective map between connected components and when restricted to each component is a homotopy equivalence.

If M is parallelizable then

$$\Gamma(M, \partial M; \mathcal{E}_M, \mathcal{E}_{\partial M}) \sim \mathrm{Maps}(M, \partial M; Top_n/O_n, Top_{n-1}/O_{n-1})$$

$$\Gamma_K(M, \partial M; \mathcal{E}_M, \mathcal{E}_{\partial M}, s) \sim \mathrm{Maps}(M, \partial M, K; Top_n/O_n, Top_{n-1}/O_{n-1}, *)$$

3.About proofs and open questions.

Concerning Theorem 1: The proof of Theorem 1 is contained in [BLR] section 1, and is based on Morlet disjunction lemma (whose complete proof is also contained in [BLR] section 1).

Concerning Theorem 2: Strictly speaking, in the published literature, Theorem 2 is proven only for manifolds which admit a smooth structure. (One produce a stability range $\omega_T(n)$ apriory valid for topological manifolds which admit at least one smooth structure. This range is derived via [BL1] from the stability range $\omega_D(n)$. It is possible (but not trivial) to show that the stability theorem for concordances holds for all topological manifolds but at half of the above mentioned mentioned range.

It is important to provide a proof for stability theorem inside the category of topological manifolds. Most likely such proof will lead to the right value of the stability range $\omega_T(n)$ which, most likely, is much better than what we know at this time.

The proof of part 1 of Theorem 1 is given in [I], the proof of part (2) is implicit in [BL2] and is based on the results of [BL1] and the proof of part 3 is contained in [G1]. Part 3 and Part 1 for $M = D^n$ imply Part 1 for an arbitrary manifold M. In the sequel I will explain Igusa's proof of Part 1 in the case $M^n = D^n$.

Steps in the proof of Stability for Smooth Concordances of D^{n-1} :

Step 1: Establish

$$Emb(D^n)//Diff(D^n) \sim BC(D^{n-1})$$

Step 2: Define

$\mathcal{H}(R^n) := \{h : R^n \to R | \text{so that (i),(ii),(iii) are satisfied } \}$

(i)0 is an absolute minimum, all critical values are strictly smaller than 1,

(ii) all critical points are either Morse or Birth-Death type[11].

(iii) $h^{-1}([0,1])$ is diffeomorphic to D^n.

One can show that the projection

$$p : \mathcal{H}(R^n) \to Emb(D^n)//Diff(D^n) \sim BC(D^{n-1})$$

is a smooth bundle with fiber $F(R^n)$, and one can prove the existence of a natural map in

$$F(R^n) \to \Omega^\infty \Sigma^\infty BO$$

which is a $n-$connected. Indeed if we denote by $D_-^{n-1} \subset \partial D^n$ the lower hemisphere of $S^{n-1} = \partial D^n$ one can observe that:

[11]a Birth-Death critical point of index $(k-1)$ is characterized by the existence of a coordinate system in which $h(x_1, \cdots, x_n) = 1/3x_1^3 - 1/2\sum_{2\leq i\leq k} x_i^2 + 1/2\sum_{(k+1)\leq i\leq n} x_i^2$

a) $Diff(D^n, D_-^{n-1}) \subset Diff(D^n) \to Emb(D_-^{n-1}, S^{n-1})$ is a principal fibration associated with a quotient of simplicial groups, which induces the principal fibration $Diff(D^n) \to Emb(D_-^{n-1}, S^{n-1}) \to BDiff(D^n, D_-^{n-1})$,

b) there is an (obvious) homotopy equivalence $Emb(D_-^{n-1}, S^{n-1} \to Emb(D^n, \mathbb{R}^n)$

c) $BC(D^{n-1} \sim BDiff(D^n, D_-^{n-1})$.

Combining these facts together one obtains the homotopy equivalence

$Emb(D^n)//Diff(D^n) \sim BC(D^{n-1})$. The $n-$ connectivity of $F(R^n) \to \Omega^\infty\Sigma^\infty BO$ is an essential part of the proof. It is contained in [I2]

Step 3:Let $S : \mathcal{H}(R^n) \to \mathcal{H}(R^{n+1})$ be the map defined by $S(h)(x_1, \cdots, x_n, x_{n+1}) := h(x_1, \cdots, x_n) + x_{n+1}^2$. One can show that the following diagram

$$\begin{array}{ccccccc} \Omega^\infty\Sigma^\infty BO & \longleftarrow & F(R^n) & \longrightarrow & \mathcal{H}(R^n) & \longrightarrow & BC(D^{n-1}) \\ \downarrow{\scriptstyle id} & & \downarrow & & {\scriptstyle S}\downarrow & & {\scriptstyle \Sigma}\downarrow \\ \Omega^\infty\Sigma^\infty BO & \longleftarrow & F(R^{n+1}) & \longrightarrow & \mathcal{H}(R^{n+1}) & \longrightarrow & BC(D^n) \end{array}$$

is homotopy commutative and therefore it suffices to check that $S : \mathcal{H}(R^n) \to \mathcal{H}(R^{n+1})$ is $\omega_D(n)$ connected.

Step 4: Introduce the subspace $\mathcal{H}(R^n)_{k+2,k+3} \subset \mathcal{H}(R^n)$ of functions which have only one absolute minimum and all other critical points are either Morse type of index $(k+2)$ and $(k+3)$ or birth -death type of index $(k+2)$ and verify that $S(\mathcal{H}(R^n)_{k+2,k+3}) \subset \mathcal{H}(R^{n+1})_{k+2,k+3}$ and the inclusion $\mathcal{H}(R^n)_{k+2,k+3} \subset \mathcal{H}(R^n)$ is $k-$connected provided $3k+2 \leq n$ and $2k+5 \leq n$.

Step 5: Observe that $\mathcal{H}(R^n)_{k+2,k+3}$ is stratified space whose strata are indexed by symbols $\{r; (k_1, p_1), \cdots, (k_r, p_r)\}$. Such strata consists of functions: with one absolute minima 0 and exactly r additional critical values $c_1 < c_2 < \cdots c_r$ and for each critical value c_i exactly k_i nondegenerate critical points of index $k+2$, k_i nondegenerate critical points of index $(k+3)$ and p_i birth-death critical points of index $(k+2)$.

One verifies that S preserves the strata and under the same dimensional restrictions as in in step 4 is $k-$ connected when restricted to each strata. This implies the result.

Concerning Theorem 3: The proof of Theorem 3 except part 3) is contained in [BL3]. In [BL3] an involution is constructed for any vector bundle and it is implicit in the arguments that up to homotopy it depends only on the stable spherical class of the bundle.

The construction of the involution associated with an arbitrary spherical fibration requires the work of Waldhausen [W]; details of this construction were given in the thesis of his students, W. Vogel.

Concerning Theorem 4: The proof uses cyclic homology and rational homotopy theory and is contained in [BFG]. Part of it is however implicit in the work of

Burghelea Vigue-Poirier.

Concerning Theorem 5: Much of this theorem is a consequence of the work of Waldhausen. The last parts require in addition the relation between Waldhausen K-theory, and cyclic homology and free loop spaces which was done by Burghelea, Fiedorowicz, Goodwillie. For this long story made it short consult [B1]

Concerning Theorems 6,7: The proofs are contained in [BL3]

Concerning Theorem 9: The proof is contained in [BL2]. The theorem was stated first by C.Morlet, however the first proof of this result is probably the probably the one given in [BL3].

Challenges and open problems.

1). Part two and three of Theorem 2 and the stability (of concordances) for D^n in the smooth case imply the stability (for concordances) in both topological and smooth category. The derivation of smooth stability for arbitrary manifolds from the stability for D^n is implicit in Goodwillie's thesis and the derivation of stability in Top-category from from stability in Diff-category is implicit in [BL1]. A self contained presentation of such derivations is inexistent and will be very useful.

2)So far the best ranges of stability are not yet established. I expect $\omega_D \sim n$. I have no reasons to believe that the topological stability range be the same but I do believe that the best rank can be obtain from a the proof of stability within the framework of topological manifolds. So I consider important to

3)Provide a direct proof of stability in topological category.

There are two possible approaches:

One is on the lines of Hatcher, reconsidered in early 80's by myself but unfinished. In that work I have established the stability theore subject to a good theory of regular neighborhoods with parameters (which extend the theorem of tubular neighborhoods with parameters from submanifolds to ANR's inside of a manifold.) The $n-$connectivity of a map can be formulated as the property of that map to induce isomorphisms for homotopy groups in dimension $\leq (n-1)$ and surjectivity in dimension n. The surjectivity part in my conjectural topological stability range, (considerably better than the one we know) can be derived from the published work of Chapman even without full developement of the theory of regular neighborhoods.

The other is on the lines of Goodwillie and will actually require the derivation of relative stability for topological manifolds from a "multiple disjunction lemma" in topological category,for which I can not see conceptual obstructions. (A disjunction lemma was proven in this category in [BLR])

4) Use analytic methods , Chern Weill theory and higher analytic torsion of Bismut -Lott to define the relevant characteristic classes for smooth bundles with fiber D^n.

5)Extend the results about $\tilde{A}(M)/A(M)$ to prime 2 (i.e. drop the *odd*$-$ localization from theorems 6,7).

6) Use geometry to provide a natural map $H(M)/Diff(M) \to \mathcal{C}(X^{S^1}//S^1, BS^1)$. This issue will be discussed in a forthcoming paper of mine.

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