REMOVING METRIC ANOMALIES FROM RAY SINGER TORSION

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Abstract.

Ray Singer torsion is a numerical invariant associated with a compact Riemannian manifold equipped with a flat bundle and a Hermitian structure on this bundle. In this note we show how one can remove the dependence on the Riemannian metric and on the Hermitian structure with the help of a base point and of an Euler structure, in order to obtain a topological invariant. A numerical invariant for an Euler structure and additional data is also constructed.

Contents

0. Introduction.

In this Note all vector spaces V will be over the field $\mathbb{R}$ of real numbers and the scalar products are always positive definite. If V has dimension one then the scalar products on V can be identified to the elements of the quotient space $V/\{-1, +1\}$ of V by the antipodal action.

Let (M^n, m_0) be a base pointed closed smooth manifold with fundamental group $\Gamma = \pi_1(M, m_0)$, and $\rho : \Gamma \to GL(V)$ a finite dimensional representation.

Denote by $\mathcal{F}_\rho = (\mathcal{E}_\rho, \nabla_\rho)$ the associated vector bundle $\mathcal{E}_\rho \to M$ equipped with the canonical flat connection whose holonomy representation at m_0 is given by ρ. The fiber $\mathcal{E}_{m_0}$ of $\mathcal{E}$ above m_0 identifies canonically to V. Note that the data

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(M, m_0, ρ) and $(\mathcal{F} = (\mathcal{E} \to M, \nabla), m_0)$ with ∇ a flat connection in the bundle $\mathcal{E} \to M$ are equivalent.

Denote by $H^*(M;\rho) = H^*(M;\mathcal{F}_\rho)$ the singular cohomology of M with coefficients in ρ and by $\det(H(M;\rho))$ the one dimensional vector space

$$\det(H(M;\rho)) := \bigotimes_{i=1}^{n} (\bigwedge^{b_i} H^i(M;\rho))^{\epsilon(i)},$$

where $b_i = \dim H^i(M;\rho)$ and $W^{\epsilon(i)} = W$ resp. W^*, if i is even resp. odd.

To a Riemannian metric g on M and a Hermitian structure (smooth fiberwise scalar product) μ on $\mathcal{E}_\rho$ one can associate:

(a) The scalar product $\mathbb{T}_{met}(M, \rho, g, \mu)$ on $\det H(M;\rho)$, obtained by Hodge theory. Indeed the Hodge theory identifies the singular cohomology vector spaces $H^q(M;\mathcal{F})$ to the space of harmonic q-forms. The scalar product provided by g and μ on $\Omega^q(M;\rho)$ restricted to harmonic forms induces a scalar product on each $H^q(M;\mathcal{F})$, and therefore on $\det H(M;\rho)$.

(b) A positive real number $T_{an}(M\rho, g, \mu)$, the Ray Singer torsion, obtained from the zeta regularized determinants of the Laplacians Δ_q induced by the pair (g, μ).

The metric anomaly for T_{an}, cf [BZ] section iv, permits to show that in the case of odd dimensional manifolds, the scalar product $(T_{an}(M\rho, g, \mu))^{-1} \cdot \mathbb{T}_{met}(M, \rho, g, \mu)$ on the vector space $\det H(M;\rho)$, referred to as Ray Singer metric, is independent of g and μ, hence is a topological invariant[1]. This is not the case for an even dimensional manifold. Note that the Euler Poincaré characteristic of an odd dimensional manifold is zero.

In [Tu] Turaev has introduced the concept of Euler structure for compact smooth manifolds with zero Euler Poincaré characteristic, and has verified that $H_1(M;\mathbb{Z})$ acts freely and transitively on the set $E(M)$ of Euler structures. He has shown that Euler structures can be also defined combinatorially, for triangulated compact spaces, and for a compact smooth manifold the analytic Euler structures identify canonically to the combinatorial Euler structures.

For a triangulated space (X, τ) with Euler Poincaré characteristic zero, an Euler structure $A \in E(X, \tau)$ and a finite dimensional representation ρ over an arbitrary field, Farber and Turaev have defined in [FT] an element in $\det H(M;\rho))/\{+1, -1\}$. and have verified that it is a combinatorial invariant, i.e independent of triangulation up to a PL (piecewise linear) equivalence. As indicated above in the case the field is $\mathbb{R}$, such element is a scalar product. They have also obtained an analytic interpretation of this element in the case the manifold is of odd dimension and the field is $\mathbb{R}$. cf [FT].

[1]Here topological invariant actually means differential invariant; i.e. an object which although defined with the help of geometric data like Riemannian metric, Hermitian structure, etc. is independent of them. It is however possible to show that this particular invariant is in fact a topological invariant in the traditional sense

Consider the one dimensional vector space $\det V^{-\chi(M)} = (\bigwedge^{\dim V} V)^{-\chi(M)}$, ($V^{-1} = V^*$, the dual of V) equipped with the scalar product $(\det \mu_0)^{-\chi(M)}$ induced from μ_0. If $\chi(M) = 0$ then $\det V^{-\chi(M)} = \mathbb{R}$ and the induced scalar product is the canonical scalar product on $\mathbb{R}$, hence independent on μ_0. In this note:

(1) We observe that Euler structures can be defined for any base pointed closed smooth manifold (M, m_0) and, as in [Tu], that $H_1(M;\mathbb{Z})$ acts freely and transitively on the set $E(M, m_0)$ of Euler structures of (M, m_0). If $\chi(M) = 0$ the choice of m_0 is superfluous, the set $E(M, m_0)$ being canonically identified to the set $E(M)$ introduced by Turaev. This set consists of the equivalence classes of nonsingular vector fields modulo the equivalence relation introduced by Turaev in [Tu].

(2) Inspired by [BZ], we define for an Euler structure $A \in E(M, m_0)$, a Riemannian metric g and a Hermitian structure μ which is parallel in an unspecified small neighborhood of m_0, a numerical invariant, $I(M, \rho, A, g, \mu) \in \mathbb{R}$, and we verify that:

Main Theorem.

(1): If M is odd dimensional then $I(M, \rho, A, g, \mu)$ is independent of g and μ.

(2): If the Hermitian structure μ induces a parallel Hermitian structure in the vector bundle $\det \mathcal{E}_\rho \to M$, then $I(M, \rho, A, g, \mu) = 0$.

(3): The scalar product on $\det H(M;\rho) \otimes (\det V)^{-\chi(M)}$, defined by

$$e^{I(M,\rho,A,g,\mu)} \cdot T_{an}^{-1}(M\rho, g, \mu) \cdot (\mathbb{T}_{met}(M, \rho, g, \mu) \otimes (\det(\mu_0)^{-\chi(M)})$$

is independent of g and μ.

(4): If the Euler Poincaré characteristic of M vanishes then the scalar product defined in (3) is the same as the Farber Turaev torsion for a smooth triangulation (cf [FT]).

Part (3) of the Main Theorem shows that the Ray-Singer metric properly modified with the help of an Euler structure $A \in E(M, m_0)$, defines a scalar product on $\det H(M;\rho) \otimes (\det V)^{-\chi(M)}$. This scalar product, although defined with the help of a Riemannian metric and of a Hermitian structure is independent of them, and therefore is a topological invariant.

Part (4) provides an analytic description of the Farber-Turaev torsion when the PL space is a triangulated smooth manifold and the field is $\mathbb{R}$.

Using the considerations in [BFK] one can routinely extend the Main Theorem from finite dimensional representations to representations $\rho : \Gamma \to GL_{\mathcal{A}}(\mathcal{W})$, with $\mathcal{W}$ an $\mathcal{A}$- Hilbertian module of finite type over a finite von Neumann algebra $\mathcal{A}$ and (M, ρ) of "determinant class" in the sense of [BFK].

I wish to thank Misha Farber for bringing to my attention the concept of Euler structure and for very helpful comments concerning this Note.

1. Euler structures .

Let (M^n, m_0) be a base pointed closed smooth manifold. Consider $\mathcal{V}(M, m_0)$ be the collection of smooth vector fields X on M with $X(x) \neq 0$ for $x \neq m_0$. $\mathcal{V}(M, m_0)$ is nonempty. We say that $X_1, X_2 \in \mathcal{V}(M, m_0)$ are equivalent iff there exists a smooth family of vector fields $X_t \in \mathcal{V}(M, m_0)$, $1 \leq t \leq 2$; equivalently iff there exists an embedded disc $D^n \subset M^n$ centered at m_0 and a continuous family of smooth vector fields $X_t \in \mathcal{V}(M, m_0)$, $1 \leq t \leq 2$, so that $X_t(x) \neq 0$ for $x \in M \setminus \mathrm{Int} D^n$. Denote the collection of equivalence classes of such vector fields by $E(M, m_0)$.

There exists a natural action

$$\lambda : H_1(M; \mathbb{Z}) \times E(M, m_0) \to E(M, m_0)$$

noticed by Turaev in the case $\chi(M) = 0$ which is free and transitive. To describe this action we proceed as follows: Choose a (generalized) triangulation τ of M in the sense of [BFKM] section 5, $\tau = (h, g')$ with h a selfindexing Morse function and g' a Riemannian metric so that the vector field $-grad_{g'} h$ satisfies the Morse Smale condition, (cf. [BFKM]).The cells =simplexes of this triangulation are the unstable manifolds of the critical points of h with respect to the vector field $-grad_{g'} h$. It is known that any simplicial triangulation can be obtained in this way.

An "embedded spray" will be given by a smooth embedding $\alpha : K \to M$, of the base point union $K = \bigvee_{y \in Cr(h)} I_y$, ($I_y$ denotes the based pointed interval $([0, 1], 0)$) with $\alpha_y := \alpha : I_y \to M$ a smooth embedding so that $\alpha_y(0) = m_0$ and $\alpha_y(1) = y$. Clearly a smooth regular neighborhood of $\alpha(K)$ is an embedded disc $D^n \subset M^n$.

To a given embedded spray α one can associate an Euler structure $A(\alpha) \in E(M, m_0)$ defined as follows: First one chooses an embedded disc D^n (smooth image of the unit disc in $\mathbb{R}^n$), which is a regular neighborhood of the smooth spray $\alpha(K)$. Second one defines the vector field X on $M \setminus D^n$ by $X = -grad_g(h)$. Third one extends X radially and scaled by a factor equal to the distance to the center inside D^n. The resulting vector field X will have only one zero at the center of the disc, hence it represents an Euler structure. Different regular neighborhoods lead to vector fields representing the same Euler structure, and so do embedded sprays which are isotopic by isotopy fixed on the vertices of K and m_0. By changing the spray one changes the Euler structure. In fact for any given Euler structure one can choose an embedded spray so that the vector field constructed above represents the given Euler structure. If α and β are two embedded sprays, the singular cycle $\sum_{y \in Cr(h)} (-1)^{index(y)} (\alpha_y - \beta_y)$ represents a homology class $u(\alpha, \beta) \in H_1(M; \mathbb{Z})$ and given $u \in H_1(M; \mathbb{Z})$ and α an embedded spray, there exists an embedded spray β so that $u = u(\alpha, \beta)$. We put

$$\lambda(u(\alpha, \beta), A(\alpha)) = A(\beta).$$

The verification that any Euler structure is of the form $A(\alpha)$ and that λ is a well defined action, free and transitive, can be done as in [Tu]. It can be also derived by using elementary obstruction theory[2].

As already stated embedded sprays which are isotopic by isotopy which are fixed on the vertices of K and m_0 give rise to the same Euler structure. Consequently one can consider (when $dimM > 2$) continuous sprays by replacing smooth embedding of I by continuous maps (since a continuous map of I can be approximated arbitrary close in C^0 topology by a smooth embedding unique up to an isotopy). In particular if $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1)$ represents the element $[\gamma] \in H_1(M;\mathbb{Z})$ and α^γ denotes the continuous spray defined by $\alpha^\gamma_y = \gamma(2t)$ for $0 \le t \le 1/2$ and $\alpha_y(2t-1)$ for $1/2 \le t \le 1$, one can verify (cf [FT]) that

$$A(\alpha^\gamma) = \lambda(\chi(M)[\gamma], A(\alpha)). \tag{1.1}$$

It is not hard to see that starting with an embedded spray α one can produce the embedded spray α^γ which satisfy (1.1)even in the case $dimM = 2$.

2. The Invariant $I(M, \rho, A, g, \mu)$.

Let $\pi : E \to M$ be a vector bundle of rank k and denote by $\mathcal{O}$ the orientation bundle, which is in a natural way a flat bundle of rank one and by $\pi^*(\mathcal{O})$ its pullback on E. For forms ω with values in $\mathcal{O}$ resp. $\pi^*(\mathcal{O})$, denote by $d\omega$ the covariant differentiation with respect to the natural flat connection of $\mathcal{O}$ resp. $\pi^*(\mathcal{O})$. Consider pairs $\tilde\nabla = (\nabla, \mu)$ consisting of a connection ∇ and a $\nabla-$parallel hermitian structure μ in $\pi : E \to M$. Recall that a Hermitian structure means a scalar product in each fiber E_x, $x \in M$ depending smoothly on x. For any such $\tilde\nabla$ consider:

(1) the Euler integrand $e(\tilde\nabla) \in \Omega^k(M;\mathcal{O})$, which is a closed form,

(2) Mathai-Quillen form $\Psi(\tilde\nabla) \in \Omega^{k-1}(E \setminus 0; \pi^*(\mathcal{O}))$, cf [BZ] page 40-44.)

If $\tilde\nabla_1$ and $\tilde\nabla_2$ are two such pairs consider

(3) $c(\tilde\nabla_1, \tilde\nabla_2) \in \Omega^{k-1}(M;\mathcal{O})/d(\Omega^{k-2}(M;\mathcal{O}))$ the Chern Simon class.

It is well known that

$$dc(\tilde\nabla_1, \tilde\nabla_2) = e(\tilde\nabla_1) - e(\tilde\nabla_2) \tag{2.1}$$

and that if there exists $\phi(t) : E \to E$, $1 \le t \le 2$ a smooth family of bundle isomorphisms so that $\phi(1) = Id$ and $\phi(2)^*(\tilde\nabla_1) = \tilde\nabla_2$ then

$$c(\tilde\nabla_1, \tilde\nabla_2) = 0. \tag{2.2}$$

[2]choose a generalized triangulation τ and assume that m_0 lies in a top dimensional simplex (cell). Denote by $M(k)$ the $k-$th skeleton.There is no obstruction to construct the homotopy X_t between X_1 and X_2, ($X_1, X_2 \in \mathcal{V}(M, m_0)$) above an open neighborhood of $M(n-2)$ with $X_t(x) \neq 0$ for any $t \in [1,2]$. The obstruction to extend such homotopy to $M(n-1)$ is an $(n-1)-$cocycle in $C^*(M,\tau;\mathcal{O})$ the geometric complex provided by τ. The cocycle is cohomological to zero iff the restriction of X_t to $M(n-3)$ can be extended to a neighborhood of $M(n-1)$. Fixing a vector field $X_1 \in \mathcal{V}(M, m_0)$ one can provide X_2 and homotopy X_t realizing such cycle (cohomology class) as obstruction. Once such homotopy is constructed one can easily show that X_1 and X_2 define the same Euler class.

Here $\phi(2)^*(\tilde{\nabla}_1)$ denotes the action of $\phi(2)$ on the pair $\tilde{\nabla}_1$.

If $X : M \to E \setminus 0$ is a smooth map denote by $X^*(\Psi(\tilde{\nabla})) \in \Omega^{k-1}(M; \mathcal{O})$ the pullback of $\Psi(\tilde{\nabla})$ by X. We have (cf [BZ] pages 40-44)

$$d\Psi(\tilde{\nabla}) = \pi^* e(\tilde{\nabla}) \tag{2.3}$$

$$\Psi(\tilde{\nabla}_1) - \Psi(\tilde{\nabla}_2) = \pi^* c(\tilde{\nabla}_1, \tilde{\nabla}_2), \tag{2.4}$$

with $\pi^* c(\tilde{\nabla}_1, \tilde{\nabla}_2) \in \Omega^{k-1}(E \setminus 0; \pi^*(\mathcal{O}))/d(\Omega^{k-2}(E \setminus 0; \pi^*(\mathcal{O}))$. In particular we have

$$dX^*(\Psi(\tilde{\nabla})) = e(\tilde{\nabla}) \tag{2.3'}$$

and

$$X^*(\Psi(\tilde{\nabla}_1)) - X^*(\Psi(\tilde{\nabla}_2)) = c(\tilde{\nabla}_1, \tilde{\nabla}_2) \in \Omega^{k-1}(M; \mathcal{O})/d(\Omega^{k-2}(M; \mathcal{O})) \tag{2.4'}$$

If $f : M \to \mathbb{R}_+$ is a smooth function then

$$(fX)^*(\Psi(\tilde{\nabla})) = X^*(\Psi(\tilde{\nabla})). \tag{2.5}$$

Suppose X_1 and X_2 are two smooth maps as above and suppose that there exists $\phi(t) : E \to E$, $1 \leq t \leq 2$, a smooth family of bundle isomorphisms so that $\phi(1) = Id$ and $\phi(2) \cdot X_1 = X_2$. Then

$$\begin{aligned} X_1^*(\Psi(\tilde{\nabla})) - X_2^*(\Psi(\tilde{\nabla})) &= X_1^*(\Psi(\tilde{\nabla})) - X_1^*(\phi(2)^*\Psi(\tilde{\nabla})) = \\ &= c(\tilde{\nabla}, \phi(2)^*\tilde{\nabla}) = 0. \end{aligned} \tag{2.6}$$

The vanishing of $c(\tilde{\nabla}, \phi(2)^*\tilde{\nabla})$ follows from the definition of secondary characteristic classes.

These considerations will be applied to $\pi : E \to M$ the tangent bundle of $M \setminus D^n(1/2)$ where D^n will be an embedded disc centered at m_0, $D^n(1/2)$ the disc of radius half of the radius of D^n and $\tilde{\nabla}(g) := (\nabla(g), g)$, where $\nabla(g)$ is the Levi- Civita connection associated with a Riemannian metric g and the Hermitian structure is given by g.

Let $(\mathcal{E} \to M, \nabla)$ be a flat bundle. Given two Hermitian structures μ_1 and μ_2 one defines the smooth real valued function $\log\det(\mu_1, \mu_2) \in \Omega^0(M)$ by the formula

$$\log\det(\mu_1, \mu_2)(x) := \log \mathrm{Vol}(Id : (\mathcal{E}_x, \mu_1(x)) \to (\mathcal{E}_x, \mu_2(x))) \tag{2.7}$$

where $\mathrm{Vol}(Id : (\mathcal{E}_x, \mu_1(x)) \to (\mathcal{E}_x, \mu_2(x)))$ is the $\mu_2(x)-$volume of a parallelepiped generated by a $\mu_1(x)-$ orthonormal base. Equivalently $\log \mathrm{Vol}(Id : (\mathcal{E}_x, \mu_1(x)) \to$

$(\mathcal{E}_x, \mu_2(x))) = 1/2 \log\det(Id^* \cdot Id)$ where Id^* is the adjoint of $Id : (\mathcal{E}_x, \mu_1(x)) \to (\mathcal{E}_x, \mu_2(x))$.

Given a Hermitian structure μ one defines the closed form $\theta(\mu) \in \Omega^1(M)$, first introduced by Kamber-Tondeur (cf [BZ]), as follows: For any $x \in M$ choose U a contractible open neighborhood and denote by $\tilde{\mu}_x$ the Hermitian structure in $(\mathcal{E}|_U \to U, \nabla|_U)$ obtained by parallel transport of the scalar product μ_x in $\mathcal{E}_x$ with respect to the flat connection $\nabla|_U$. Define $\theta(\mu)|_U$ by

$$\theta(\mu)|_U := d \log\det(\mu, \tilde{\mu}_x). \tag{2.8}$$

One can verify that the definition is independent of the choice of U, and that it leads to a globally defined closed one form. It follows from (2.8) that $\theta(\mu)$ vanishes on open sets $V \subset M$ if $\mu|_V$ is parallel and

$$\theta(\mu_1) - \theta(\mu_2) = d \log\det(\mu_1, \mu_2). \tag{2.9}$$

Consider now a Riemannian metric g on M and a Hermitian structure μ in $\mathcal{E}_\rho$ which is parallel in an open neighborhood of m_0. The form $\theta(\mu)$ vanishes in that neighborhood. Let $X \in \mathcal{V}(M, m_0)$. The quantity

$$I(M, \rho, X, g, \mu) := 1/2 \int_{M\setminus m_0} \theta(\mu) \wedge X^*\Psi(\tilde{\nabla}(g)) \in \mathbb{R} \tag{2.10}$$

is well defined since the integrand has compact support.

Proposition 2.1. *Let X_1 and X_2 be two vector fields in $\mathcal{V}(M, m_0)$ which represent the same Euler structure $A \in E(M, m_0)$. Then $I(M, \rho, X_1, g, \mu) = I(M, \rho, X_2, g, \mu)$.*

Proof: Since $I(M, \rho, X, g, \mu)$ is a continuous function in X with respect to the C^1- topology, it suffices to prove the result for X_1 and X_2 close enough in this topology. In view of (2.3) one can suppose that the tangent $X_1(x)$ and $X_2(x)$ have the same length (say equal to one) for $x \in M \setminus D^n(1/2)$. Since we can suppose X_1 and X_2 are sufficiently closed one can find the bundle isomorphism $\phi(t) : TM|_{M\setminus d^n(1/2)} \to TM|_{M\setminus D^n(1/2)}$, $1 \leq t \leq 2$ so that $\phi(1) = Id$, $\phi(2) \cdot X_2 = X_1$, and the result follows from (2.5) and (2.6).

In view of the above proposition we will write $I(M, \rho, A, g, \mu)$ for $I(M, \rho, X, g, \mu)$ when $X \in A$, $A \in E(M, m_0)$.

Proposition 2.2. *Suppose g, g' are Riemannian metrics on M and μ, μ' are Hermitian structures parallel in some neighborhood of m_0. Then*

$$I(M, \rho, A, g, \mu) - I(M, \rho, A, g', \mu) = 1/2 \int_M \theta(\mu) \wedge c(\tilde{\nabla}(g), \tilde{\nabla}(g')) \tag{2.11}$$

and

$$(2.12)\, I(M,\rho,A,g,\mu) - I(M,\rho,A,g,\mu') = -1/2 \int_M \log\det(\mu,\mu')) \cdot e(\tilde{\nabla}(g)) +$$

$$+1/2\chi(M)\log\det(\mu,\mu')(m_0).$$

Proof: (2.11) follows from (2.4'). To prove (2.12) we proceed as follows: Choose D^n an embedded disc centered at m_0 so that μ and μ' are parallel above D^n, and choose a vector field $X \in \mathcal{V}(M, m_0)$ representing A. Denote by $D^n(1/2)$ the disc of radius half of the radius of D^n. In view of the definition (2.10), and because μ, μ'' are parallel on D^n, and because of (2.9),

$$2(I(M,\rho,A,g,\mu) - I(M,\rho,A,g,\mu')) = \exp(\int_{M\setminus D^n(1/2)} (\theta(\mu)-\theta(\mu'))\wedge X^*(\Psi(\nabla(g))),$$

$$= \exp(\int_{M\setminus D^n(1/2)} d\log\det(\mu,\mu') \wedge X^*(\Psi(\tilde{\nabla}(g))$$

which in view of Stokes Theorem and of (2.3) is equal to

$$\exp(\int_{\partial D^n(1/2)} \log\det(\mu,\mu') X^*(\Psi(\tilde{\nabla}(g))) - \int_{M\setminus D^n(1/2)} \log\det(\mu,\mu') e(\tilde{\nabla}(g)).$$

Because $\log\det(\mu,\mu')$ is constant on D^n, the last quantity equals to

$$\exp(\log\det(\mu,\mu')(m_0) \int_{\partial D^n(1/2)} X^*(\Psi(\tilde{\nabla}(g))) - \int_{M\setminus D^n(1/2)} \log\det(\mu,\mu') e(\tilde{\nabla}(g))),$$

which by (2.3) and by Stokes Theorem is equal to

$$\exp(\log\det(\mu,\mu')(m_0) \int_{M\setminus D^n(1/2)} e(\tilde{\nabla}(g)) - \int_{M\setminus D^n(1/2)} \log\det(\mu,\mu') e(\tilde{\nabla}(g))).$$

The result follows by adding and subtracting inside exp(....) the quantity

$$\int_{D^n(1/2)} \log\det(\mu,\mu') e(\tilde{\nabla}(g))$$

and by using the fact that $\int_M e(\tilde{\nabla}(g)) = \chi(M)$.

3. Proof of the Main Theorem.

Proof of (1): Since for an odd dimensional manifold $e(\tilde{\nabla}(g))$ and $c(\tilde{\nabla}(g), \tilde{\nabla}(g'))$ are both zero, the invariant $I(M, \rho, A, g, \mu)$ is independent of g and μ.

Remark 3.1: If $[\theta(\rho)] \in H^1(M; \mathbb{R})$ is the cohomology class represented by the form[3] $\theta(\mu)$ and $\gamma \in H^{n-1}(M \setminus m_0; \mathbb{R}) = H^{n-1}(M; \mathbb{R})$ is the Euler class of the bundle normal to $X \in \mathcal{V}(M, m_0)$, for X representing A, then $I(M, \rho, A, g, \mu)$ is $[\theta(\rho)] \cup \gamma[M]$ evaluated on the fundamental class cf [FT].

Proof of (2): follows from the definition (2.10).

Proof of (3): For a pair (g, μ) with μ parallel in the neighborhood of m_0 consider the scalar product

$$T_{an}(M, \rho, g, \mu) \cdot \mathbb{T}_{met}(M, \rho, g, \mu).$$

Note that:

(a) The quotients for the scalar products corresponding to (g, μ) and (g', μ) is by [BZ] (page 10) and Proposition 2.2 (1) equal to

$$\exp(I(M, \rho, A, g, \mu) - I(M, \rho, A, g', \mu)).$$

This proves the independence of g.

(b) The quotients for the scalar products corresponding to (g, μ) and (g, μ') is by [BZ] (page 10),

$$-1/2 \exp(\int_M \log \det(\mu, \mu') e(\nabla(g)).$$

By Proposition 2.2 (2) the difference $I(M, \rho, A, g, \mu) - I(M, \rho, A, g, \mu')$ is equal to

$$-1/2 \int_M \log \det(\mu, \mu')) \cdot e(\tilde{\nabla}(g)) + 1/2\chi(M) \log \det(\mu, \mu')(m_0).$$

This implies the independence of μ.

Proof(4): We choose a generalized triangulation $\tau = (h, g)$, a spray α representing the Euler structure A with respect to τ, $(e_1, e_2, \cdots, e_k)$ a base of $\mathcal{E}_{m_0}$ and D^n an embedded disc centered at m_0 which is a regular neighborhood of the spray α. We will take as representative of A a vector field $X = -grad_g(h)$ on $M \setminus D^n$. We take as metric g the one provided by τ and as a Hermitian structure μ one which is parallel above D^n and at m_0 makes the base $(e_1, e_2, \cdots, e_k)$ orthonormal. We will calculate our invariant (scalar product defined in the Main Theorem (3)) with the help of these data and compare the result with the Farber-Turaev torsion which we recall below.

[3] the notation is justified by (2.9) which implies that the cohomology class of $\theta(\mu)$ depends only on ρ; this cohomology class is given by the composition of $\log\det : GL(V) \to \mathbb{R}$ with the holonomy representation $\rho : \pi_1(M, m_0) \to GL(V)$.

The FT(Farber Turaev) torsion. Let (M^n, m_0) be a base pointed closed smooth manifold and $\tau = (g, h)$ a generalized triangulation.

Choose a base $(e_1, e_2, \cdots, e_k)$ for $\mathcal{E}_{m_0}$ and a spray α representing the Euler structure $A \in E(M, m_0)$ with respect to τ. Use the parallel transport to provide a base in each fiber $\mathcal{E}_y$, $y \in Cr(h)$. Once this is done we have a base in each component of the geometric complex $C^*(M, \tau; \rho)$. These bases define an element $\omega_1 \in \det H(M; \rho)$ as described in [FT] and the base $(e_1, e_2, \cdots, e_k)$ gives an element $\omega_2 \in \det V$, $\omega_2 = e_1 \wedge e_2, \cdots \wedge e_k$. The element $\omega_1 \otimes \omega_2^{-\chi(M)} \in (\det H(M; \rho) \otimes (\det V)^{\chi(M)}/\{-1, +1\}$ is the Farber Turaev torsion, and is independent of all choices but the PL structure of M and the Euler structure $A \in E(M, m_0)$. Strictly speaking in [FT] only the case $\chi(M) = 0$ was considered, but the above definition is implicit in that paper.

We observe that μ_y is exactly the scalar product which makes orthonormal the base obtained from $(e_1, e_2, \cdots, e_k)$ by parallel transport along α_y and therefore the scalar products induced by the bases constructed in each component of $C^*(M, \tau; \rho)$ are the same as those determined by τ and μ.

We use the combinatorial Laplacians associated with τ and μ to identify the harmonic elements in $C^*(M, \tau; \rho)$ with the singular cohomology and to obtain a scalar product in $\det H(M; \rho)$. We multiply this scalar product by the positive real number $(T_{comb}(M, \tau, \rho, \mu))^{-1}$ (as defined in [BFKM]) and tensor by the scalar product induced from ω_2; we obtain a scalar product on $\det H(M; \rho) \otimes V^{-\chi(M)}$. Elementary linear algebra permits to show that this scalar product is the same as Farber Turaev torsion.

It is immediate that the quotient of the scalar product defined in (3) and the scalar product described above is the same as the quotient between Ray-Singer torsion and Reidemeister torsion and therefore, by [BZ], is exactly $1/2 \int_{M \setminus m_0} \theta(\mu) \wedge X^* \Psi(\tilde{\nabla}(g)) = I(M, \rho, A, g, \mu)$. This finishes the proof of statement (4).

Remark 3.1 show that our analytic interpretation is consistent with the one given by Farber Turaev for odd dimensional manifolds.

6. References.

References

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