Witten deformation of the analytic torsion and the Reidemeister torsion

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To Professor Selim Grigorjevich Krein on the occasion of his eightieth birthday

0. Introduction. Index and Determinant. Euler characteristic and Torsion.

Let us consider finite-dimensional complex vector bundles E and F over a closed manifold M of dimension n. If A is an elliptic differential (or pseudodifferential) operator of, say, order $m > 0$ that maps sections of E into sections of F then it is well known that both $\ker A$ and $\mathrm{coker} A = L^2(M, F)/\overline{\mathrm{Range} A}$ are finite-dimensional, and, therefore, the index of A,

$$\mathrm{ind} A := \dim \ker A - \dim \mathrm{coker} A$$

is defined. The most important property of the index is that it is stable under deformations. This means that if $A(t)$ is a continuous family of elliptic operators of the same order, then $\mathrm{ind} A(t)$ does not depend on t. Stability of the index implies that it depends on the principal symbol of A only, and the celebrated Atiyah–Singer theorem provides us with an exact formula for $\mathrm{ind} A$.

There are several proofs of the Atiyah–Singer theorem. One of them, due to Atiyah–Bott–Patodi [ABP], is based on the heat trace expansions. Let us briefly recall how the heat trace expansion is used. We choose a Riemannian metric on M and Hermitian structures on E and on F in order to define A^*, the adjoint of A, which is again a differential (pseudodifferential) operator. Both operators $\exp(-tA^*A)$ and $\exp(-tAA^*)$ are of trace class when $t > 0$. The function

$$\theta(t) := \theta_+(t) - \theta_-(t),$$

with $\theta_+(t) = \mathrm{tr} e^{-tA^*A}$ and $\theta_-(t) = \mathrm{tr} e^{-tAA^*}$, has the following properties:

(P1) $\theta(t)$ is independent of t, or equivalently $\frac{d}{dt}\theta(t) = 0$;

(P2) $\lim_{t\to\infty} \theta(t) = \mathrm{ind} A$.

Property (P2) follows from the fact that $\lim_{t\to\infty} \exp(-tA^*A)$ is the orthogonal projector onto $\ker A$, and $\lim_{t\to\infty} \exp(-tAA^*)$ is the orthogonal projector onto $\ker A^*$, which is isomorphic to $\mathrm{coker} A$.

It remains to study $\lim_{t\to 0^+}\theta(t)$. According to Minakshisundaram and Pleijel, the functions $\theta_\pm$ admit asymptotic expansions

$$\theta_\pm(t) \sim \sum_{k=0}^{\infty} c_k^\pm t^{(-n+2k)/2m} \tag{0.1}$$

as $t \to 0^+$. The Minakshisundaram–Pleijel expansion in (0.1) is stated for differential operators. If A is a pseudodifferential operator, additional terms would appear.

All coefficients $c_k^\pm$ can be computed in terms of the complete symbol of A. In particular, one can compute $c_{n/2}^+ - c_{n/2}^-$, which equals $\lim_{t\to 0^+}\theta(t)$ and thus obtains

$$\mathrm{ind}A = c_{n/2}^+ - c_{n/2}^-. \tag{0.2}$$

The formula for $c_{n/2}^+ - c_{n/2}^-$, due to Minakshisundaram–Pleijel, provides an algorithm for computing $\mathrm{ind}A$ and is in fact the starting point for the proof of the Atiyah–Singer formula. Formula (0.2) and general results about the structure of $c_{n/2}$ imply that if the dimension n is odd then $\mathrm{ind}A = 0$.

To compute $c_{n/2}^+ - c_{n/2}^-$, one can use the zeta-function instead of the heat trace expansion. Let H be a positive elliptic operator of order $m > 0$ that acts on sections of a vector bundle E over a compact manifold M of dimension n. Then the function

$$\zeta_H(s) = \mathrm{tr}H^{-s}$$

is holomorphic in the half-plane $\Re s > n/m$, and it can be extended analytically to a meromorphic function in the whole complex plane, with simple poles at the points $(n-j)/m$, $j = 1, 2, \ldots$ ([Se]). Some of these points are, in fact, regular for $\zeta_H(s)$, and, most importantly, $s = 0$ is a regular point. The ζ-function and the heat trace

$$\theta_H(t) = \mathrm{tr}e^{-tH}$$

are related to each other by

$$\zeta_H(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\theta_H(t)dt, \tag{0.3}$$

and the coefficients of the expansion of $\theta_H(t)$ (see (0.1)) are proportional to the residues of the zeta-function and its values at some special points, 0 included. In particular, the index of an elliptic operator A can be represented as

$$\mathrm{ind}A = \zeta_{A^*A+\epsilon}(0) - \zeta_{AA^*+\epsilon}(0)$$

for any $\epsilon > 0$. One needs to add $\epsilon > 0$ to insure that the operators $AA^* + \epsilon$ and $A^*A + \epsilon$ are positive. The assumption $H > 0$ is, in fact, not important. It suffices to assume that H admits an *Agmon angle*, i.e. that there exists a solid angle in the complex plane, centered at the origin, that does not intersect the spectrum of H. In particular, 0 is not in the spectrum of H. An Agmon angle is needed for defining complex powers of H, and the value of the zeta-function depends on the choice of this angle.

There are several geometric quantities that can be interpreted as indices of certain elliptic operators. For example, the Euler characteristic of a manifold (the superindex of the DeRham differential d) equals the index of

$$d + d^* : \Lambda^{ev}(M) \to \Lambda^{odd}(M)$$

where $\Lambda^{ev(odd)}(M)$ is the space of even(odd) forms on M, d is the DeRham differential, and d^* is its adjoint with respect to the scalar product induced by a Riemannian metric on M.

Next we define the analytic torsion as the "absolute value of the superdeterminant" of d. First, we recall how the regularized determinant of a positive elliptic operator H is defined. By the definition of Ray–Singer ([RS]),

$$\log\det H = -\zeta'_H(0). \tag{0.3'}$$

The most important property of this determinant is that the variational formula

$$\frac{d}{dt}\log\det H(t) = \mathrm{tr}\dot{H}(t)H(t)^{-1} \tag{0.4}$$

holds. Here $H(t)$ is a differentiable family of elliptic operators. Formula (0.4) has perfect sense if the operator on the right is of the trace class. Otherwise, the right hand side of (0.4) is understood in the following sense: The "generalized zeta-function"

$$\zeta(s;t) = \mathrm{tr}\dot{H}(t)H(t)^{-s}$$

is meromorphic in the whole complex plane, with at most simple poles. By F.P.$\zeta(1;t)$ we denote the free term in the Laurent expansion of this function at $s = 1$. The trace on the right in (0.4) is defined to be F.P.$\zeta(1;t)$. Again, the assumption that H is positive is not important; it is sufficient to assume that there exists an Agmon angle for H. Note that this implies that ind$H = 0$. The determinant (as well as the modified determinant defined below) is defined for operators with index 0 only.

The analytic torsion of a closed Riemannian manifold is a counterpart of the Euler characteristic: it is the superdeterminant of the DeRham differential d. To define it properly, let us take the Hodge decomposition

$$\Lambda^k(M) = \Lambda^k_+(M) \oplus \Lambda^k_-(M) \oplus \mathcal{H}^k(M)$$

where $\Lambda^k_+(M)$ is the space of exact k-forms, $\Lambda^k_-(M)$ is the space of coexact k-forms, and $\mathcal{H}^k(M)$ is the space of harmonic k-forms (with respect to an arbitrary, but fixed Riemannian metric g on M). Denote by $\Delta^{\pm}_k$ the restriction of the Laplacian Δ_k to the subspace $\Lambda^{\pm}_k(M)$. The operators $\Delta^{\pm}_k$ are positive, and one can define a regularized determinant of them using their spectra. Define

$$\log T(M,g) = \frac{1}{2}\sum_{k=0}^{n}(-1)^k \log\det\Delta^-_k. \tag{0.5}$$

Note that the operator Δ_k^- is given by the formula $d_k^* d_k$, so $\det \Delta_k^-$ can be thought of as the square of the absolute value of the determinant of d. One can use the fact that operators Δ_k^+ and Δ_{k+1}^- are unitarily equivalent to show that (0.4) implies the more familiar formula for the analytic torsion [RS]:

$$\log T(M,g) = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1} k \log \det{}' \Delta_k. \tag{0.5'}$$

Here $\det'$ denotes the modified determinant (zero modes are discarded) defined by

$$\log \det{}' \Delta_k := \lim_{\epsilon \to 0}(\log \det(\Delta_k + \epsilon) - \log \epsilon \dim(\ker \Delta_k)). \tag{0.3''}$$

The Poincaré duality implies that the analytic torsion is trivial for even-dimensional manifolds (as opposed to the Euler characteristic, which is trivial for odd-dimensional manifolds).

The analytic torsion can actually be defined for a closed Riemannian manifold and a flat Hermitian bundle $E \to M$. The analytic torsion $T(M,g)$ corresponds to the trivial one-dimensional Hermitian bundle. Given a flat connection, the DeRham differential can be extended to an operator d_E taking smooth E-valued k-forms to smooth E-valued $(k+1)$-forms. In the case when the fibers of E are finite-dimensional, the above construction can be carried out with no changes, and one obtains the analytic torsion of the complex $(\Lambda(M;E), d_E)$. A classical example is when E is associated to an irreducible unitary representation ρ of the fundamental group $\pi_1(M)$. Another possibility is a *Hilbert bundle* of $\mathcal{A}$-Hilbert modules. This means that the fibers of E are finite Hilbert modules over a finite von Neumann algebra $\mathcal{A}$, and the bundle is equipped with a flat connection that makes the $\mathcal{A}$-module structure of the fibers, as well as the scalar products in the fibers, parallel with respect to the connection. Let us recall the definitions of the above used notions (e.g., see [BFKM], [CM], [Co], [Di], [GS], [LR]).

A finite von Neumann algebra $\mathcal{A}$ is a unital C^*-algebra with a $*$-operation and a faithful trace $\mathrm{tr} : \mathcal{A} \to C$ that satisfies the following properties:
1. $\langle a, b\rangle = \mathrm{tr}(ab^*)$ is a scalar product, and the completion $\mathcal{A}_2$ of $\mathcal{A}$ with respect to this scalar product is a separable Hilbert space.
2. The left regular representation gives an embedding of $\mathcal{A}$ into $L(\mathcal{A}_2)$, the algebra of all bounded, linear operators in $\mathcal{A}_2$. If viewed as a subalgebra of $L(\mathcal{A}_2)$, $\mathcal{A}$ is weakly closed.
3. The trace is normal. This means that for any monotone increasing net $(a_i)_{i\in I}$ such that $a_i \geq 0$, and $a = \sup_{i\in I} a_i$ exists in $\mathcal{A}$, one has $\mathrm{tr} a = \sup_{i \in I} \mathrm{tr} a_i$.

A Hilbert space $\mathcal{W}$ is an $\mathcal{A}$-*Hilbert module* if $\mathcal{W}$ is a left $\mathcal{A}$-module, $\langle a^* v, w\rangle = \langle v, aw\rangle$ ($a \in \mathcal{A}$, $v, w \in \mathcal{W}$), and $\mathcal{W}$ is isomorphic to a closed submodule of $\mathcal{A}_2 \otimes V$ where V is a separable Hilbert space, and the tensor product is taken in the category of Hilbert spaces. If the space V is finite-dimensional then $\mathcal{W}$ is called a *Hilbert module of finite type.* The main example of a Hilbert bundle is the canonical bundle associated to the universal covering $\pi : \tilde{M} \to M$ of the closed manifold M whose fiber above a point $m \in M$ is the Hilbert space $l^2(\pi^{-1}(m)) = l^2(\Gamma)$ where Γ is the fundamental group of M, and the algebra $\mathcal{A}$ is the weak closure of the group algebra $C(\Gamma)$ acting from the left by convolution on $l^2(\Gamma)$

([At]). Here the $*$-operation is induced by $g \to g^{-1}$, and the trace is given by $\mathrm{tr}(f) = f(e)$ with e the unital element in Γ.

In the case when E is an infinite-dimensional flat Hilbert bundle, defining the analytic torsion by using formulae (0.3), (0.3'), (0.3") and (0.5) needs (besides the replacement of "trace" by "von Neumann trace") additional work. For example to make sense of (0.3), one writes the right hand side of (0.3) as the sum of the integral from 0 to 1 and the integral from 1 to ∞. The integral from 0 to 1 admits an analytic continuation to a meromorphic function, with 0 being a regular point. The problem that arises in the infinite-dimensional case is that, in general, there is no reason, why the second term can be extended beyond $\Re s < 0$. A sufficient condition for this to happen for H given by the Laplacians Δ_k, (and therefore to have the analytic torsion defined), is that the *Novikov–Shubin invariants* of (M, g, E) are positive. Let us explain this condition in terms of the spectral distribution function.

Denote by $N_k(\lambda)$ the lowest upper bound for the von Neumann dimensions of $\mathcal{A}$-invariant subspaces $\mathcal{M} \subset \Lambda^k_-(M; E)$ such that $|d\omega|^2 < \lambda|\omega|^2$, $\omega \in \mathcal{M}$. Here $\Lambda^k_-(M; E)$ is the image of d^* in the space of E-valued k-forms. Introducing

$$\beta_k := \overline{\lim_{\lambda \to 0}} \frac{\log N_k(\lambda)}{\log \lambda}.$$

the Novikov–Shubin invariants are defined by ([NS], [GS])

$$\alpha_k := \min\{\beta_k, \beta_{k-1}\}.$$

Positivity of all Novikov–Shubin invariants implies that the analytic torsion can be defined. In fact, a milder assumption suffices. We say that E, together with all introduced structures, is of *a-determinant class* if

$$\int_0^1 \log \lambda dN_k(\lambda) > -\infty$$

for all values of k. Equivalently, one can formulate this condition in terms of the heat trace: the corresponding integral from 1 to infinity should converge (cf. Proposition 2.12 in [BFKM]).

Both the Novikov–Shubin invariants and the property of being or not being of determinant class, depend only on the topology of the pair (M, E). In particular, in the case of the canonical flat Hilbert bundle that was mentioned above, they depend on the homotopy type of M only.

There exists a combinatorial counter-part to the above construction. Let (h, g') be a pair consisting of a Riemannian metric g' and a self-indexing Morse function h on M (selfindexing means that $h(x)$ is equal to the index of x for any critical point x of h) such that the g'-gradient ∇h satisfies the Morse–Smale condition (i.e. for any two critical points x and y of h, the stable manifold W^+_x of the flow, generated by $-\nabla h$, is transversal to the unstable manifold W^-_y). The unstable manifolds W^-_x provide open cells of a CW-structure on M. Note that the partition of M generated by the open simplexes of a smooth triangulation can be obtained in such a way. This partition gives rise to a cochain complex on M with values in E, and one defines the combinatorial torsion T_c in the same way as the analytic torsion, with the DeRham differential replaced by the coboundary operator. In the case when the rank of E is finite, T_c is a combination of determinants of finite-dimensional matrices. In the case of infinite-dimensional flat Hilbert bundles, one has to make the assumption that E is of *c-determinant class*:

$$\int_0^1 \log \lambda dN_k^c(\lambda) > -\infty$$

where $N_k^c(\lambda)$ is defined in the same way as $N_k(\lambda)$, with the DeRham differential replaced by the coboundary operator. It follows from [GS] that the conditions of being of a-determinant class and of c-determinant class are equivalent. Therefore, we will just use the notion of being of the determinant class.

The analytic torsion and the combinatorial torsion are not equal, in general. The reason is that the analytic torsion depends on the Riemannian metric, and the Reidemeister torsion depends on the cell partition provided by (h, g'). To connect these two torsions, one needs to introduce a term that accounts for the choice of the Riemannian metric and of (h, g'). Integration of smooth k-forms over k-cells provides an isomorphism θ_k^{-1} from the space of harmonic forms to the space of harmonic cochains. The correction term is given by the following formula

$$\log T_m = \frac{1}{2} \sum_k (-1)^k \log \det(\theta_k^* \theta_k).$$

The Reidemeister torsion τ is defined as the product of T_c and T_m. A celebrated theorem of Cheeger and Müller ([Ch], [Mü]) says that, in the case of a flat bundle of finite rank, the analytic torsion and the Reidemeister torsion are equal. In [BFKM] this theorem was proved for von Neumann bundles of determinant class.

The goal of this paper is to explain the main ideas that are used in the proof of

Theorem $T = \tau$.

The complete proof of this theorem can be found in [BFKM]. We point out that the ideas used in the proof have other applications as well, but they will not be discussed here.

One of the main tools for our treatment of the analytic torsion is the Witten deformation of the DeRham complex

$$d(t) = e^{-th} d e^{th} = d + t dh \wedge \tag{0.6}$$

where $h(x)$ is a "good" Morse function on M. One can define the analytic torsion $T(t)$ of the complex $d(t)$. Following an idea proposed in [T], and implemented in [BZ], [BFK1], [BFKM], we use the function $\log T(t)$ for torsions in a similar way as Atiyah, Bott and Patodi [ABP] have used the function $\theta(t)$ for the index. Although $\frac{d}{dt} \log T(t)$ is not zero in general, it is computable (cf. Lemma 1.4 below). As $T(0) = T$, it remains to study the behavior of $T(t)$ when $t \to \infty$. (In the index theorem the analogue of this last step is the analysis, due to Minakshisundaram-Plejel, of the heat trace when $t \to 0^+$.)

It turns out that, as t gets large, a gap in the spectrum of the deformed Laplacian $\Delta_k(t)$ develops: a part of the spectrum is exponentially close to 0 as $t \to \infty$; the remaining part is bounded away from 0 by a factor proportional to t, and the deformed DeRham complex splits into the direct sum of two complexes: the small complex and the large complex. We study the small and the large complex separately. Although the small complex contains most of the topological information it turns out that the most delicate part of the analysis is the one of the large complex. We use two technical tools for investigating the behavior of $T(t)$ for large t. The first one concerns Mayer–Vietoris type formulae for determinants (see [BFK2], [L]) that are used when making "surgery". The second one is an asymptotic analysis of determinants of elliptic operators with parameter.

The plan of the paper is the following one. In section 1 we discuss the Witten deformation of the analytic torsion. In section 2 we present Mayer–Vietoris type formulae. Comparison theorems, which allow to compare torsions of two different manifolds, and the derivation of the Theorem are treated in section 3.

1. Witten deformation

Witten [Wi] introduced the deformation (0.6) of the DeRham complex induced by a Morse function h. This deformation does not change the cohomology of the complex. In particular, for the computation of the Euler characteristic, one can take any value of the parameter t. It turns out that it is most beneficial to take the limit $t \to \infty$. Let us recall some basic facts about the Witten deformation (e.g., see [CFKS]).

Choose an arbitrary but fixed pair (h, g') consisting of a Morse function h and a Riemannian metric g'. The Laplacians on k-forms associated to the deformed differential have the form

$$\Delta_k(t) = \Delta_k + tB_k + t^2|\nabla h|^2 \tag{1.1}$$

where Δ_k is the Laplace–Beltrami operator associated with the metric g' and a flat connection on E, B_k is a zero order differential operator that is an endomorphism of $E\otimes\Lambda^k(TM)$, and ∇h denotes the gradient of h with respect to g'. Let us assume that the Riemannian metric is Euclidean in (small) coordinate neighborhoods of the critical points of h where the function h has the standard Morse form

$$h(x) = h(0) - \frac{1}{2}\sum_{j=1}^{q} x_j^2 + \frac{1}{2}\sum_{j=q+1}^{n} x_j^2.$$

Here q is the index of the corresponding critical point. We assume also that the gradient vector field ∇h satisfies the Morse–Smale condition and that the Morse function h is self-indexing (although this last condition is merely a convenience). Let $e(x)$ be a section of E in a neighborhood of a critical point of h. Then, for any multi-index $I = (1 \le i_1 < i_2 < \cdots < i_k \le n)$,

$$B_k(e(x)\otimes dx^I) = \omega(I)e(x)\otimes dx^I \tag{1.2}$$

where $dx^I = dx^{i_1}\wedge\cdots\wedge dx^{i_k}$, and $\omega(I)$ is a number. What is important is that $\omega(I) \ge -n$, and $\omega(I) = -n$ if and only if $k = q$ and $I = (1, 2, \ldots, q)$.

It follows from (1.1) that

$$\langle \Delta_k(t)\alpha, \alpha\rangle \ge C_1t^2 - C_2t$$

if $\alpha \in \Lambda^k(M;E)$ is supported outside of (small) neighborhoods of the critical points of h. In a neighborhood of a critical point of index q, the operator $\Delta_k(t)$ is a direct sum of shifted harmonic oscillators. If $q \ne k$, then all eigenvalues of these oscillators grow linearly as functions of t. In the case when $q = k$, one eigenvalue is 0, and its multiplicity equals the dimension of E (the von Neumann dimension of a fiber of E if E is a flat Hilbert bundle over M). The lower bound for the positive part of the spectrum is growing linearly with t. This analysis leads to the following

Theorem 1.1. *There exist constants t_0, C_1, C_2, and C_3 so that the interval $(C_1e^{-C_2t}, C_3t)$ does not intersect the spectrum of $\Delta_k(t)$ when $t > t_0$.*

Let $P_k(t)$ be the spectral projector associated to $\Delta_k(t)$ that corresponds to the interval $[0, C_1e^{-C_2t}]$. Then the von Neumann trace, $\mathrm{tr}P_k(t)$, equals the product of the number of critical poins of h of index k and the von Neumann dimension of a fiber of E.

Denote by $\Lambda^k_{\text{sm}}(M;E)$ the image $P_k(t)$, and by $\Lambda^k_{\text{la}}(M;E)$ the image of $I - P_k(t)$ where, in these notations, we suppress t. Theorem 1.1 implies that, for sufficiently large values of t, the deformed DeRham complex splits into the direct sum of $(\Lambda_{\text{sm}}(M;E), d(t))$ and $(\Lambda_{\text{la}}(M;E), d(t))$. One defines the torsions $T(t)$, $T_{\text{sm}}(t)$, and $T_{\text{la}}(t)$ of the deformed DeRham complex, its small part, and its large part. Clearly, they depend on (h, g'), whereas $T(0)$ depends on g' only.

The key for the proof of Theorem 1.1 is a simple variational lemma.

Lemma 1.2. *Let A be a selfadjoint operator in a Hilbert space H. Let H_1 and H_2 be subspaces of H such that $H = H_1 \dot{+} H_2$. Assume that $(Ax, x) \le a||x||^2$ when $x \in H_1$, and $(Ax, x) \ge b||x||^2$ when $x \in H_2$ where $a < b$. Then the open interval (a, b) does not intersect the spectrum of A.*

Proof. Suppose that a number $\lambda \in (a, b)$ belongs to the spectrum of A. Then there exists a sequence $x_n \in H$ such that $||x_n|| = 1$, and $w_n = (A - \lambda)x_n \to 0$. Decompose x_n into the sum $y_n \in H_1$ and $z_n \in H_2$. Then

$$\Re(w_n, y_n) = ((A-\lambda)y_n, y_n) + \Re((A-\lambda)z_n, y_n) \le (a-\lambda)||y_n||^2 + \Re((A-\lambda)z_n, y_n)$$

and

$$\Re(w_n, z_n) = ((A-\lambda)z_n, z_n) + \Re((A-\lambda)y_n, z_n) \ge (b-\lambda)||z_n||^2 + \Re((A-\lambda)z_n, y_n).$$

Subtract the second inequality from the first one:

$$\Re(w_n, y_n - z_n) \le (a-\lambda)||y_n||^2 + (\lambda - b)||z_n||^2 \le -c(||y_n||^2 + ||z_n||^2) \qquad (1.3)$$

where $c = \min\{\lambda - a, b - \lambda\}$. Note that norms of y_n and z_n are uniformly bounded, and $w_n \to 0$; so the left hand side of (1.3) approaches 0 when $n \to \infty$. Therefore $||y_n|| \to 0$ and $||z_n|| \to 0$, which contradicts $||y_n + z_n|| = 1$.

○

Helffer and Sjöstrand [HS1,2] (see also [BZ]) analyzed the "small" complex in a finite-dimensional setting. Their analysis was extended in [BFKM] to flat Hilbert bundles over M. Their conclusion is that, up to rescaling, the small subcomplex is asymptotically isometric to the cochain complex (C^*, δ) associated to (h, g'). The rescaling of the small subcomplex consists in replacing $d_q(t)$ by $(\pi/t)^{1/2}e^t d_q(t)$ and the scalar product $\langle\cdot,\cdot\rangle_q$ in $\Lambda^q_{\text{sm}}(M;E)$ by $\left(\frac{\pi}{t}\right)^{(2q-d)/2} e^{2tq}\langle\cdot,\cdot\rangle_q$.

More precisely, let $\text{Int} : (\Lambda^*(M;E), d) \to (C^*, \delta)$ be the cochain complex map provided by integration over the unstable manifolds of $-\nabla h$, and let $e^{th} : (\Lambda^*(M;E), d(t)) \to (\Lambda^*(M;E); d)$ be the cochain complex map induced by multiplication by e^{th}. Helffer and Sjöstrand have constructed isometries $b_q : C^q \to \Lambda^q_{\text{sm}}(M;E)$ so that

$$\text{Int} e^{th} \left(\frac{\pi}{t}\right)^{(d-2q)/4} e^{-tq} = b_q^{-1} + O\left(\frac{1}{t}\right)$$

(cf. [BFKM, section 5). The comparison between the torsion of $(\Lambda^*(M;E), d(t))$ and its rescaled version leads to the following asymptotic expansion:

Theorem 1.3. (Theorem A(3) from [BFKM]). *The function* $\log T_{\mathrm{sm}}(t)$ *admits an asymptotic expansion for* $t \to \infty$ *of the form*

$$\log T_c + \frac{1}{2}\left(\sum_{k=0}^{n}(-1)^{k+1}(k\beta_k - km_k l)\right)(2t - \log t + \log \pi) + o(1)$$

where β_k *is the* k*-th Betti number,* m_k *is the number of critical points of* h *of index* k*, and* l *is the rank (von Neumann dimension of a fiber) of* E*.*

Another part of the analysis of the Witten deformation consists of deriving a variational formula for the analytic torsion. In the finite-dimensional case, it was done by Tangermann [T]. The idea is the same as in [RS]. The variational formula (0.4) is the key ingredient of this analysis, and the problem is, how to interpret the right hand side of it in geometrical terms. We will follow [BFKM], section 6.

Lemma 1.4 (Proposition 6.2 from [BFKM]). *Let* $Q_k(t)$ *be the orthogonal projector onto the kernel of* $\Delta_k(t)$*. Then*

$$\frac{d}{dt}\log T(t) = \sum_{k=0}^{n}(-1)^{k+1}\mathrm{tr}(Q_k(t)hQ_k(t)).$$

Note that Lemma 1.4 implies that if the DeRham complex is acyclic then $T(t)$ is independent of t. Let us sketch a formal proof of Lemma 1.4. Formula (0.5) says that

$$\log T(t) = \frac{1}{2}\sum_{k=0}^{n}(-1)^k \log\det \Delta_k^-(t)$$

where the operator $\Delta_k^-(t) = d^*(t)d(t)$ acts on $\Lambda_-^k(M;E)(t) := d^*(t)\Lambda^{k+1}(M;E)$. One has $\Delta_k^-(t) = e^{th}d^*e^{-2th}de^{th}$, and one considers $H_k(t) = e^{-th}\Delta_k^-(t)e^{th}$ acting on $\Lambda_-^k(M;E) = d^*\Lambda^{k+1}(M;E)$. One has $\dot{H}_k(t) = -2d^*e^{-th}he^{-th}de^{2th} + 2d^*e^{-2th}de^{2th}$, and therefore

$$\begin{aligned}
&\mathrm{tr}\dot{\Delta}_k^-(t)\Delta_k^-(t)^{-s-1} = \mathrm{tr}\dot{H}_k(t)H_k(t)^{-s-1}\\
=&-2\mathrm{tr}d^*e^{-th}he^{-th}de^{th}\Delta_k^-(t)^{-s-1}e^{th} + 2\mathrm{tr}d^*e^{-2th}de^{th}h\Delta_k^-(t)^{-s-1}e^{th}\\
=&-2\mathrm{tr}e^{th}d^*e^{-th}he^{-th}de^{th}\Delta_k^-(t)^{-s-1} + 2\mathrm{tr}e^{th}d^*e^{-th}e^{-th}de^{th}h\Delta_k^-(t)^{-s-1}\\
=&-2\mathrm{tr}d^*(t)hd(t)\Delta_k^-(t)^{-s-1} + 2\mathrm{tr}d^*(t)d(t)h\Delta_k^-(t)^{-s-1}\\
=&-2\mathrm{tr}hd(t)\Delta_k^-(t)^{-s-1}d^*(t) + 2\mathrm{tr}h\Delta_k^-(t)^{-s}.
\end{aligned}$$

One easily verifies that $d(t)\Delta_k^-(t)^{-s-1}d^*(t) = \Delta_{k+1}^+(t)^{-s}$. Hence,

$$\frac{d}{dt}\log T(t) = \sum_{k=0}^{n}(-1)^k\mathrm{tr}h(\Delta_k^-(t) + \Delta_k^+(t))^{-s}|_{s=0}. \tag{1.4}$$

Further, $\mathrm{tr}h(\Delta_k^-(t) + \Delta_k^+(t))^{-s} = \lim_{\epsilon\to 0}\mathrm{tr}h(\Delta_k(t)+\epsilon)^{-s} - \mathrm{tr}Q_k(t)hQ_k(t)$ and

$$\mathrm{tr}h(\Delta_k(t)+\epsilon)^{-s} = \mathrm{tr}h(\Delta_{n-k}(t)+\epsilon)^{-s}.$$

Because the sum (1.4) is alternating, and n is odd, the last two equalities, together with (1.4), imply the statement of Lemma 1.4. This proof is almost complete in a finite-dimensional situation. In the case when E is a flat Hilbert bundle, the essential spectrum of Laplacians can be nonempty, and it may include 0, so one has to be careful in dealing with inverses, complex powers, etc.

The next step is to find a geometric interpretation for $\mathrm{tr} Q_k(t) h Q_k(t)$. To formulate the result, we introduce some additional notations. Denote by $\mathcal{H}_t^k(M;E)$ the space of t-harmonic k-forms in $\Lambda^k(M;E)$, i.e. is the kernel of the operator $\Delta_k(t)$. Consider the operator $K_k(t) : \mathcal{H}_t^k(M;E) \to \mathcal{H}_0^k(M;E)$ defined by

$$K_k(t)(\omega) = Q_k(0) e^{th} \omega.$$

Then

$$\mathrm{tr}(Q_k(t) h Q_k(t)) = \frac{1}{2} \frac{d}{dt} \log\det(K_k(t) K_k(t)^*). \tag{1.5}$$

The proof of formula (1.5) is based on the Hodge decomposition and (0.4). Note that $\log\det(K_k(0)K_k(0)^*) = 0$. Therefore, Lemma 1.4 and (1.5) imply

$$\log T(t) = \log T(0) + \frac{1}{2} \sum_{k=0}^{n} (-1)^{k+1} \log\det(K_k(t) K_k(t)^*). \tag{1.6}$$

Now we will relate the asymptotic expansion of $\log T(t)$ when $t \to \infty$ to the metric part T_m of the Reidemeister torsion, defined with the help of the pair (h, g') (cf. section 0), in the case where $g = g'$. Denote by $\mathcal{H}^k_{\mathrm{comb}}$ the kernel of the combinatorial Laplacian Δ_k^{comb} of the complex (C^*, δ). Recall that in section 0 we introduced the isomorphism

$$\theta_k = \mathcal{H}^k_{\mathrm{comb}} \to \mathcal{H}^k(M;E)$$

the inverse of which is given by the composition of the operator Int^k of integration over k-chains with the orthogonal projection π_k onto $\mathcal{H}^k_{\mathrm{comb}}$. Stokes' theorem implies a decomposition

$$K_k(t) = \theta_k K_k'(t) = \theta_k \pi_k K_k''(t) I_k(t)$$

where $I_k(t)$ is the inclusion of $\mathcal{H}_t^k$ into $\Lambda^k(M;E)_{\mathrm{sm}}$, and $K_k''(t)\omega$ is the k-cochain defined by integrating of $e^{th}\omega$ over k-chains. Note that

$$\log\det(K_k(t)K_k(t)^*) = \log\det(\theta_k \theta_k^*) + \log\det(K_k'(t) K_k'(t)^*). \tag{1.7}$$

Recall that

$$\log T_m = \frac{1}{2} \sum_{k=0}^{n} (-1)^k \log\det(\theta_k \theta_k^*).$$

The analysis of the small part of the deformed DeRham complex leads to asymptotics for $\log\det(K_k'(t)K_k'(t)^*)$. If combined with (1.6) and (1.7), it implies

$$\log T(t) = \log T - \log T_m + \sum_{k=0}^{n} (-1)^{k+1} k \beta_k t + \frac{1}{2} \sum_{k=0}^{n} (-1)^k k \beta_k \log\left(\frac{t}{\pi}\right) + O(t^{-1}). \tag{1.8}$$

One can combine (1.8) and Theorem 1.3 to obtain an asymptotic expansion for the torsion of the large part of the deformed DeRham complex:

$$\log T_{\mathrm{la}}(t) = \log T - \log \tau + \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1} k m_k l(2t - \log t + \log \pi) + o(1). \tag{1.9}$$

One can try to complete the proof of $T = \tau$ (for the metric g') by deriving an asymptotic expansion for $\log T_{\mathrm{la}}(t)$ independently from the above analysis. We use a different approach involving surgery and a comparison theorem. This will be explained in the next section.

2. Mayer–Vietoris type formula

We discuss a formula of Mayer–Vietoris type [BFK2] that relates the determinant of an elliptic differential operator of second order on a closed manifold M to the determinant of the operator given by the same differential expression, with the Dirichlet boundary condition along a hypersurface in M. This formula is a particular case of a more general Mayer–Vietoris type formula established in [BFK2] in the case of a finite-dimensional bundle. We first describe the result in more detail, then sketch its proof, and, at the end, explain how it is used to analyze (1.9).

In this section, our main object is a positive differential operator of second order that acts on sections of a Hilbert bundle $\mathcal{E}$ over a closed manifold M. This operator is assumed to be elliptic, and, moreover, of *Laplace–Beltrami type*, that is its principal symbol is scalar $||\xi||^2 Id_{\mathcal{E}_x}$. The main example is $\Delta_k(t) + \epsilon$ (cf (1.1)) where $\epsilon > 0$ and $\mathcal{E} = E \otimes \Lambda^k(TM)$. Note that we add a positive number ϵ to the Laplacian to make the operator positive. Let Γ be a smooth embedded hypersurface in M with trivial normal bundle. Denote by M_Γ the manifold with boundary, the interior of which is $M \setminus \Gamma$, and the boundary of which consists of two copies of Γ, Γ^+ and Γ^-. The bundle $\mathcal{E}$ can be extended to a bundle over M_Γ that will be also denoted by $\mathcal{E}$. By $\mathcal{E}_\Gamma$ we denote the restriction of $\mathcal{E}$ to Γ. Let A_D be an operator acting on sections of $\mathcal{E} \to M_\Gamma$ defined by the differential expression A and the Dirichlet boundary conditions, i.e. the closure of the restriction of A to sections with compact support in $M \setminus \Gamma$. It is well known that the operator A_D is elliptic and positive. Further we need to introduce the *Dirichlet-to-Neumann operator*. Let ϕ be a section of $\mathcal{E}_\Gamma$. Let the section u be the unique solution of $Au = 0$, $u(x) = \phi(x)$ $(x \in \Gamma)$. It is smooth everywhere on M_Γ. We denote by $\psi = R\phi$ the jump of the normal derivative of u along Γ. The operator R maps a section of $\mathcal{E}_\Gamma$ into a section of $\mathcal{E}_\Gamma$ and is called the Dirichlet-to-Neumann operator. R is a positive pseudo-differential operator of order 1. The easiest way to see this, is to note that $Au = \psi(x)\delta_\Gamma$ where δ_Γ is the Dirac distribution supported on Γ. Therefore,

$$\phi = R^{-1}\psi = rA^{-1}(\psi\delta_\Gamma)$$

where r is the operator of restricting a section to Γ. A^{-1} is a pseudo-differential operator, and one can use local coordinates to see that R^{-1} is also a pseudo-differential operator. In this way one also obtains the following formula for the principal symbol of R^{-1}:

$$\sigma(R^{-1})(x', \xi') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sigma(A^{-1})(x', 0; \xi', \eta) d\eta$$

where (x', y) are coordinates on M near Γ such that Γ is given by $y = 0$, and $\partial/\partial y$ is the field of unit vectors normal to Γ, and (ξ', η) are dual coordinates. A Mayer–Vietoris formula for determinants is given in the following

Theorem 2.1 *If all the assumptions above are satisfied then*

$$\det A = \bar{c} \det A_D \det R \tag{2.1}$$

where

$$\bar{c} = \exp\left(\int_\Gamma c(x')\right).$$

The homogeneous components of the symbol of R and the density $c(x')$ depend only on finitely many jets of the coefficients of A on Γ.

Let us briefly explain how Theorem 2.1 is proved. The complete proof can be found in [BFKM, section 3] (cf. also [L]). Theorem 2.1 is a special case of more general Mayer-Vietoris type formulae, valid for arbitrary elliptic operators, which were established earlier in [BFK2]. First, we will give supporting arguments in the case when the rank of E is finite, and, therefore, the spectrum of A is discrete. Consider two functions

$$f(\lambda) = \frac{\det(A + \lambda)}{\det(A_D + \lambda)} \quad \text{and} \quad g(\lambda) = \det R(\lambda)$$

where $R(\lambda)$ is the Dirichlet-to-Neumann operator associated to $A + \lambda$. One can show that both $f(\lambda)$ and $g(\lambda)$ are meromorphic functions. The zeroes of $f(\lambda)$ are eigenvalues of $-A$, and the poles of $f(\lambda)$ are eigenvalues of $-A_D$. The multiplicity of a zero or a pole of $f(\lambda)$ equals exactly the multiplicity of the eigenvalue of the corresponding operator. On the other hand, the operator $R(\lambda)$ has zero modes when λ belongs to the spectrum of $-A$, and it is not defined when λ belongs to the spectrum of $-A_D$. Therefore, $g(\lambda)$ has the same zeroes and the same poles as $f(\lambda)$. Further it is easy to see that the multiplicity of the zeroes and poles of $f(\lambda)$ and $g(\lambda)$ are the same. Of course, the above argument does not prove (2.1) but it gives an idea, why one can expect a formula like (2.1) to hold.

In a general situation, when the fibers of E are Hilbert modules, the functions $f(\lambda)$ and $g(\lambda)$ are holomorphic outside the union of the spectrum of $-A$ and $-A_D$. In particular, they are holomorphic everywhere away from the ray $(-\infty, -\epsilon]$ for some $\epsilon > 0$. One can use formula (0.4) for computing the derivatives of $\log f(\lambda)$ and $\log g(\lambda)$. These computations show ([BFK2], see also [BFKM]) that these derivatives coincide. Therefore,

$$\log\det(A + \lambda) = \log c + \log\det(A_D + \lambda) + \log\det R(\lambda) \tag{2.2}$$

where c is a constant.

The last step in the proof of Theorem 2.1 concerns the computation of the constant c in (2.2). For this purpose we study the asymptotic behavior of both sides when $\lambda \to +\infty$. It was established in [V], [Fr] that, in the discrete-spectrum-case, both $\log\det(A + \lambda)$ and $\log\det(A_D + \lambda)$ admit a complete asymptotic expansion when $\lambda \to +\infty$. One can show that $R(\lambda)$ is an elliptic pseudodifferential operator with parameter (for the definition, see e.g. [Sh]), and a somewhat tedious but rather standard analysis in [BFK2] leads to a complete asymptotic expansion for $\log\det R(\lambda)$. What is important is that the coefficients in any of these expansions are local expressions, i.e. they are integrals of certain polynomials in the components of the complete symbol of A and their derivatives. For $\log\det(A + \lambda)$, these integrals are taken over M, for $\log\det R(\lambda)$, the integrals are taken over Γ; in the case of $\log\det(A_D + \lambda)$, terms, which are integrals over M or integrals over Γ are present. Finally, the constant c is determined from comparing the free terms in the asymptotic expansions of both sides of (2.2) when $\lambda \to +\infty$. This analysis was extended in [BFKM] to operators acting on sections of Hilbert bundles.

In fact, in [BFK2] (cf. also [L]), Theorem 2.1 is derived from a corresponding result for operators of order $m > 2n$, which is easier to prove, but more elaborate to state.

We consider the following special case. Let A be a *non-negative* differential operator of Laplace–Beltrami type, and consider $A^p + \alpha$ where $p \geq n+1$ is a positive integer, and $\alpha > 0$. By $(A^p + \alpha)_D$ we denote the operator acting on sections of $E \to M_\Gamma$ defined by the differential expression $A^p + \alpha$ and boundary conditions $A^l u|_\Gamma = 0$, $l = 0, \ldots, p-1$. In this situation, the Dirichlet-to-Neumann operator $R_p(\alpha)$ acts on sections of the direct sum of p copies of $E_\Gamma \to \Gamma$. If $\phi_0, \ldots, \phi_{p-1}$ are sections of E_Γ then $(\psi_0, \ldots, \psi_{p-1}) = R_p(\alpha)(\phi_0, \ldots, \phi_{p-1})$ is constructed in the following way. First, one solves the boundary value problem $A^p u + \alpha u = 0$ in $M \setminus \Gamma$, subject to boundary conditions $A^l u|_\Gamma = \phi_l$, $l = 0, \ldots, p-1$. Then one defines ψ_l as the jump of the normal derivative of $A^l u$ across Γ. The operators $R_p(\alpha)$ are elliptic in the sense of Agmon–Douglis–Nirenberg (see [ADN]), and

$$\log\det(A^p + \alpha) = \log c + \log\det(A^p + \alpha)_D + \log\det R_p(\alpha) \tag{2.3}$$

where $\log c$ is given by a local expression.

To conclude this section, let us explain how formula (2.3) (rather than Theorem 2.1) will be used to analyze formula (1.9). As we have seen in (1.9), the difference $\log T - \log\tau$ appears in the asymptotic expansion of the large part of the analytic torsion. This large part is defined by

$$\log T_{\mathrm{la}}(t) = \frac{1}{2}\sum_{k=0}^{n}(-1)^{k+1} k \log\det\tilde{\Delta}_k(t)$$

where $\tilde{\Delta}_k(t)$ is the restriction of $\Delta_k(t)$ to the image of $P_k(t)$, the spectral projector corresponding to $[1, +\infty)$ (t is assumed to be sufficiently large). Then

$$\log T_{\mathrm{la}}(t) = \frac{1}{2p}\sum_{k=0}^{n}(-1)^{k+1} k \log\det(\tilde{\Delta}_k(t))^p.$$

As $p \geq n+1$, the Fredholm determinant of the operator $I + \tilde{\Delta}_k(t)^{-p}$ exists (for the von Neumann case, see [FK]), and

$$\log\det(I + \tilde{\Delta}_k(t)^{-p}) = o(1)$$

when $t \to \infty$. The reason for this is that a lower bound for the spectrum of $\tilde{\Delta}_k(t)$ grows linearly in t (cf. Theorem 1.1). Therefore,

$$\log\det\tilde{\Delta}_k(t)^p = \log\det(\tilde{\Delta}_k(t)^p + 1) + o(1).$$

On the other hand,

$$\log\det(\tilde{\Delta}_k(t)^p + 1) = \log\det(\Delta_k(t)^p + 1) + o(1)$$

because the operator $\Delta_k(t)^p + 1$ is the direct sum of $\tilde{\Delta}_k(t)^p + 1$ and an operator, acting in a finite-dimensional (in the von Neumann sense) space, the spectrum of which is exponentially close to 1. Finally,

$$\log T_{\mathrm{la}}(t) = \frac{1}{2p}\sum_{k=0}^{n}(-1)^{k+1} k \log\det(\Delta_k(t)^p + 1) + o(1). \tag{2.4}$$

To analyze (2.4), formula (2.3) will be applied with $A := \Delta_k(t)$ and $\alpha := 1$.

3. Comparison Theorems

Let $E_1 \to M_1$ and $E_2 \to M_2$ be two flat $\mathcal{A}$-Hilbert bundles over the compact Riemannian manifolds M_1 and M_2 of the same odd dimension n. Denote by T_i (τ_i), $i = 1, 2$, their analytic (Reidemeister) torsions. Assume that the fibers of E_1 and E_2 are isomorphic as $\mathcal{A}$-Hilbert modules. Let (h_1, g_1') and (h_2, g_2') be two pairs, each consisting of a Morse function and a Riemannian metric that satisfy the hypotheses of section 1. We make the assumption that

(C) *the number of critical points of h_1 and h_2 of any given index are the same.*

We will sketch the proof of the following comparison theorem.

Theorem 3.1. *Suppose that $E_1 \to M_1$ and $E_2 \to M_2$ are of determinant class. If the assumption (C) holds, then*

$$\log T_1 - \log \tau_1 = \log T_2 - \log \tau_2. \tag{3.1}$$

We will first sketch the proof in the particular case when $g_i = g_i'$, $i = 1, 2$. Denote by $p_1, \dots, p_N \in M_1$ the critical points of h_1, and by $q_1, \dots, q_N \in M_2$ the critical points of h_2. We enumerate them in such a way that the index of p_j equals the index of q_j, $j = 1, \dots, N$. Choose for each each critical point in M_i, $i = 1, 2$, a small coordinate neighborhood such that these neighborhoods are mutually disjoint, the bundle $E_i \to M_i$ is trivial over each of these neighborhoods, and h_i has the standard Morse form if expressed in local coordinates (see the beginning of sect. 1). One can assume that each neighborhood, in local coordinates, is given by $\{x : |x| < \epsilon\}$ for some $\epsilon > 0$. Denote by U_j the chosen neighborhood of p_j, and by V_j the one of q_j. For both bundles, we consider the Witten deformation of the twisted DeRham complex. Formula (1.9) implies that large parts of the deformed analytic torsion are related to each other:

$$\log T_{2,\mathrm{la}}(t) - \log T_{1,\mathrm{la}}(t) = (\log T_2 - \log \tau_2) - (\log T_1 - \log \tau_1) + o(1).$$

Therefore, to prove Theorem 3.1, one has to show that

$$\mathrm{C.T.}(\log T_{2,\mathrm{la}}(t) - \log T_{1,\mathrm{la}}(t)) = 0 \tag{3.2}$$

where C.T. denotes the constant term in the asymptotic expansion. In view of (2.4), equality (3.2) follows from (with $p > n = \dim M_i$)

$$\mathrm{C.T.}(\log\det(\Delta_{2,k}(t)^p + 1) - \log\det(\Delta_{1,k}(t)^p + 1)) = 0. \tag{3.3}$$

We will apply the Mayer–Vietoris type formula (2.3) to both $\log\det(\Delta_{1,k}(t)^p + 1)$ and $\log\det(\Delta_{2,k}(t)^p + 1)$. Let Γ_1 (Γ_2) be the union of boundaries of neighborhoods U_j (V_j). Denote by M_{1+} (M_{2+}) the union of closures of U_j (V_j), and let $M_{i-} = \overline{M_i \setminus M_{i+}}$. Let $R_{i,k}(t)$, $i = 1, 2$, be the Dirichlet-to-Neumann operator that corresponds to $A = \Delta_{i,k}(t)$, $\alpha = 1$, and p as in (3.3) (cf. also(2.3)), and let $c_{i,k}(t)$ be the corresponding constants in (2.3).

Our first observation is that

$$c_{1,k}(t) = c_{2,k}(t) \tag{3.4}$$

because the constant $\log c$ in (2.3) is given by local expressions, and the operators $\Delta_{i,k}(t)^p + 1$, restricted to small neighborhoods of Γ_i are isomorphic.

Secondly, the operators $R_{i,k}(t)$ are elliptic pseudodifferential operators with parameter, and, therefore, $\log\det R_{i,k}(t)$ admit asymptotic expansions as $t \to \infty$ ([BFK], [L]). Coefficients in these expansions are determined by the complete symbols of $R_{i,k}(t)$. These symbols are equal. Hence

$$\log\det R_{1,k}(t) - \log\det R_{2,k}(t) = o(1). \tag{3.5}$$

Next, for $i = 1, 2$,

$$\log\det(\Delta_{i,k}(t)^p + 1)_D = \log\det(\Delta^+_{i,k}(t)^p + 1)_D + \log\det(\Delta^-_{i,k}(t)^p + 1)_D \tag{3.6}$$

where the superscripts $\pm$ indicate that the corresponding operator is restricted to $M_{i\pm}$. As the operators $(\Delta^+_{i,k}(t)^p + 1)_D$, $i = 1, 2$, are unitarily equivalent, their determinants are equal.

Summarizing (3.4)–(3.6), and using (2.3), one obtains

$$\begin{aligned}\log\det(\Delta_{2,k}(t)^p + 1) - \log\det(\Delta_{1,k}(t)^p + 1) &= \\ \log\det(\Delta^-_{2,k}(t)^p + 1)_D - \log\det(\Delta^-_{1,k}(t)^p + 1)_D &+ o(1),\end{aligned}$$

so (3.3) is equivalent to

$$\text{C.T.}(\log\det(\Delta^-_{2,k}(t)^p + 1)_D - \log\det(\Delta^-_{1,k}(t)^p + 1)_D) = 0. \tag{3.7}$$

Both terms in (3.7) have an asymptotic expansion when $t \to \infty$ ([BFK]). The coefficients of these expansions are given as a sum of two terms. The first term is the integral over the cosphere bundle S^*M_{i-} of a polynomial in components of the complete symbol of $\Delta^-_{i,k}$ and their derivatives. ¿From the formula for the constant term in the asymptotic expansion of $\log\det$ ([BFK]) one can easily derive that the contribution to the constant term from the interior equals 0 for manifolds of odd dimension as in this case, the integrand is an odd function in the dual variables. The second term is the integral over Γ_i of an expression that is determined by the complete symbol of $\Delta^-_{i,k}$ in an infinitesimal neighborhood of Γ_i. Because the operators $\Delta^-_{i,k}$ are isomorphic in small neighborhoods of Γ_i, the contributions from Γ_i to $\text{C.T.}(\log\det(\Delta^-_{i,k}(t)^p + 1)_D)$ are equal for $i = 1$ and 2. This shows that (3.7) holds. This concludes the proof of Theorem 3.1 in the special case where $g_i = g'_i$.

To prove Theorem 3.1 in full generality one needs the metric anomaly of the analytic torsion. Let $E \to M$ be a flat Hilbert bundle, g_1 and g_2 be two Riemannian metrics on M, and (h, g') be a pair consisting of a Morse function and a Riemannian metric that satisfies all assumptions of section 1. Denote by T_i $(i = 1, 2)$ the analytic torsions of (M, E, g_i) and by $T_{m,i}$ $(i = 1, 2)$ the correction terms (metric part of the Reidemeister torsion) introduced in section 0.

Using similar arguments as in section 1, one can show that

$$\log T_1 - \log T_2 = \log T_{m,1} - \log T_{m,2}.$$

This result is known as the metric anomaly of the analytic torsion and was first established for flat Hermitian bundles by Ray–Singer [RS]. Their proof works for flat Hilbert bundles as well (cf. [BFKM], Lemma 6.11, Appendix).

To finish the proof of the equality of the analytic torsion and the Reidemeister torsion, we use the product formulae for the analytic and the Reidemeister torsion (to simplify the writing we suppress any reference to bundles)

$$\log T(M_1 \times M_2) = \chi(M_2) \log T(M_1) + \chi(M_1) \log T(M_2)$$

and

$$\log \tau(M_1 \times M_2) = \chi(M_2) \log \tau(M_1) + \chi(M_1) \log \tau(M_2)$$

where $\chi(M_j)$ is the Euler characteristic of $E_j \to M_j$. We apply these formulae when $M_1 := M$, and M_2 is either S^6 or $S^3 \times S^3$ with E_2 being the trivial line bundle. One derives

$$\log T(M \times S^6) = 2 \log T(M), \quad \log \tau(M \times S^6) = 2 \log \tau(M) \tag{3.8}$$

and

$$\log T(M \times S^3 \times S^3) = \log \tau(M \times S^3 \times S^3) = 0. \tag{3.9}$$

As $M \times S^6$ and $M \times S^3 \times S^3$ are manifolds of the same odd dimension, one can construct self-indexing Morse functions h_1 and h_2 on them so that they have equal numbers of critical points for each index (cf. [Mi]). Choose pairs (h_1, g_1') and (h_2, g_2') that satisfy all assumptions of section 1. Theorem 3.1, (3.8), and (3.9) imply that $T(M) = \tau(M)$.

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